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On Remoteness Functions of k-NIM with $k+1$ Piles in Normal and in Misère Versions

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Abstract: Given integer n and k such that $0 < k \leq n$ and n piles of stones, two players alternate turns. On each move, a player is allowed to choose any k piles and remove exactly one stone from each. The player who has to move but cannot is the loser in the normal version of the game and (s)he is the winner in the misère version. Cases $k = 1$ and $k = n$ are trivial. For $k = 2$, the game was solved for $n \leq 6$. For $n \leq 4$, the Sprague–Grundy function was efficiently computed (for both versions). For $n = 5, 6$, a polynomial algorithm computing P-positions was obtained for the normal version. Then, for the case $k = n - 1$, a very simple explicit rule that determines the Smith remoteness function was found for the normal version of the game: the player who has to move keeps a pile with the minimum even number of stones; if all piles have an odd number of stones, then (s)he keeps a maximum one, while the $n - 1$ remaining piles are reduced by one stone each in accordance with the rules of the game. Computations show that the same rule works efficiently for the misère version too. The exceptions are sparse. We list some. Denote a position by $x = (x_1, \ldots, x_n)$. Due to symmetry, we can assume wlog that $x_1 \leq \ldots \leq x_n$. Our computations partition all exceptions into the following three families: x_1 is even, $x_1 = 1$, and odd $x_1 \geq 3$. In all three cases, we suggest formulas covering all found exceptions, but it is not proven that there are no others.

Keywords: impartial game theory; Sprague–Grundy and remoteness functions; exact slow NIM

MSC: 91A05; 91A46; 91A68

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1. Introduction

We assume that the reader is familiar with basic concepts of impartial game theory (see e.g., [1–4] for an introduction) and also with the recent paper [5], where the normal version of game NIM(n, k), the exact slow NIM, was analyzed for the case $n = k + 1$. Here, we consider the misère version for this case.

1.1. Exact Slow NIM

Game Exact Slow NIM was introduced in [6] as follows: Given two integers n and k such that $0 < k \leq n$, and n piles containing $x_1, \ldots, x_n$ stones each. On each move, a player is allowed to reduce any k piles by exactly one stone each. Two players alternate turns. A player who has to move but cannot is the loser in the normal version of the game and (s)he is the winner in the misère version. In [6], this game was denoted NIM$^1_=(n, k)$. Here, we will simplify this notation to NIM(n, k).

Game NIM(n, k) is trivial if $k = 1$ or $k = n$. In the first case, it ends after $x_1 + \ldots + x_n$ moves and in the second one—after $\min(x_1, \ldots, x_n)$ moves. In both cases, nothing depends on the players' skills. All other cases are more complicated.

The game was solved for $k = 2$ and $n \leq 6$. In [7], an explicit formula for the Sprague–Grundy (SG) function was found for $n \leq 4$, for both the normal and misère versions. This formula allows us to compute the SG function in linear time. Then, in [8], the P-positions of the normal version were found for $n \leq 6$. For the subgame where $x_1 + \ldots + x_n$ is even, a simple formula for the P-positions was obtained, allowing verifification in linear time if x is a P-position and, if not, finding a move from it to a P-position. The subgame with odd $x_1 + \ldots + x_n$ is more difficult. Still, a (more sophisticated) formula for the P-positions was found, providing a linear time recognition algorithm.

Further generalizations of exact slow NIM were considered in [9].

1.2. Case $n = k + 1$, the Normal Version

In [5], the normal version was solved in case $n = k + 1$ by the following simple rule:

- (o) if all piles are odd, keep a largest one and reduce all others;
- (e) if there exist even piles, keep the smallest one of them and reduce all others.

This rule is well-defined and it uniquely determines a move in every position x. (Obviously, permuting the piles with the same number of stones, we keep the game unchanged.) The rule and the corresponding moves are called the *M-rule* and *M-moves*; the sequence of successive M-moves is called the M-sequence.

Obviously, $n > 1$ is required. If $n = 1$, then x_1 will reach an even value in at most one M-move, after which it will stop. Since this case is trivial, we can assume that $n > 1$ without any loss of generality (wlog).

It is also easily seen that no M-move can result in a position whose entries are all odd. Hence, for an M-sequence, part (o) of the M-rule can be applied at most once, at the beginning; after this, only part (e) works.

Given a position $x = (x_1, \ldots, x_n)$, assume that both players follow the M-rule and denote by $\mathcal{M}(x)$ the number of moves from x to a terminal position. In [5], it was proven that $\mathcal{M} = \mathcal{R}$, where $\mathcal{R}$ is the classical *remoteness function* introduced by Smith [10]. Thus, the M-rule solves the game and, furthermore, allows a player to win as quickly as possible in an N-position and to resist as long as possible in a P-position.

A polynomial algorithm computing $\mathcal{M} = \mathcal{R}$ (and in particular, the P-positions) is given, even if n is a part of the input and integers are presented in binary form.

Let us also note that an explicit formula for the P-positions is known only for $n \leq 4$, and for $n = 3$, it is already quite complicated [7] and the Appendix in [5].

1.3. Related Versions of NIM

By definition, the present game NIM(n, k) is the exact slow version of the famous Moore's NIM$_k$ [11]. In the latter game, a player, by one move, reduces arbitrarily (not necessarily by one stone) at most k piles from n.

The case $k = 1$ corresponds to the classical NIM whose P-position was found by Bouton [12] for both the normal and misère versions.

Remark 1. *Actually, the Sprague–Grundy (SG) values of NIM were also computed in Bouton's paper, although were not defined explicitly in general. This was completed later by Sprague* [13] *and Grundy* [14] *for arbitrary disjunctive compounds of impartial games; see also* [3,10].

In fact, the concept of a P-position was also introduced by Bouton in [12], *but only for the (acyclic) digraph of NIM, not for all impartial games. In its turn, this is a special case of the concept of a kernel, which was introduced for arbitrary digraphs by von Neumann and Morgenstern* [15].

Also the misère version was introduced by Bouton in [12], *but only for NIM, not for all impartial games. The latter was completed by Grundy and Smith* [16]*; see also* [3,10].

Moore [11] obtained an elegant explicit formula for the P-positions of NIM$_k$ generalizing the Bouton's case $k = 1$. Even more generally, the positions of the SG-values 0 and 1 were efficiently characterized by Jenkins and Mayberry [17]; see also Section 4 in [18].

Also in [17]; the SG function of NIM_k was computed explicitly for the case $n = k + 1$ (in addition to the case $k = 1$). In general, no explicit formula, nor even a polynomial algorithm, computing the SG-values (larger that 1) is known. The smallest open case: 2-values for $n = 4$ and $k = 2$.

The remoteness function of k-NIM was recently studied in [19].

Let us also mention the exact (but not slow) game $NIM^{=}(n, k)$ [18] in which exactly k from n piles are reduced (by an arbitrary number of stones) in a move. The SG-function was efficiently computed in [18] for $n \leq 2k$. Otherwise, even a polynomial algorithm looking for the P-positions is not known (unless $k = 1$, of course). The smallest open case is $n = 5$ and $k = 2$.

2. Case $n = k + 1$, Misère Version

Computations show that the same M-rule works pretty efficiently also for the misère version of the considered game $NIM(n, k)$ with $n = k + 1$, yet, not always. A position $x = (x_1, \ldots, x_n)$ is called an *exception* if the M-move is not optimal, in other words, if $\mathcal{R}(x) - \mathcal{R}(x') \neq 1$ for the M-move $x \to x'$.

In fact, $\mathcal{R}(x)$ is odd and $\mathcal{R}(x) - \mathcal{R}(x')$ takes values 0 or 2 for all known exceptions. ($\mathcal{R}(x)$ and $\mathcal{R}(x')$ can be equal, but both cannot be even.) The exceptions are sparse and satisfy a regular pattern based on two parameters: n and $\min(x_1, \ldots, x_n)$. However, the complete description of this pattern is open.

2.1. Monotonicity for the Entries of Positions

Recall that $x_1 \leq \ldots \leq x_n$ is assumed for any position $x = (x_1, \ldots, x_n)$ (We order the entries x_i just for convenience; their permutations do not change the game.).

However, even if this monotonicity holds for x, it may fail for x', after a move $x \to x'$. In this case, we have to restore it by permuting entries of x'.

Alternatively, we can make the M-rule slightly stricter, as follows. Given a position $x = (x_1, \ldots, x_n)$ for which the M-rule is "ambiguous"; that is, x contains

- (o) several smallest even entries, or
- (e) several largest odd entries, provided all x_i are odd.

In both cases, among these equal entries, keep one with the largest index reducing all others by 1. We will call such an M-move (as well as the corresponding M-sequence and M-rule) *strict*. It is easily seen that a strict M-move $x \to x'$ respects the non-decreasing monotonicity of the entries; that is, $(x'_1 \leq \ldots \leq x'_n)$ whenever $(x_1 \leq \ldots \leq x_n)$. In contrast, every non-strict M-move breaks this monotonicity.

2.2. Monotonicity of Exceptions

A position is called an *exception* if the M-move is not optimal in it. An optimal move in such a position is called *exceptional*.

Proposition 1. *Given integer $m \geq n$, a position $x' = (x_1, \ldots, x_n, \ldots x_m)$ is an exception whenever $x = (x_1, \ldots, x_n)$ exists; of course, not vice versa. Moreover, the exceptional moves coincide in x and x'. (More precisely, the entry x_i that is kept unchanged by an optimal move is the same for x and x'; furthermore, $1 \leq i \leq n$)*

Listing the exceptional position below, by default, we do not include x' if x is already listed; in other words, we include only the minimal exceptions.

2.3. General Properties of Exceptions

All found minimal monotone exceptions share the following properties:

- [x_n-monotone] $x_n > x_{n-1}$; if position $x^i = (x_1, \ldots, x_{n-1}, x_{n-1} + i)$ is an exception for some $i > 0$, then x^i is an exception for each $i > 0$.
- $x_n - x_{n-1} = 1$ in every minimal exception.

- [x_{n-1}-determining] $\mathcal{R}(x) = f(x_{n-1}) + 1$, where $f(\ell) = 2\lceil \ell/2 \rceil$; that is, $f(\ell) = \ell$ if ℓ is even and $f(\ell) = \ell + 1$ if ℓ is odd, for all integer $\ell \geq 0$.
  Thus, $\mathcal{R}(x)$ (and $\mathcal{R}(x')$) are odd (and, hence, the first player wins) in every exception. However, the M-move is losing. (It could win but would require a larger number of moves. Yet, such a case is not realized in any found exception.)
- In every exception x, the optimal move keeps the entry x_n if x_{n-1} is even and keeps x_{n-1} if it is odd. In contrast, the strict M-move, vice versa, keeps x_{n-1} if it is even and keeps x_n if x_{n-1} is odd.
- [$0 \leq \mathcal{R}(x) - \mathcal{R}(x') \leq 2$] By definition of the remoteness function, we have $\mathcal{R}(x) - \mathcal{R}(x') = 1$ for each optimal move $x \to x'$ in every impartial game, in particular, for every M-move in the normal version of NIM$(n, n-1)$. In contrast, for its misère version, in every found *minimal* exception, $\mathcal{R}(x) - \mathcal{R}(x')$ takes only values 0, when x_{n-1} is even, or 2, when x_{n-1} is odd.

2.4. Even x_1

Given $x_1 = 2i$, a position $x = (x_1, \ldots, x_n, \ldots x_m)$ is an exception if and only if

$$2i = x_1 = \ldots = x_{i+2} < x_{i+3} = x_n \leq \ldots \leq x_m \text{ where } 1 \leq i \leq n-3 \text{ and } n \geq 4.$$

Since x_{n-1} is always even, the optimal move keeps x_n, while the M-move keeps x_j for any fixed $j < n$. Since $\mathcal{R}(x) = 2i + 1$ is odd, the first player always wins. Note that both properties agree with Section 2.3.

Examples for $x_1 = 2, 4, 6, 8$ are given below

$$\begin{array}{lll} 2 = x_1 = x_2 = x_3 < x_4 = x_n \leq \cdots \leq x_m, & \mathcal{R}(x) = 3, & n = 4; \\ 4 = x_1 = \ldots = x_4 < x_5 = x_n \leq \cdots \leq x_m, & \mathcal{R}(x) = 5, & n = 5; \\ 6 = x_1 = \ldots = x_5 < x_6 = x_n \leq \cdots \leq x_m, & \mathcal{R}(x) = 7, & n = 6; \\ 8 = x_1 = \ldots = x_6 < x_7 = x_n \leq \cdots \leq x_m, & \mathcal{R}(x) = 9, & n = 7, \text{ etc.} \end{array}$$

2.5. $x_1 = 1$

A position $x = (x_1, \ldots, x_n, \ldots, x_m)$ with $x_1 = 1$ is an exception if and only if

$$1 = x_1 \leq x_2 < x_3 \leq \ldots \leq x_m, \text{ here } 3 = n \leq m.$$

Furthermore, if x_2 is even, then $\mathcal{R}(x) = x_2 + 1$, and the only optimal move keeps x_3, while the M-move keeps x_2; if x_2 is odd, then $\mathcal{R}(x) = x_2 + 2$, and, in contrast, the only optimal move keeps x_2, while the M-move keeps x_i, for some $i > 2$. In both cases, $\mathcal{R}(x)$ is odd and, hence, the first player always wins. It is easily seen that all these properties agree with Section 2.3.

Wlog, we could restrict ourselves by $n = 3$. All exceptions with larger n are implied by monotonicity.

Thus, it remains to consider odd values of $x_1 \geq 3$. Our computer analysis includes only $x_1 = 3, 5, 7, 9, 11, 13, 15, 17$. In each case, we observe a pattern; however, its extension to arbitrary odd x_1 remains an open problem.

2.6. Odd $x_1 \geq 5$ with $n = 4$

For any odd $x_1 \geq 5$, fix an integer $i \geq 0$ to obtain the following two exceptions $x = (x_1, x_2, x_3, x_4)$:

$$x_1, \; x_2 = x_1 + 2i, \; x_3 = 2(x_1 + i - 2), \; x_4 = 2x_1 + 2i - 3 = x_3 + 1;$$

$$x_1, \; x_2 = x_1 + 2i, \; x_3 = 2x_1 + 2i - 3, \; x_4 = 2(x_1 + i - 1) = x_3 + 1.$$

In the first case, $\mathcal{R}(x) = x_4$, and the unique optimal move keeps x_4, while the unique M-move keeps x_3; in contrast, in the second case, $\mathcal{R}(x) = x_4 + 1$, and the unique optimal move keeps x_3, while the unique M-move keeps x_4.

Furthermore, the remoteness function is given by formula $\mathcal{R}(x) = f(x_3 + 1)$, where $f(m) = 2\lfloor m/2 \rfloor + 1$; that is, $f(m) = m$ if m is odd and $f(m) = m + 1$ if m is even, for all $m \geq 0$.

It is easily seen that $\mathcal{R}(x)$ is odd for each i; hence, the first player always wins.

Examples for $x_1 = 5, 7, 9, 11$ are given below. Notation $y+$ means "any number that is greater than or equal to y".

$x_1 = 5$	$\mathcal{R}$	$x_1 = 7$	$\mathcal{R}$	$x_1 = 9$	$\mathcal{R}$	$x_1 = 11$	$\mathcal{R}$
(5, 5, 6, 7+)	7	(7, 7, 10, 11+)	11	(9, 9, 14, 15+)	15	(11, 11, 18, 19+)	19
(5, 5, 7, 8+)	9	(7, 7, 11, 12+)	13	(9, 9, 15, 16+)	17	(11, 11, 19, 20+)	21
(5, 7, 8, 9+)	9	(7, 9, 12, 13+)	13	(9, 11, 16, 17+)	17	(11, 13, 20, 21+)	21
(5, 7, 9, 10+)	11	(7, 9, 13, 14+)	15	(9, 11, 17, 18+)	19	(11, 13, 21, 22+)	23
(5, 9, 10, 11+)	11	(7, 11, 14, 15+)	15	(9, 13, 18, 19+)	19	(11, 15, 22, 23+)	23
(5, 9, 11, 12+)	13	(7, 11, 15, 16+)	17	(9, 13, 19, 20+)	21	(11, 15, 23, 24+)	25

By monotonicity, any such exception $x = (x_1, x_2, x_3, x_4)$ can be extended to the exceptions $x' = (x'_1, \ldots, x'_m)$, with $m \geq 5$ and $x_i = x'_i$ for $i \leq 4$, while $x'_5, \ldots x'_m$ can be chosen arbitrary such that $x_4 \leq x'_5 \leq \ldots \leq x'_m$. Note also that case $x_1 = 3$ is considered in Section 2.9.

2.7. *Odd $x_1 \geq 7$ with $n = 5$*

The following families of exceptions were found:

$x_1 = 7$	$x_1 = 9$	$x_1 = 11$	$x_1 = 13$
(7, 7, 8, 8, 9)	(9, 9, 12, 12, 13)	(11, 11, 16, 16, 17)	(13, 13, 20, 20, 21)
(7, 7, 9, 9, 10)	(9, 9, 13, 13, 14)	(11, 11, 17, 17, 18)	(13, 13, 21, 21, 22)
(7, 9, 10, 10, 11)	(9, 11, 14, 14, 15)	(11, 13, 18, 18, 19)	(13, 15, 22, 22, 23)
(7, 9, 11, 11, 12)	(9, 11, 15, 15, 16)	(11, 11, 19, 19, 20)	(13, 15, 23, 23, 24)
(7, 11, 12, 12, 13)	(9, 13, 16, 16, 17)	(11, 15, 20, 20, 21)	(13, 17, 24, 24, 25)

$x_1 = 9$	$x_1 = 13$	$x_1 = 17$	$x_1 = 21$
(9, 9, 9, 10, 11+)	(13, 13, 13, 16, 17+)	(17, 17, 17, 22, 23+)	(21, 21, 21, 28, 29+)
(9, 9, 9, 11, 12+)	(13, 13, 13, 17, 18+)	(17, 17, 17, 23, 24+)	(21, 21, 21, 29, 30+)
(9, 11, 11, 12, 13+)	(13, 15, 15, 18, 19+)	(17, 19, 19, 24, 25+)	(21, 23, 23, 30, 31+)
(9, 11, 11, 13, 14+)	(13, 15, 15, 19, 20+)	(17, 19, 19, 25, 26+)	(21, 23, 23, 31, 32+)
(9, 13, 13, 14, 15+)	(13, 17, 17, 20, 21+)	(17, 21, 21, 26, 27+)	(21, 25, 25, 32, 33+)

$x_1 = 11$	$x_1 = 13$	$x_1 = 15$	$x_1 = 17$
(11, 11, 13, 14, 15+)	(13, 13, 17, 18, 19+)	(15, 15, 21, 22, 23+)	(17, 17, 25, 26, 27+)
(11, 11, 13, 15, 16+)	(13, 13, 17, 19, 20+)	(15, 15, 21, 23, 24+)	(17, 17, 25, 27, 28+)
(11, 13, 15, 16, 17+)	(13, 15, 19, 20, 21+)	(15, 17, 23, 24, 25+)	(17, 19, 27, 28, 29+)
(11, 13, 15, 17, 18+)	(13, 15, 19, 21, 22+)	(15, 17, 23, 25, 26+)	(17, 19, 27, 29, 30+)
(11, 15, 17, 18, 19+)	(13, 17, 21, 22, 23+)	(15, 19, 25, 26, 27+)	(17, 21, 29, 30, 31+)

$x_1 = 15$	$x_1 = 17$	$x_1 = 19$	$x_1 = 21$
(15, 15, 17, 20, 21+)	(17, 17, 21, 24, 25+)	(19, 19, 25, 28, 29+)	(21, 21, 29, 32, 33+)
(15, 15, 17, 21, 22+)	(17, 17, 21, 25, 26+)	(19, 19, 25, 29, 30+)	(21, 21, 29, 33, 34+)
(15, 17, 19, 22, 23+)	(17, 19, 23, 26, 27+)	(19, 21, 27, 30, 31+)	(21, 23, 31, 34, 35+)
(15, 17, 19, 23, 24+)	(17, 19, 23, 27, 28+)	(19, 21, 27, 31, 32+)	(21, 23, 31, 35, 36+)
(15, 19, 21, 24, 25+)	(17, 21, 25, 28, 29+)	(19, 23, 29, 32, 33+)	(21, 25, 33, 36, 37+)

2.8. *Odd $x_1 \geq 9$ with $n = 6$*

The following families of exceptions were found:

$x_1 = 9$	$x_1 = 11$	$x_1 = 13$	$x_1 = 15$
(9, 9, 10, 10, 10, 11+)	(11, 11, 14, 14, 14, 15+)	(13, 13, 18, 18, 18, 19+)	(15, 15, 22, 22, 22, 23+)
(9, 9, 11, 11, 11, 12+)	(11, 11, 15, 15, 15, 16+)	(13, 13, 19, 19, 19, 20+)	(15, 15, 23, 23, 23, 24+)
(9, 11, 12, 12, 12, 13+)	(11, 13, 16, 16, 16, 17+)	(13, 15, 20, 20, 20, 21+)	(15, 17, 24, 24, 24, 25+)
(9, 11, 13, 13, 13, 14+)	(11, 13, 17, 17, 17, 18+)	(13, 15, 21, 21, 21, 22+)	(15, 17, 25, 25, 25, 26+)
(9, 13, 14, 14, 14, 15+)	(11, 15, 18, 18, 18, 19+)	(13, 17, 22, 22, 22, 23+)	(15, 19, 26, 26, 26, 27+)

$x_1 = 11$	$x_1 = 15$	$x_1 = 19$	$x_1 = 23$
(11, 11, 11, 12, 12, 13+)	(15, 15, 15, 18, 18, 19+)	(19, 19, 19, 24, 24, 25+)	(23, 23, 23, 30, 30, 31+)
(11, 11, 11, 13, 13, 14+)	(15, 15, 15, 19, 19, 20+)	(19, 19, 19, 25, 25, 26+)	(23, 23, 23, 31, 31, 32+)
(11, 13, 13, 14, 14, 15+)	(15, 17, 17, 20, 20, 21+)	(19, 21, 21, 26, 27, 27+)	(23, 25, 25, 32, 32, 33+)
(11, 13, 13, 15, 16, 16+)	(15, 17, 17, 21, 21, 22+)	(19, 21, 21, 27, 27, 28+)	(23, 25, 25, 33, 33, 34+)
(11, 15, 15, 16, 16, 17+)	(15, 19, 19, 22, 22, 22+)	(19, 23, 23, 28, 29, 29+)	(23, 27, 27, 34, 34, 35+)

$x_1 = 15$	$x_1 = 17$	$x_1 = 19$
(15, 15, 17, 17, 18, 19+)	(17, 17, 21, 21, 22, 23+)	(19, 19, 25, 25, 26, 27+)
(15, 15, 17, 17, 19, 20+)	(17, 17, 21, 21, 23, 24+)	(19, 19, 25, 25, 27, 28+)
(15, 17, 19, 19, 20, 21+)	(17, 19, 23, 23, 24, 25+)	(19, 21, 27, 27, 28, 29+)
(15, 17, 19, 19, 21, 22+)	(17, 19, 23, 23, 25, 26+)	(19, 21, 27, 27, 29, 30+)
(15, 19, 21, 21, 22, 23+)	(17, 21, 25, 25, 26, 27+)	(19, 23, 29, 29, 30, 31+)

$x_1 = 13$	$x_1 = 15$	$x_1 = 17$	$x_1 = 19$
(13, 13, 15, 16, 16, 17+)	(15, 15, 19, 20, 20, 21+)	(17, 17, 23, 24, 24, 25+)	(19, 19, 27, 28, 28, 29+)
(13, 13, 15, 17, 17, 18+)	(15, 15, 19, 21, 21, 22+)	(17, 17, 23, 25, 25, 26+)	(19, 19, 27, 29, 29, 30+)
(13, 15, 17, 18, 18, 19+)	(15, 17, 21, 22, 22, 23+)	(17, 19, 25, 26, 26, 27+)	(19, 21, 29, 30, 30, 31+)
(13, 15, 17, 19, 19, 20+)	(15, 17, 21, 23, 23, 24+)	(17, 19, 25, 27, 28, 28+)	(19, 21, 29, 31, 31, 32+)
(13, 17, 18, 20, 20, 21+)	(15, 19, 21, 24, 24, 25+)	(17, 21, 27, 28, 28, 29+)	(19, 23, 31, 32, 32, 33+)

$x_1 = 13$	$x_1 = 19$	$x_1 = 25$
(13, 13, 13, 13, 14, 15+)	(19, 19, 19, 19, 22, 23+)	(25, 25, 25, 25, 30, 31+)
(13, 13, 13, 13, 15, 16+)	(19, 19, 19, 19, 23, 24+)	(25, 25, 25, 25, 31, 32+)
(13, 15, 15, 15, 16, 17+)	(19, 21, 21, 21, 24, 25+)	(25, 27, 27, 27, 32, 33+)
(13, 15, 15, 15, 17, 18+)	(19, 21, 21, 21, 25, 26+)	(25, 27, 27, 27, 33, 34+)
(13, 17, 17, 17, 18, 19+)	(19, 23, 23, 23, 26, 27+)	(25, 29, 29, 29, 34, 35+)

Note that the second and last families are defined only for $x_1 = 4m + 3$ and $x_1 = 6m + 1$, respectively, where $m \geq 3$. This complicated pattern shows that it is hardly possible to combine all exceptions by a formula.

2.9. Odd $x_1 \geq 3$ with $n = \frac{1}{2}(x_1 + 1) + 2$

For $n - \frac{1}{2}(x_1 + 1) > 2$, no exceptions were found, while in the considered case, the exceptions are as follows. Given an integer $i \geq 0$, a position $x = (x_1, \ldots, x_n)$ is an exception if and only if

$$x_1 + i = x_2 = \ldots = x_{n-1} < x_n \leq \cdots \leq x_m, \; i = 0, 1, \ldots$$

Furthermore, the remoteness function is given by the formula $\mathcal{R}(x) = f(x_1 + i + 1)$, where function f was defined in Section 2.3.

Note that $x_2 = \ldots = x_{n-1}$ and this number is even if and only if i is odd.

Examples for $x_1 = 3, 5, 7, 9, 11$ and, respectively, $n = 4, 5, 6, 7, 8$ are given below.

$x_1 = 3, n = 4$	$\mathcal{R}$	$x_1 = 5, n = 5$	$\mathcal{R}$	$x_1 = 7, n = 6$	$\mathcal{R}$
(3, 3, 3, 4+)	5	(5, 5, 5, 5, 6+)	7	(7, 7, 7, 7, 7, 8+)	9
(3, 4, 4, 5+)	5	(5, 6, 6, 6, 7+)	7	(7, 8, 8, 8, 8, 9+)	9
(3, 5, 5, 6+)	7	(5, 7, 7, 7, 8+)	9	(7, 9, 9, 9, 9, 10+)	11
(3, 6, 6, 7+)	7	(5, 8, 8, 8, 9+)	9	(7, 10, 10, 10, 10, 11+)	11
(3, 7, 7, 8+)	9	(5, 9, 9, 9, 10+)	11	(7, 11, 11, 11, 11, 12+)	13

$x_1 = 9, n = 7$	$\mathcal{R}$	$x_1 = 11, n = 8$	$\mathcal{R}$
(9, 9, 9, 9, 9, 9, 10+)	11	(11, 11, 11, 11, 11, 11, 11, 12+)	13
(9, 10, 10, 10, 10, 10, 11+)	11	(11, 12, 12, 12, 12, 12, 12, 13+)	13
(9, 11, 11, 11, 11, 11, 12+)	13	(11, 13, 13, 13, 13, 13, 13, 14+)	15
(9, 12, 12, 12, 12, 12, 13+)	13	(11, 14, 14, 14, 14, 14, 14, 15+)	15
(9, 13, 13, 13, 13, 13, 14+)	15	(11, 15, 15, 15, 15, 15, 15, 16+)	17

2.10. Odd $x_1 \geq 5$ with $n = \frac{1}{2}(x_1 + 1) + 1$

Given an integer $i \geq 0$, a position $x = (x_1, \ldots, x_n, \ldots, x_m)$ is an exception if and only if one of the following two cases holds:

$$x_1, \; x_2 = x_1 + 2i, \; x_3 = \cdots = x_{n-1} = x_2 + 1 < x_n \leq \cdots \leq x_m,$$

$$x_1, \; x_2 = x_1 + 2i, \; x_3 = \cdots = x_{n-1} = x_2 + 2 < x_n \leq \cdots \leq x_m.$$

Furthermore, the remoteness function is given by formula $\mathcal{R}(x) = f(x_n)$, where function f is defined above. Again, it is easily seen that $\mathcal{R}(x)$ is odd for each i; hence, the first player wins in every exceptional position, but (s)he loses if (s)he follows the M-rule.

Finally, the unique optimal move in x is to keep x_n when x_{n-1} is even and x_{n-1} when it is odd. In contrast, the unique M-move in x is to keep x_n when x_{n-1} is odd and

x_{n-1} when it is even. Thus, the sets of optimal moves and M-moves are disjoint in every exceptional position.

Note that $x_3 = \ldots = x_{n-1}$ and this number is even if and only if i is odd.

Examples for $x_1 = 5, 7, 9, 11, 13$ and, respectively, $n = 4, 5, 6, 7, 8$ are given below.

$x_1 = 5, n = 4$	$\mathcal{R}$	$x_1 = 7, n = 5$	$\mathcal{R}$	$x_1 = 9, n = 6$	$\mathcal{R}$
(5, 5, 6, 7+)	7	(7, 7, 8, 8, 9+)	9	(9, 9, 10, 10, 10, 11+)	11
(5, 5, 7, 8+)	9	(7, 7, 9, 9, 10+)	11	(9, 9, 11, 11, 11, 12+)	13
(5, 7, 8, 9+)	9	(7, 9, 10, 10, 11+)	11	(9, 11, 12, 12, 12, 13+)	13
(5, 7, 9, 10+)	11	(7, 9, 11, 11, 12+)	13	(9, 11, 13, 13, 13, 14+)	15
(5, 9, 10, 11+)	11	(7, 11, 12, 12, 13+)	13	(9, 13, 14, 14, 14, 15+)	15

$x_1 = 11, n = 7$	$\mathcal{R}$	$x_1 = 13, n = 8$	$\mathcal{R}$
(11, 11, 12, 12, 12, 12, 13+)	13	(13, 13, 14, 14, 14, 14, 14, 15+)	15
(11, 11, 13, 13, 13, 13, 14+)	15	(13, 13, 15, 15, 15, 15, 15, 16+)	17
(11, 13, 14, 14, 14, 14, 15+)	15	(13, 15, 16, 16, 16, 16, 16, 17+)	17
(11, 13, 15, 15, 15, 15, 16+)	17	(13, 15, 17, 17, 17, 17, 17, 18+)	19
(11, 15, 16, 16, 16, 16, 17+)	17	(13, 17, 18, 18, 18, 18, 18, 19+)	19

2.11. *Odd $x_1 \geq 7$ with $n = \frac{1}{2}(x_1 + 1)$*

Given an integer $i \geq 0$, a position $x = (x_1, \ldots, x_n, \ldots, x_m)$ is an exception if and only if one of the following two cases holds:

$$x_1,\ x_2 = x_1 + 2i,\ x_3 = \ldots = x_{n-1} = x_2 + 3 < x_n \leq \ldots \leq x_m,$$

$$x_1,\ x_2 = x_1 + 2i,\ x_3 = \ldots = x_{n-1} = x_2 + 4 < x_n \leq \ldots \leq x_m.$$

Interestingly, all further arguments can be copied from the previous subsection without any changes; however, we should remember that n is reduced by 1.

Examples for $x_1 = 7, 9, 11, 13, 15$ and, respectively, $n = 4, 5, 6, 7, 8$ follow.

$x_1 = 7, n = 4$	$\mathcal{R}$	$x_1 = 9, n = 5$	$\mathcal{R}$	$x_1 = 11, n = 6$	$\mathcal{R}$
(7, 7, 10, 11+)	11	(9, 9, 12, 12, 13+)	13	(11, 11, 14, 14, 14, 15+)	15
(7, 7, 11, 12+)	13	(9, 9, 13, 13, 14+)	15	(11, 11, 15, 15, 15, 16+)	17
(7, 9, 12, 13+)	13	(9, 11, 14, 14, 15+)	15	(11, 13, 16, 16, 16, 17+)	17
(7, 9, 13, 14+)	15	(9, 11, 15, 15, 16+)	17	(11, 13, 17, 17, 17, 18+)	19
(7, 11, 14, 15+)	15	(9, 13, 16, 16, 17+)	17	(11, 15, 18, 18, 18, 19+)	19

$x_1 = 13, n = 7$	$\mathcal{R}$	$x_1 = 15, n = 8$	$\mathcal{R}$
(13, 13, 16, 16, 16, 16 ,17+)	17	(15, 15, 18, 18, 18, 18, 18, 19+)	19
(13, 13, 17, 17, 17, 17, 18+)	19	(15, 15, 19, 19, 19, 19, 19, 20+)	21
(13, 15, 18, 18, 18, 18, 19+)	19	(15, 17, 20, 20, 20, 20, 20, 21+)	21
(13, 15, 19, 19, 19, 19, 20+)	21	(15, 17, 21, 21, 21, 21, 21, 22+)	23
(13, 17, 20, 20, 20, 20, 21+)	21	(15, 19, 22, 22, 22, 22, 22, 23+)	23

There exists another family of exceptions for $x_1 \geq 9$ with $n = \frac{1}{2}(x_1 + 1)$.

$$x_1,\ x_2 = x_3 = x_1 + 2i,\ x_4 = \ldots = x_{n-1} = x_2 + 3 < x_n \leq \ldots \leq x_m,$$

$$x_1,\ x_2 = x_3 = x_1 + 2i,\ x_4 = \ldots = x_{n-1} = x_2 + 4 < x_n \leq \ldots \leq x_m.$$

Examples for $x_1 = 9, 11, 13$ and, respectively, $n = 5, 6, 7$ follow.

$x_1 = 9, n = 5$	$\mathcal{R}$	$x_1 = 11, n = 6$	$\mathcal{R}$	$x_1 = 13, n = 7$	$\mathcal{R}$
(9, 9, 9, 10, 11+)	11	(11, 11, 11, 12, 12, 13+)	13	(13, 13, 13, 14, 14, 14, 15+)	15
(9, 9, 9, 11, 12+)	13	(11, 11, 11, 13, 13, 14+)	15	(13, 13, 13, 15, 15, 15, 16+)	17
(9, 11, 11, 12, 13+)	13	(11, 13, 13, 14, 14, 15+)	15	(13, 15, 15, 16, 16, 16, 17+)	17
(9, 11, 11, 13, 14+)	15	(11, 13, 13, 15, 15, 16+)	17	(13, 15, 15, 17, 17, 17, 18+)	19
(9, 13, 13, 14, 15+)	15	(11, 15, 15, 16, 16, 17+)	17	(13, 17, 17, 18, 18, 18, 19+)	19

2.12. *Odd $x_1 \geq 9$ with $n = \frac{1}{2}(x_1 + 1) - 1$*

Exceptions for $x_1 = 9, 11, 13, 15$ and, respectively, $n = 4, 5, 6, 7$ follow:

$x_1 = 9, n = 4$	$\mathcal{R}$	$x_1 = 11, n = 5$	$\mathcal{R}$	$x_1 = 13, n = 6$	$\mathcal{R}$
(9, 9, 14, 15+)	15	(11, 11, 13, 14, 15+)	15	(13, 13, 13, 13, 14, 15+)	15
(9, 9, 15, 16+)	17	(11, 11, 13, 15, 16+)	17	(13, 13, 13, 13, 15, 16+)	17
(9, 11, 16, 17+)	17	(11, 13, 15, 16, 17+)	17	(13, 15, 15, 15, 16, 17+)	17
(9, 11, 17, 18+)	19	(11, 13, 15, 17, 18+)	19	(13, 15, 15, 15, 17, 18+)	19
(9, 13, 18, 19+)	19	(11, 15, 17, 18, 19+)	19	(13, 17, 17, 17, 18, 19+)	19

$x_1 = 13, n = 6$	$\mathcal{R}$	$x_1 = 15, n = 7$	$\mathcal{R}$
(13, 13, 15, 16, 16, 17+)	17	(15, 15, 15, 15, 16, 16, 17+)	17
(13, 13, 15, 17, 17, 18+)	19	(15, 15, 15, 15, 17, 17, 18+)	19
(13, 15, 17, 18, 18, 19+)	19	(15, 17, 17, 17, 18, 18, 19+)	19
(13, 15, 17, 19, 19, 20+)	21	(15, 17, 17, 17, 19, 19, 20+)	21
(13, 17, 19, 20, 20, 21+)	21	(15, 19, 19, 19, 20, 20, 21+)	21

2.13. Odd $x_1 \geq 11$ with $n = \frac{1}{2}(x_1 + 1) - 2$

For $n = (x_1 + 1)/2 - 2$, we obtained the following exceptions:

$x_1 = 11, n = 4$	$\mathcal{R}$	$x_1 = 13, n = 5$	$\mathcal{R}$	$x_1 = 15, n = 6$
(11, 11, 18, 19+)	19	(13, 13, 20, 20, 21+)	21	(15, 15, 22, 22, 22, 23+)
(11, 11, 19, 20+)	21	(13, 13, 21, 21, 22+)	23	(15, 17, 23, 23, 23, 24+)
(11, 13, 20, 21+)	21	(13, 15, 22, 22, 23+)	23	(15, 17, 24, 24, 24, 25+)
(11, 13, 21, 22+)	23	(13, 15, 23, 23, 24+)	25	(15, 19, 25, 25, 25, 26+)
(11, 15, 22, 23+)	23	(13, 17, 24, 24, 25+)	25	(15, 19, 26, 26, 26, 27+)

In addition, the following exceptions were found:

$x_1 = 13, n = 5$	$\mathcal{R}$
(13, 13, 13, 16, 17+)	17
(13, 13, 13, 17, 18+)	19
(13, 15, 15, 18, 19+)	19
(13, 15, 15, 19, 20+)	21
(13, 17, 17, 20, 21+)	21

$x_1 = 15, n = 6$
(15, 15, 23, 23, 23, 24+)
(15, 17, 24, 24, 24, 25+)
(15, 17, 25, 25, 25, 26+)
(15, 19, 26, 26, 26, 27+)
(15, 19, 27, 27, 27, 28+)

$x_1 = 17, n = 7$
(17, 17, 17, 17, 17, 18, 19+)
(17, 17, 17, 17, 17, 19, 20+)

2.14. Odd $x_1 \geq 13$ with $n = \frac{1}{2}(x_1 + 1) - 3$

Exceptions for $x_1 = 13, 15, 17$ and, respectively, $n = 4, 5, 6$ follow:

$x_1 = 13, n = 4$	$x_1 = 15, n = 5$	$x_1 = 17, n = 6$
(13, 13, 22, 23+)	(15, 15, 24, 24, 25+)	(17, 17, 26, 26, 26, 27+)
(13, 13, 23, 24+)	(15, 15, 25, 25, 26+)	(17, 17, 27, 27, 27, 28+)
(13, 15, 24, 25+)	(15, 17, 26, 26, 27+)	(17, 19, 28, 28, 28, 29+)
(13, 15, 25, 26+)	(15, 17, 27, 27, 28+)	(17, 19, 29, 29, 29, 30+)
(13, 17, 26, 27+)	(15, 19, 28, 28, 29+)	(17, 21, 30, 30, 30, 31+)

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References

1. Albert, M.H.; Nowakowski, R.J.; Wolfe, D. *Lessons in Play: An Introduction to Combinatorial Game Theory*, 2nd ed.; A. K. Peters Ltd.: Wellesley, MA, USA, 2007.
2. Berlekamp, E.R.; Conway, J.H.; Guy, R.K. *Winning Ways for Your Mathematical Plays*, 2nd ed.; A.K. Peters: Natick, MA, USA, 2001–2004; Volumes 1–4.
3. Conway, J.H. *On Numbers and Games*; Academic Press: London, UK; New York, NY, USA; San Francisco, CA, USA, 1976.
4. Siegel, A.N. *Combinatorial Game Theory*; American Mathematical Society: Providence, RI, USA, 2023; Volume 146.

5. Gurvich, V.; Martynov, D.; Maximchuk, V.; Vyalyi, M. On remoteness functions of exact slow k-NIM with $k+1$ piles. *Integers* **2024**, #G1. [CrossRef]
6. Gurvich, V.; Ho, N.B. Slow k-Nim. *RUTCOR Res. Rep.* **2015**, *3*, 5777. [CrossRef]
7. Gurvich, V.; Heubach, S.; Ho, N.B.; Chikin, N. Slow k-nim. *Integers* **2020**, *20*, #G3.
8. Chikin, N.; Gurvich, V.; Knop, K.; Paterson, M.; Vyalyi, M. More about exact slow k-nim. *Integers* **2021**, *21*, #G4.
9. Gurvich, V.; Naumova, M. Screw Discrete Dynamical Systems and their Applications to Exact Slow NIM. *Discrete Appl. Math.* **2024**, *358*, 382–394. [CrossRef]
10. Smith, C.A.B. Graphs and composite games. *J. Comb. Theory* **1966**, *1*, 51–81. [CrossRef]
11. Moore, E.H. A generalization of the game called Nim. *Ann. Math.* **1910**, *11*, 93–94. [CrossRef]
12. Bouton, C.L. Nim, a Game with a Complete Mathematical Theory. *Ann. Math.* **1901–1902**, *3*, 35–39. [CrossRef]
13. Sprague, R. Über mathematische Kampfspiele. *Tohoku Math. J.* **1936**, *41*, 438–444.
14. Grundy, P.M. Mathematics of games. *Eureka* **1939**, *2*, 6–8.
15. von Neumann, J.; Morgenstern, O. *Theory of Games and Economic Behavior*; Princeton University Press: Princeton, NJ, USA, 1944.
16. Grundy, P.M.; Smith, C.A.B. Disjunctive games with the last player losing. *Math. Proc. Camb. Philos. Soc.* **1956**, *52*, 527–533. [CrossRef]
17. Jenkyns, T.A.; Mayberry, J.P. The skeletion of an impartial game and the nim-function of Moore's nim_k. *Int. J. Game Theory* **1980**, *9*, 51–63. [CrossRef]
18. Boros, E.; Gurvich, V.; Ho, N.B.; Makino, K.; Mursič, P. Sprague–Grundy function of matroids and related hypergraphs. *Theor. Comput. Sci.* **2019**, *799*, 40–58. [CrossRef]
19. Boros, E.; Gurvich, V.; Makino, K.; Vyalyi, M. Computing Remoteness Functions of Moore, Wythoff, and Euclid's Games. *Int. J. Game Theory* **2023**, *22*, 02685. [CrossRef]