games

Article

On the Stability of Couples

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Abstract: In this article, we analyze the stability of couples on the marriage market. Recent developments in cooperative game theory allow a new model that uses team games which make it possible to model the marriage market. Coalition structures can model couples. We analyze two cases: a symmetrical one with only one type of men and one type of women; and one with several types of women and men.

Keywords: team games; χ-value; stability of couples; marriage market

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1. Introduction

In this article, we analyze the stability of couples on the marriage market. Recent developments in cooperative game theory allow a new model that uses team games [1] which make it possible to model the marriage market. Coalition structures can be used to model couples. We analyze two marriage market cases: a symmetrical one with only one type of men and one type of women; and one with several types of women and men. This differentiation could be interpreted in terms of economic success and/or attractiveness of the players.

Team games can have a non-zero worth only for coalitions with certain cardinality. For example in football, teams have eleven players. Only teams with this size could get a higher score than zero [2,3]. In our model, we assume that there are more than two players on the market, but only groups with two players (couples) get a higher worth then zero. To model couples, coalition structures are used. These divide players into disjointed components. The most popular value for games with a coalition structure (CS games) was introduced by [4]. According to them, components are active groups.[1] The Aumann-Drèze value assumes component efficiency, meaning that the worth of the component—the happiness of the couple—is divided between the players of the component. However, the outside options of players have no bearing on their payoffs. This is unrealistic. With this in mind, the Wiese value [6] was introduced. This value is component efficient, reflecting the outside options of the players on the market. The better a player's outside options (possibilities to other couples), the higher its influence in the couple. Inspired by the Wiese value, the χ-value [7] was presented. The main advantage of the χ-value with respect to the Wiese value is a "nicer" axiomatization and a more intuitive definition of the players' payoffs [7]. The Wiese value and the χ-value are the only values for CS games that interpret components as active groups and account for outside options of the players. In [8], the χ-value is used for a first analysis of team games.[2]

[1] In contrast, the Owen value [5] interprets components as bargaining unions.

[2] Beyond the application to team games, the χ-value has been used to analyse the distribution of power in government coalitions [9–11] and to answer the question why firms train more employees than needed [12].

To the best our knowledge, our article is the first to use coalition structures of cooperative game theory and a component efficient value accounting for outside options to analyze the stability of couples. We replicate the results of the literature. This could be a starting point for further theoretical developments of our model framework.

For our two marriage market cases, we obtain the following results. In the symmetric case, couples with one woman and one man are part of stable structures. In the asymmetric case with an excess of men, the least attractive and the least economically successful men are in single components without a partner.

In the literature, the article by [13] presented an algorithm, based on the preferences of the players, that yields stable couples. In our article, we do not take on the process of forming stable structures; rather, we analyze exogenous given structures on the marriage market and indicate whether they are stable or not. The problem of stable couples on the marriage market is analyzed using the core in [14]. The drawbacks of using this value are extreme payoffs for players on the long side of the market. Additionally, it is not possible to model structures like couples. A review of literature based on these two articles is presented by [15].[3] A theme of both articles, the literature and our own approach is to find stable couples in the marriage market.

In his seminal article, ref. [23] applied basic microeconomic considerations to the marriage market. He addressed the main questions: who marries whom (men tend to marry women with similar characteristics; called assortative matching) and how to divide the gains of a marriage (among others, depending on the sex ratio). The first result is in line with our findings on asymmetric marriage markets. The second finding is crucial for our model as it takes into consideration the outside options. A review of literature on assortative matching is presented by [24].

Deviating to the predominant view of the literature, our model assumes transferable utility (TU game), i.e., the payoff of the couple can be freely distributed among the members of the coalition in any way desired. The joint assessment for income tax in Germany is one example for a transfer of economic success of spouses. Another deviation to literature is that stable couples stay on the marriage market. The spouses are possible outside options for other market participants. Analogously, the spouses divide the success of the couple with respect to their possible outside options.

The remainder of the article considers the basic notations of cooperative game theory in Section 2. In Section 3, the main results are presented. Section 4 concludes the article.

2. Preliminaries

The player set is called $N = \{1, 2, ..., n\}$. The set of women is denoted by W, the set of men by M, $W \cup M = N$. The coalitional function v assigns every subset K of N a certain worth $v(K)$, $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The cardinality of N is denoted n or $|N|$. A game – more precisely a game with transferable utility (TU game) – is a pair (N, v). A team game has non-zero worth only in coalitions with certain cardinality t, $t < n$. Since we analyze couples, we have $t = 2$. The worth of a couple could be interpreted as the economic abilities of the couple or as happiness of the couple.

A value is an operator ϕ that assigns payoff vectors to all games (N, v). One important value is the Shapley value. To calculate the player's payoffs, rank orders ρ on N are used. They are written as $(\rho_1, \ldots, \rho_n)$ where ρ_1 is the first player in the order, ρ_2 the second player etc. The set of these orders is denoted by $RO(N)$; $n!$ rank orders exist. The set of players before i in rank order ρ together with player i is called $K_i(\rho)$. The Shapley value is the average of the marginal contributions of i taken over all rank orders of the players [25]:

$$\mathrm{Sh}_i(N, v) = \frac{1}{n!} \sum_{\rho \in RO(N)} v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\}). \tag{1}$$

[3] Important developments of both articles mentioned are done by [16–22], for example.

Important players – players with better options on the marriage market – get positive payoffs. For team games, it holds $\sum_{i \in N} \mathrm{Sh}_i(N, v) = 0 = v(N)$.

A coalition structure $\mathcal{P}$ is a partition of N into non-empty components $C_1, \ldots, C_m$:

$$\mathcal{P} = \{C_1, \ldots, C_m\}, \tag{2}$$

with $C_i \cap C_j = \emptyset$, $i \neq j$ and $N = \bigcup C_j$. The component containing player i is denoted by $\mathcal{P}(i)$. The set of partitions of N is $\mathfrak{P}(N)$.

A CS game is a game with a coalition structure, $(N, v, \mathcal{P})$. A CS value is an operator φ that assigns payoff vectors to all games $(N, v, \mathcal{P})$. The χ-value is one CS value. It divides the worth of a component, $v(\mathcal{P}(i))$, among its members, $j \in \mathcal{P}(i)$. The χ-value can be interpreted as a measure for the power distribution within a couple / the sharing of the economic success of the couple. In contrast to the Aumann-Drèze value, the χ-value accounts for outside options. Roughly speaking, the greater a players outside options, the higher its share within the couple. The formula for computing the χ-payoff of player $i \in N$ is [7]:[4]

$$\chi_i(N, v, \mathcal{P}) = \mathrm{Sh}_i(N, v) + \frac{v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i)} \mathrm{Sh}_j(N, v)}{|\mathcal{P}(i)|}. \tag{3}$$

The main idea underlying the χ-value is that splitting a couple affects the members of a component equally.

Later on, we will use the concept of stability. A coalition structure $\mathcal{P}$ for (N, v) is χ-stable iff for all $\varnothing \neq K \subseteq N$ there is some $i \in K$ such that [6,26]

$$\chi_i(N, v, \mathcal{P}) \geq \chi_i(N, v, \{K, N \backslash K\}). \tag{4}$$

Hence, starting from $\mathcal{P}$ it is not possible to raise the payoffs of all $i \in K$, if K is separated into one component. In other words, when starting from a structure $\mathcal{P}$, separations for new couples are not worthwhile for both new partners.

3. Results

3.1. Symmetric Case

In this section, we analyze symmetrical marriage markets. To model this, we use the glove game [27]. In this game, each player owns either one left glove or one right glove. A single glove is worthless; a pair of gloves has a worth of one. The coalitional function for this game in the team game context with $t = 2$ is given by

$$v(K) = \begin{cases} \min\{|K \cap L|, |K \cap R|\}, & k = 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{with } N = R \cup L,\ R \cap L = \emptyset, \tag{5}$$

where L (R) denotes the set of left (right) glove owners. In the context of symmetrical marriage markets, we denote $L = W$ and $R = M$, i.e., a couple between a woman and a man has a worth of one.[5]

In a first step, we assume an equal number of women and men; $|W| = |M|$. Hence, with $v(N) = 0$ we have $\mathrm{Sh}_i(N, v) = \mathrm{Sh}_j(N, v) = 0$, $i \in W$ and $j \in M$.

[4] In a similiar way, the expected payoff for an agent $i \in N$ is computed in [22].

[5] We assume this definition of a couple for a simplification of our model. We do not discriminate against other types of partnerships.

Theorem 1. *In a symmetric marriage market with* $|W| = |M|$ *only partitions with* $\mathcal{P} = \{C_1, \ldots, C_p\}$, $|C_l| = 2$ *with* $|C_l \cap W| = |C_l \cap M| = 1$, $l = 1, ..., p$, *are* χ*-stable.*

The proof of this Theorem is in the Appendix A. In each component there is one woman and one man. The players obtain the payoff $\chi_i(N, v, \mathcal{P}) = \frac{1}{2}$.

Now we assume that there are fewer women than men, $|W| < |M|$. From Theorem 5 in [8] we know $0 < \mathrm{Sh}_i(N, v) < \frac{1}{2}$ and $-\frac{1}{2} < \mathrm{Sh}_j(N, v) < 0$, $i \in W$ and $j \in M$. The shorter side of the marriage market has higher payoffs.

From Theorem 1 and Theorem 6 in [8] we deduce the following Corollary:

Corollary 1. *In a symmetric marriage market with* $|W| < |M|$ *only partitions with* $\mathcal{P} = \{C_1, \ldots, C_h, C_{h+1}, \ldots, C_p\}$, $\bigcup_{1,\ldots,h} C_i \cap W = \varnothing$, $\left|\bigcup_{1,\ldots,h} C_i\right| = |M| - |W|$, *and* $|C_l| = 2$ *with* $|C_l \cap W| = |C_l \cap M| = 1$, $l = h + 1, ..., p$, *are* χ*-stable.*

Some comments to this Corollary are in the Appendix A. The results of this subsection are in line with an intuitive conjecture about stable coalition structures in these games – components with one woman and one man are χ-stable. In the case of $|W| = |M|$, no player is in a single player component.

3.2. Asymmetric Case

In this section, we analyze marriage markets with different types of women and men. This differentiation could be interpreted as economic success or attractiveness of the players. In a first step, we assume a particular coalitional function to exemplify the main idea. In a second step, the results are generalized for (symmetric) supermodular functions.

For the coalitional function in the particular case, we assume

$$v(K) = \begin{cases} i + j, & K = \{w_i, m_j\} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

with $W = \{w_1, ..., w_w\}$ and $M = \{m_1, ..., m_m\}$; $N = W \cup M$; $m, w > 2$. The number of a man/woman represents the economic success of the player. This measure is only realized in coalitions with one player of the opposite gender – money doesn't buy happiness.

For the Shapley payoffs, we have $\mathrm{Sh}_{w_{i+1}}(N, v) > \mathrm{Sh}_{w_i}(N, v)$, $i = 1, ..., w - 1$, and $\mathrm{Sh}_{m_{j+1}}(N, v) > \mathrm{Sh}_{m_j}(N, v)$, $j = 1, ..., m - 1$. For the next theorems, we need the following Lemma.

Lemma 1. *For* $\mathrm{Sh}_{w_{i+1}}(N, v) - \mathrm{Sh}_{w_i}(N, v)$, $i = 1, ..., w - 1$, *we obtain*

$$\frac{m}{(m + w)^2 - 3(w + m) + 2} < \frac{1}{2}. \tag{7}$$

Analogously, we have $\mathrm{Sh}_{m_{j+1}}(N, v) - \mathrm{Sh}_{m_j}(N, v) < \frac{1}{2}$.

The proof of this Lemma is in the Appendix A.

In the first step of the analysis, we assume $|W| = |M|$. Hence, in each level of "quality" exists one woman and one man, i.e., it holds $\mathrm{Sh}_{w_l}(N, v) = \mathrm{Sh}_{m_l}(N, v)$.

Theorem 2. *In an asymmetric marriage market with* $|W| = |M|$ *only partitions with* $\mathcal{P} = \{C_1, \ldots, C_p\}$, $C_l = \{w_l, m_l\}$, $l = 1, ..., p$, *are* χ*-stable.*

The proof of Theorem 2 is given in the Appendix A. Hence, couples with spouses of the same "quality" are the only stable structures in asymmetric marriage markets with $|W| = |M|$, i.e., each player has an economically equivalent partner. Again, all players are paired.

In a next step, we analyze asymmetric marriage markets with $|W| < |M|$, i.e., there are more men than women in the market. In our model, the most economic successful man has no equally successful women in the market. The results are the same, if we increase the number of women and we have $|M| - |W|$ men at the lower end who do not have an equivalent woman in the marriage market. From Theorem 2 we deduce:

Corollary 2. *In an asymmetric marriage market with $|W| < |M|$ only partitions with $\mathcal{P} = \{C_1, ..., C_s, C_{s+1}, \ldots, C_p\}$, with $C_r = \{m_r\}$ with $r = 1, ..., s$ with $s = |M| - |W|$ and $C_l = \{w_l, m_{l+s}\}$, $l = s+1, ..., p$, are χ-stable.*

An additional comment is in the Appendix A. All women are paired with men. The least economically successful men do not get a spouse. Finally, the most successful man is paired with the most successful woman.

Now we generalize our results for symmetric supermodular functions. Again, we define $W = \{w_1, ..., w_w\}$, $M = \{m_1, ..., m_m\}$, $N = W \cup M$ and $m, w > 2$. We assume a symmetric supermodular coalitional function

$$v(K) = \begin{cases} f(w_i, m_j), & K = \{w_i, m_j\} \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

with $v(K) = v(K')$ for $K' = W' \cup M'$, $W' = K \cap M$, $M' = K \cap W$ (symmetry) and

$$f(w_{i+2}, m_{j+1}) - f(w_{i+1}, m_{j+1}) \geq f(w_{i+1}, m_j) - f(w_i, m_j) \quad \text{(supermodularity).} \tag{9}$$

Supermodularity means that the marginal contribution of woman to a couple increases as the "economic success" of the man increases.

Again, we assume first $|W| = |M|$. Since $v(K)$ is symmetric, we have $\text{Sh}_{w_l}(N, v) = \text{Sh}_{m_l}(N, v)$.

Theorem 3. *In an asymmetric marriage market with $|W| = |M|$ and a symmetric supermodular function $v(K)$ only partitions with $\mathcal{P} = \{C_1, \ldots, C_p\}$, $C_l = \{w_l, m_l\}$, $l = 1, ..., p$, are χ-stable.*

The proof of Theorem 3 is given in the Appendix A.

For asymmetric marriage markets with $|W| < |M|$, we deduce from Theorem 3:

Corollary 3. *In an asymmetric marriage market with $|W| < |M|$ and a symmetric supermodular function $v(K)$ only partitions with $\mathcal{P} = \{C_1, ..., C_s, C_{s+1}, \ldots, C_p\}$, with $C_r = \{m_r\}$ with $r = 1, ..., s$ with $s = |M| - |W|$ and $C_l = \{w_l, m_{l+s}\}$, $l = s+1, ..., p$, are χ-stable.*

The results for asymmetric marriage markets are in line with reasonable conjectures about stable coalition structures. Again, components with one woman and one man are χ-stable. Additionally, in the case of $|W| = |M|$ we find that women and men of the same economic success form couples (assortative matching). When men outnumber women in the market, the least economically successsful men remain single.

4. Conclusions

In this paper, we analyze the stability of couples in marriage markets. The results are in line with an intuitive conjecture about stable coalition structures in these games. These results show that modelling and analyzing the stability of couples within the marriage market with team games and CS values seems an appropriate research strategy.

This article could be the starting point for fruitful further research. One line of research is the development of marriage market models. One question could be for example, how does the stability of couples changes, if new men (or women) enter the market? Another theoretical development could be using a modified version of the χ-value that is based on other value-like solution concepts of cooperative game theory [28] to analyze the stability of couples. A further possible field of research should examine the formation of couples using non-cooperative models of couple formation with the χ-payoffs of the players as possible outcomes of the players' decisions [22]. Another application could be the integration of the model into evolutionary games[6] [29–31]. In evolutionary games, our approach could replace the matching function and the detailed modelling of strategies.

The last field of research involves experimental research. With these studies, it is possible to verify the statements on χ-payoffs and stability of couples by comparing them to their actual behavior.

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Appendix A

Proof of Theorem 1

First, we have to prove that the partitions mentioned are χ-stable. This result can be deduced directly from Corollary 4 in [8]. In the second step, we show no other possible partitions are χ-stable. Components with $|C| \neq 2$ are worth nothing. Also, components with $|C| = 2 = |C \cap W|$ or $|C| = 2 = |C \cap M|$ have the worth zero. Since the players are symmetrical ($\mathrm{Sh}_i\,(N, v) = 0$ for all $i \in N$) they obtain the χ-payoff zero. Since we have an even number of players, it is always possible to form at least one new component with worth 1 and increase the players' χ-payoffs to $\frac{1}{2}$. More formally, starting from a partition $\mathcal{P}$ with at least one component C with $|C| \neq 2$ or $|C \cap W| \neq |C \cap M|$ it is possible to form an alternative partition $\mathcal{P}' = \{K, N \backslash K\}$ with $|K \cap W| = |K \cap M| = 1$ and $K \subseteq \bigcup_{C_i \in \mathcal{P}} C_i$ with $|C_i| \neq 2$ or $|C \cap W| \neq |C \cap M|$.

Comment to Corollary 1

The men in $C_1, \ldots, C_h$ obtain the χ-payoffs zero – the worth of their components. It is not possible to form an alternative partition $\mathcal{P}' = \{K, N \backslash K\}$ with $|K| = 2$, $v\,(K) > 0$ and $(C_1, \ldots, C_h) \cap K \neq \varnothing$ without at least one player from W. The χ-payoff of this player is unchanged. Now we take a look at components $C_{h+1}, \ldots, C_p$. From Theorem 6 in [8], we know that $\chi_i(N, v, \mathcal{P}) > \chi_j(N, v, \mathcal{P}) > 0$, $i \in W$, $j \in M$. Adding or removing a player would reduce the χ-payoff of the remaining player. Also, substituting a player $i \in W$ by a player $j \in M$ does not raise the χ-payoffs of the players in at least one new component. Hence, $\mathcal{P}$ is χ-stable.

Proof of Lemma 1

We argue with the rank order calculation of the Shapley value. At position 2 (probability $\frac{1}{n}$) the difference 1 between w_l and w_{l+1} occurs, if a man is in position $t = 1$ (probability $\frac{m}{n-1}$). Hence, the Shapley difference based on this position is $\frac{1}{n} \cdot 1 \cdot \frac{m}{n-1}$. At position 3 $\left(\text{probability } \frac{1}{n}\right)$ the difference 1 between w_l and w_{l+1} occurs, if a couple with w_{l+1} is formed by the players on position 1 and

[6] In these games, different types of individuals with respect to their strategies exist. A matching rule draws them into groups. In these groups, the individuals carry out their strategies and hence the players' payoffs are determined. Finally, a replicator dynamic describes the proportion of the different types of individuals in the next generation.

2 (probability $\frac{1}{n-1}\cdot\frac{m}{n-2}+\frac{1}{n-2}\cdot\frac{m}{n-1}$) , meaning that the Shapley difference based on position 3 is $\frac{1}{n}\cdot 1\cdot\frac{2m}{(n-1)(n-2)}$. The total difference is

$$\begin{aligned}\Delta &= \frac{1}{n}\cdot\frac{m}{n-1}+\frac{1}{n}\cdot\frac{2m}{(n-1)(n-2)} \\ &= \frac{m}{n^2-3n+2}=\frac{m}{(m+w)^2-3(w+m)+2}.\end{aligned} \tag{A1}$$

At $m = w = 2$ we obtain $\frac{1}{3}$. Plotting Δ for $m > 2$ and $w > 2$ shows that the function is decreasing in m and w.

Proof of Theorem 2

Again, first we prove that the partitions mentioned are χ-stable. In the second step, we show that all other possible partitions are not χ-stable.

The players within a component are symmetric in their outside options. Hence, the player's χ-payoffs are:

$$\begin{aligned}\chi_{w_l}(N,v,\mathcal{P}) &= \chi_{m_l}(N,v,\mathcal{P}) \\ &= \mathrm{Sh}_{w_l}(N,v)+\frac{l+l-\mathrm{Sh}_{w_l}(N,v)-\mathrm{Sh}_{m_l}(N,v)}{2} \\ &= \mathrm{Sh}_{w_l}(N,v)+\frac{l+l-\mathrm{Sh}_{w_l}(N,v)-\mathrm{Sh}_{w_l}(N,v)}{2}=l.\end{aligned} \tag{A2}$$

Forming an alternative partition $\mathcal{P}'$ with $C_l = \{w_{l+1}, m_l\}$ and $C_{l+1} = \{w_l, m_{l+1}\}$, we obtain:

$$\chi_{w_{l+1}}(N,v,\mathcal{P}')=\mathrm{Sh}_{w_{l+1}}(N,v)+\frac{l+l+1-\mathrm{Sh}_{w_{l+1}}(N,v)-\mathrm{Sh}_{m_l}(N,v)}{2}. \tag{A3}$$

And hence:

$$\chi_{w_{l+1}}(N,v,\mathcal{P}')-\chi_{w_{l+1}}(N,v,\mathcal{P}) \tag{A4}$$

$$= \mathrm{Sh}_{w_{l+1}}(N,v)+\frac{l+l+1-\mathrm{Sh}_{w_{l+1}}(N,v)-\mathrm{Sh}_{w_l}(N,v)}{2}-(l+1)$$

$$= \frac{\mathrm{Sh}_{w_l+1}(N,v)-\mathrm{Sh}_{w_l}(N,v)}{2}-\frac{1}{2}<0. \tag{A5}$$

The last line is derived from Lemma 1. Hence, in $C_l \in \mathcal{P}'$ not all players could raise their χ-payoffs. Analogously, the χ-payoffs of player m_{l+1} decreases from $\mathcal{P}$ to $\mathcal{P}'$.

Components with $|C| = 1$ are worth nothing. Players in this components obtain the χ-payoff zero, i.e., deviating to a partition with single player components is not meaningful. Also, components with $|C| = 2 = |C \cap W|$ or $|C| = 2 = |C \cap M|$ have the worth zero. Raising the χ-payoff of at least one player to 1 involves a negative χ-payoff for the second player in the component, so deviating to such partitions is not meaningful. The same argument is true for components with $|C| > 2$. Hence, $\mathcal{P}$ is χ-stable.

Now, we argue that no other partitions are χ-stable. From "$<$" in line (A5), we know that all partitions with $\mathcal{P} = \{C_1, \ldots, C_f, C_{f+1}, ..., C_p\}$, $C_l = \{w_l, m_l\}$, $l = 1, ..., p$, $l \neq f, l \neq f+1$ and $C_f = \left\{w_f, m_{f+1}\right\}$, $C_{f+1} = \left\{w_{f+1}, m_f\right\}$ (alternatively $C_f = \left\{w_{f+1}, m_f\right\}$, $C_{f+1} = \left\{w_f, m_{f+1}\right\}$) are not χ-stable, because an alternative partition $\mathcal{P}' = \{K, N\backslash K\}$ with $K = \left\{w_{f+1}, w_{m+1}\right\}$ raises the χ-payoffs of both players.

From these arguments, it is clear that partitions with components of cardinality 1 are not χ-stable, because the players in atomistic-components obtain the χ-payoffs zero. Hence, the deviation of a man

m_l and a woman w_l to $\mathcal{P}'$ with $\mathcal{P}' = \{K, N\backslash K\}$ with $K = \{w_l, m_l\}$ increases their payoffs to $l > 0$. The same argument holds for the partition $\mathcal{P} = N$.

We also deduce from our arguments above that all partitions with $\mathcal{P}' = \{C_1, \ldots, C_p\}$ with $\bigcup_{|C_l| \neq 2} C_l \neq \varnothing$, $l = 1, \ldots, p$, are not χ-stable. Also, all partitions with $\mathcal{P} = \{C_1, \ldots, C_p\}$ with $\bigcup_{|C_l \cap W| = 2} C_l \neq \varnothing$ or $\bigcup_{|C_l \cap M| = 2} C_l \neq \varnothing$, $l = 1, \ldots, p$, are not χ-stable. Two men (or women) in one component yields to zero χ-payoffs for both players. Since another component must have a surplus of women, both men could deviate and raise their χ-payoffs and the payoffs of the women.

Comment to Corollary 2

First, we look at the χ-payoffs for players in components $C_{s+1}, \ldots, C_p$. For example in C_l we obtain

$$\chi_{w_l}(N, v, \mathcal{P}) = \mathrm{Sh}_{w_l}(N, v) + \frac{l + l + s - \mathrm{Sh}_{w_l}(N, v) - \mathrm{Sh}_{m_{l+s}}(N, v)}{2} \tag{A6}$$
$$\chi_{m_l+s}(N, v, \mathcal{P}) = \mathrm{Sh}_{m_l+s}(N, v) + \frac{l + l + s - \mathrm{Sh}_{w_l}(N, v) - \mathrm{Sh}_{m_{l+s}}(N, v)}{2}.$$

Forming an alternative partition $\mathcal{P}'$ with $C_{l-1} = \{w_l, m_{l-1+s}\}$ and $C_l = \{w_{l-1}, m_{l+s}\}$ gives for w_l

$$\chi_{w_l}(N, v, \mathcal{P}') = \mathrm{Sh}_{w_l}(N, v) + \frac{l + l + s - 1 - \mathrm{Sh}_{w_l}(N, v) - \mathrm{Sh}_{m_{l-1+s}}(N, v)}{2} \tag{A7}$$

and

$$\begin{aligned} &\chi_{w_l}(N, v, \mathcal{P}') - \chi_{w_l}(N, v, \mathcal{P}) \\ &= \mathrm{Sh}_{w_l}(N, v) + \frac{l + l + s - 1 - \mathrm{Sh}_{w_l}(N, v) - \mathrm{Sh}_{m_{l-1+s}}(N, v)}{2} \\ &- \mathrm{Sh}_{w_l}(N, v) + \frac{l + l + s - \mathrm{Sh}_{w_l}(N, v) - \mathrm{Sh}_{m_{l+s}}(N, v)}{2} \\ &= \frac{\mathrm{Sh}_{m_{l+s}}(N, v) - \mathrm{Sh}_{m_{l-1+s}}(N, v)}{2} - \frac{1}{2} < 0 \end{aligned} \tag{A8}$$

and for m_{l+s}

$$\chi_{m_l+s}(N, v, \mathcal{P}') = \mathrm{Sh}_{m_l+s}(N, v) + \frac{l + l + s - 1 - \mathrm{Sh}_{w_{l-1}}(N, v) - \mathrm{Sh}_{m_{l+s}}(N, v)}{2}, \tag{A9}$$

$$\begin{aligned} &\chi_{m_l+s}(N, v, \mathcal{P}') - \chi_{m_l+s}(N, v, \mathcal{P}) \\ &= -\frac{1}{2} + \frac{\mathrm{Sh}_{w_l}(N, v) - \mathrm{Sh}_{w_{l-1}}(N, v)}{2} < 0. \end{aligned} \tag{A10}$$

Hence, by deviating to $C_{l-1}, C_l \in \mathcal{P}'$ not all players could raise their χ-payoffs. Deviating from $C_{s+1}, \ldots, C_p$ to single player components or components with $|C| = 2 = |C \cap W|$ or $|C| = 2 = |C \cap M|$ is not meaningful (see proof of Theorem 2). The players in $C_1, \ldots, C_s$ obtain the χ-payoffs zero. It is not possible to form an alternative partition $\mathcal{P}' = \{K, N\backslash K\}$ with $|K| = 2$, $v(K) > 0$ and $\{C_1, \ldots, C_s\} \cap K \neq \varnothing$ without at least one player from W. The χ-payoff of this player decreases. Also, deviating to components with two or more men is not meaningful. Consider a partition $\mathcal{P}' = \{C', N\backslash C'\}$ with $C' = \{m_{l+1}, m_l\}$. From $\mathrm{Sh}_{m_l+1}(N, v) > \mathrm{Sh}_{m_l}(N, v)$ we have $\chi_{m_l+1}(N, v, \mathcal{P}') > \chi_{m_l}(N, v, \mathcal{P}')$. From $v(C') = 0$ we get $\chi_{m_l}(N, v, \mathcal{P}') < 0$. Hence, $\mathcal{P}$ is χ-stable.

Obviously, in an asymmetric marriage market with $|W| < |M|$ all partitions with:

- $\mathcal{P} = \{C_1, \ldots, C_s, C_{s+1}, \ldots, C_p\}$, with $C_r = \{m_r\}$ with $r = 1, \ldots, s$ with $s = |M| - |W|$ and $\bigcup_{|C_l| \neq 2} C_l \neq \varnothing$, $l = s + 1, \ldots, p$,

- $\mathcal{P} = \{C_1, ..., C_s, C_{s+1}, \ldots, C_p\}$, with $C_r = \{m_r\}$ with $r = 1, ..., s$ with $s = |M| - |W|$ and $\bigcup_{|C_l \cap W|=2} C_l \neq \varnothing$ or $\bigcup_{|C_l \cap M|=2} C_l \neq \varnothing, l = s+1, ..., p,$
- $\mathcal{P} = \{C_1, ..., C_s, C_{s+1}, \ldots, C_p\}$, with $C_r = \{m_r\}$ with $r = 1, ..., s$ with $s = |M| - |W|$ and $C_l = \{w_l, m_{l+s}\}, l = s+1, ..., p, l \neq f, f+1$ and $C_f = \left\{w_f, m_{f+s+1}\right\}, C_{f+1} = \left\{w_{f+1}, m_{f+s}\right\}$ (alternatively $C_f = \left\{w_{f+1}, m_{f+s}\right\}, C_{f+1} = \left\{w_f, m_{f+s+1}\right\}$)
- $\mathcal{P} = \{C_1, ..., C_s, C_{s+1}, \ldots, C_p\}$, with $\bigcup_{1,...,s} C_i \cap W = \varnothing$, $\left|\bigcup_{1,...,s} C_i\right| = |M| - |W| = \Delta$, $s < \Delta$ and $C_l = \{w_l, m_{l+\Delta}\}, l = s+1, ..., p,$

are not χ-stable.

Proof of Theorem 3

Analogously to Theorem 2, we first analyze the χ-payoffs of the players. Since players within a component are symmetric in their outside options, we have

$$\chi_{w_l}(N, v, \mathcal{P}) = \chi_{m_l}(N, v, \mathcal{P}) = \frac{f(w_l, m_l)}{2}. \tag{A11}$$

Forming an alternative partition $\mathcal{P}'$ with $C_l = \{w_{l+1}, m_l\}$ and $C_{l+1} = \{w_l, m_{l+1}\}$, we obtain:

$$\chi_{w_{l+1}}(N, v, \mathcal{P}') = \mathrm{Sh}_{w_{l+1}}(N, v) + \frac{f(w_{l+1}, m_l) - \mathrm{Sh}_{w_{l+1}}(N, v) - \mathrm{Sh}_{m_l}(N, v)}{2}.$$

Hence

$$\begin{aligned} &\chi_{w_{l+1}}(N, v, \mathcal{P}') - \chi_{w_{l+1}}(N, v, \mathcal{P}) \\ &= \frac{\mathrm{Sh}_{w_{l+1}}(N, v) - \mathrm{Sh}_{m_l}(N, v) + f(w_{l+1}, m_l) - f(w_{l+1}, m_{l+1})}{2} \\ &= \frac{\mathrm{Sh}_{w_{l+1}}(N, v) - \mathrm{Sh}_{w_l}(N, v) + f(w_{l+1}, m_l) - f(w_{l+1}, m_{l+1})}{2} < 0 \end{aligned} \tag{A12}$$

The last line is derived from $\mathrm{Sh}_{w_{l+1}}(N, v) - \mathrm{Sh}_{w_l}(N, v) < f(w_{l+1}, m_{l+1}) - f(w_{l+1}, m_l)$ using symmetry and supermodularity (only non-negative marginal contributions in Equation (1)). Hence, in $C_l \in \mathcal{P}'$ not all players could raise their χ-payoffs. Analogously, the χ-payoffs of player m_{l+1} decreases from $\mathcal{P}$ to $\mathcal{P}'$. For all other arguments, we refer to the proof of Theorem 2.

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