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A Closed-Form Solution to the Mechanism of Interface Crack Formation with One Contact Area in Decagonal Quasicrystal Bi-Materials

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Abstract: Cracks and crack-like defects in engineering structures have greatly reduced the structural strength. An interface crack with one contact area in a combined tension–shear field of decagonal quasicrystal bi-material is investigated. Based on the deformation compatibility equation and displacement potential function, the complex representation of stress and displacement is given. Using the mixed boundary conditions, the closed-form expressions for the stresses and the displacement jumps in the phonon field and phason field on the material interface are obtained. The results show that the stress intensity factor at the crack tip is zero for the phason field. The variation in the stress intensity factor and the length of the contact zone in the phonon field is given, and the result is consistent with the properties of the crystal. The design of safe engineering structures and the formulation of reasonable quality acceptance standards may benefit from the theoretical research carried out here.

Keywords: interface crack; decagonal quasicrystal; contact zone; stress intensity factor

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1. Introduction

Quasicrystals, as a lightweight, high-strength material suitable for medium-temperature operation, are becoming a new functional and structural material. With the help of various special properties of quasicrystal materials, scientists have made new breakthroughs in various research fields, which has also promoted the development of quasicrystal material applications. Different from ordinary crystals [1–3], the two-dimensional decagonal quasicrystals are periodic in the direction of the decagonal rotational symmetry axis, while the arrangement of plane atoms perpendicular to the decagonal rotational symmetry axis is quasi-periodic, which leads to the additional elastic degrees of freedom that do not exist in ordinary crystals and increases the complexity of fracture mechanics research. The quasicrystal phase of the electron diffraction pattern with 10 rotationally symmetric axes was found by Bendersky [4]. At the same time, Feng et al. found the decagonal quasicrystal phase in rapidly cooled Al-Fe alloy [5]. In 1989, two types of dislocation in decagonal quasicrystals were confirmed by the comparative analysis of electron diffraction [6]. Based on the multiplication of the basis sites' groups, Girzhon et al. proposed the model of the reciprocal lattice of decagonal quasicrystal [7]. By atomic resolution high-angle annular dark-field scanning transmission electron microscopy, Ma and He observed the largest decagonal subunits, which expanded to 5.2 nm in a decagonal shape [8].

There is a lot of the literature on defects in decagonal quasicrystals. By using the Eshelby method, the elastic field and energy of a decagonal quasicrystal with a special dislocation line are given [9]. Fan's research team used the complex solution of classical elastic theory and introduced the displacement potential function and stress potential function to transform the final governing equation of the two-dimensional decagonal quasicrystal plane elastic problem into a quadruple harmonic equation [10]. Using this theory and conformal transformation of a complex function, the elliptical notch problem is solved [11]. Wang and Zhong studied the interaction between a semi-infinite crack and a line dislocation in a decagonal quasicrystal solid using the complex variable method [12]. Li constructed the complex potential theory of two-dimensional decagonal quasicrystals and further developed Muskhelishvili's complex variable method [13]. Fan et al. studied the interface crack problem of two-dimensional decagonal quasicrystal bi-material using the propagation displacement discontinuity method [14]. Wang and Schiavone investigated the elastic field near the tip of an anti-crack in a homogeneous decagonal quasicrystal material and presented explicit and elegant expressions for the anti-crack contraction force [15]. The plane problem of a two-dimensional decagonal quasicrystal with a rigid circular inclusion under infinite tension and concentrated force is studied by Zhai et al. [16]. Based on the complex representation of stress and displacement of two-dimensional decagonal quasicrystal, the above problem is transformed into the Riemann boundary problem by using the analytic continuation principle of complex variable function. Zhao et al. extended the displacement and temperature discontinuity method to the two-dimensional decagonal quasicrystal coating structure and studied the mechanical behavior of the interface crack under thermal–mechanical loads [17]. The plane elastic problem of two asymmetric edge cracks in a two-dimensional decagonal quasicrystal elliptical hole under far-field tensile stress is considered by Yu [18]. Li et al. established a phase field framework to simulate the macroscopic brittle fracture of quasicrystal materials [19]. In this phase field model, the volume fraction parameter is introduced into the fracture toughness to reflect the phase wall effect for the first time.

The classical interface crack model [20] assumes that the crack is completely open, which leads to oscillating singularity at the crack tips. By assuming that there is a small contact zone near the crack tip, the unreal vibration singularity is eliminated [21]. Using a singular integral equation formulation, Qin and Mai investigated interface cracks with contact zones in thermo-piezoelectric materials [22]. An analytical solution for an interface crack with one contact zone in anisotropic material was studied by Herrmann and Loboda [23]. Kharun and Loboda studied the crack problem at the interface of two isotropic materials under mixed-mode loading [24]. The interface crack is assumed to be fully open, partially closed, friction-free contact zone, and fully closed. The problem is reduced to a homogeneous combined Dirichlet Riemann boundary value problem and solved in closed form. The problem of interface crack with a frictionless contact zone at the right crack tip between two semi-infinite piezoelectric/piezomagnetic spaces is considered by Herrmann et al. [25]. Saikia and Muthu studied the interface crack by using the non-intrinsic cohesive zone model to eliminate the stress singularity at the crack tip [26]. However, to the best of the authors' knowledge, the problem of interface crack with contact zone at the crack tip in decagonal quasicrystals has not been studied.

In the present study, the interface crack theory in elastic fracture mechanics is extended to the elastic fracture mechanics of decagonal quasicrystal bi-material. The interface crack problem with a contact zone in decagonal quasicrystal bi-material is considered, which has a contact zone penetrating the solid along the quasi-periodic direction. By using the method of complex variable function, the mixed boundary value problem is transformed into the Dirichlet Riemann boundary value problem, and the closed solution of the problem is obtained.

2. Basic Equations

The stress and strain of decagonal quasicrystal satisfy generalized Hooke's law [10,27,28]:

$$\begin{array}{l}
\sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + R(\omega_{xx} + \omega_{yy}) \\
\sigma_{yy} = C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} - R(\omega_{xx} + \omega_{yy}) \\
\sigma_{xy} = \sigma_{yx} = (C_{11} - C_{12})\varepsilon_{xy} + R(\omega_{yx} - \omega_{xy}) \\
H_{xx} = K_1\omega_{xx} + K_2\omega_{yy} + R(\varepsilon_{xx} - \varepsilon_{yy}) \\
H_{yy} = K_2\omega_{xx} + K_1\omega_{yy} + R(\varepsilon_{xx} - \varepsilon_{yy}) \\
H_{xy} = K_1\omega_{xy} - K_2\omega_{yx} - 2R\varepsilon_{xy} \\
H_{yx} = K_1\omega_{yx} - K_2\omega_{xy} + 2R\varepsilon_{xy}
\end{array} \tag{1}$$

From Equation (1), the strain relation expressed by stress can be written as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{4(C_{12}+C_{66})}\left(\sigma_{xx}+\sigma_{yy}\right)+\frac{K_1+K_2}{4r}\left(\sigma_{xx}-\sigma_{yy}\right)-\frac{R}{2r}\left(H_{xx}+H_{yy}\right)\\
\varepsilon_{yy} &= \frac{1}{4(C_{12}+C_{66})}\left(\sigma_{xx}+\sigma_{yy}\right)-\frac{K_1+K_2}{4r}\left(\sigma_{xx}-\sigma_{yy}\right)+\frac{R}{2r}\left(H_{xx}+H_{yy}\right)\\
\varepsilon_{xy} &= \varepsilon_{yx}=\frac{K_1+K_2}{2r}\sigma_{xy}+\frac{R}{2r}\left(H_{xy}-H_{yx}\right)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\omega_{xx} &= \frac{1}{2(K_1-K_2)}\left(H_{xx}-H_{yy}\right)+\frac{C_{66}}{2r}\left(H_{xx}+H_{yy}\right)-\frac{R}{2r}\left(\sigma_{xx}-\sigma_{yy}\right)\\
\omega_{yy} &= -\frac{1}{2(K_1-K_2)}\left(H_{xx}-H_{yy}\right)+\frac{C_{66}}{2r}\left(H_{xx}+H_{yy}\right)-\frac{R}{2r}\left(\sigma_{xx}-\sigma_{yy}\right)\\
\omega_{xy} &= \frac{R}{r}\sigma_{xy}+\frac{1}{2(K_1-K_2)}\left(H_{xy}+H_{yx}\right)+\frac{C_{66}}{2r}\left(H_{xy}-H_{yx}\right)\\
\omega_{yx} &= -\frac{R}{r}\sigma_{xy}+\frac{1}{2(K_1-K_2)}\left(H_{xy}+H_{yx}\right)-\frac{C_{66}}{2r}\left(H_{xy}-H_{yx}\right)
\end{aligned} \tag{3}$$

where σ_{ks}, ε_{ks} and $C_{ij}(k,s=x,y;i,\ j=1,2)$ are stresses, strains, and elastic constants in the phonon field, respectively; H_{ks}, ω_{ks}, and K_i are the stresses, strains, and elastic constants in the phason field, respectively; R is the phonon–phason coupling elastic constant.

The strain relation of the plane elastic problem on the quasi-periodic plane of decagonal quasicrystal can be expressed by displacements as

$$\begin{aligned}
&\varepsilon_{xx}=\frac{\partial u_x}{\partial x},\ \varepsilon_{yy}=\frac{\partial u_y}{\partial y},\ \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\\
&\omega_{xx}=\frac{\partial w_x}{\partial x},\ \omega_{yy}=\frac{\partial w_y}{\partial y},\ \omega_{xy}=\frac{\partial w_x}{\partial y},\ \omega_{yx}=\frac{\partial w_y}{\partial x}
\end{aligned} \tag{4}$$

After eliminating the displacement, the deformation coordinate equation expressed by strain is

$$\begin{aligned}
&\frac{\partial^2\varepsilon_{xx}}{\partial y^2}+\frac{\partial^2\varepsilon_{yy}}{\partial x^2}=2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}\\
&\frac{\partial\omega_{xy}}{\partial x}=\frac{\partial\omega_{xx}}{\partial y}\\
&\frac{\partial\omega_{yx}}{\partial y}=\frac{\partial\omega_{yy}}{\partial x}
\end{aligned} \tag{5}$$

Introducing stress potential functions $\phi(x,y)$, $\psi_1(x,y)$, $\psi_2(x,y)$, the stress–strain relationship can be expressed as

$$\begin{aligned}
&\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2},\ \sigma_{yy}=\frac{\partial^2\phi}{\partial x^2},\ \sigma_{xy}=\sigma_{yx}=-\frac{\partial^2\phi}{\partial x\partial y},\\
&H_{xx}=\frac{\partial\psi_1}{\partial y},\ H_{xy}=-\frac{\partial\psi_1}{\partial x},\ H_{yx}=-\frac{\partial\psi_2}{\partial y},\ H_{yy}=\frac{\partial\psi_2}{\partial x}
\end{aligned} \tag{6}$$

Substituting Equation (5) into Equations (2) and (3), one obtains

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{4(C_{12}+C_{66})}\nabla^2\phi+\frac{K_1+K_2}{4r}\left(\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial x^2}\right)-\frac{R}{2r}\left(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}\right)\\
\varepsilon_{yy} &= \frac{1}{4(C_{12}+C_{66})}\nabla^2\phi-\frac{K_1+K_2}{4r}\left(\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial x^2}\right)+\frac{R}{2r}\left(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}\right)\\
\varepsilon_{xy} &= \varepsilon_{yx}=-\frac{K_1+K_2}{2r}\frac{\partial^2\phi}{\partial x\partial y}+\frac{R}{2r}\left(\frac{\partial\psi_2}{\partial y}-\frac{\partial\psi_1}{\partial x}\right)
\end{aligned} \tag{7}$$

$$\begin{aligned}
\omega_{xx} &= \frac{1}{2(K_1-K_2)}\left(\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x}\right)+\frac{C_{66}}{2r}\left(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}\right)-\frac{R}{2r}\left(\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial x^2}\right)\\
\omega_{yy} &= -\frac{1}{2(K_1-K_2)}\left(\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x}\right)+\frac{C_{66}}{2r}\left(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}\right)-\frac{R}{2r}\left(\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial x^2}\right)\\
\omega_{xy} &= -\frac{R}{r}\frac{\partial^2\phi}{\partial x\partial y}-\frac{1}{2(K_1-K_2)}\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}\right)+\frac{C_{66}}{2r}\left(\frac{\partial\psi_2}{\partial y}-\frac{\partial\psi_1}{\partial x}\right)\\
\omega_{yx} &= \frac{R}{r}\frac{\partial^2\phi}{\partial x\partial y}-\frac{1}{2(K_1-K_2)}\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}\right)+\frac{C_{66}}{2r}\left(\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_2}{\partial y}\right)
\end{aligned} \tag{8}$$

Substituting Equations (7) and (8) into Equation (5) yields

$$\begin{cases} \left(\frac{1}{2(C_{12}+C_{66})}+\frac{K_1+K_2}{2r}\right)\nabla^2\nabla^2\phi+\frac{R}{r}\left(\frac{\partial}{\partial y}\Pi_1\psi_1-\frac{\partial}{\partial x}\Pi_2\psi_2\right)=0 \\ \left(\frac{r}{K_1-K_2}+C_{66}\right)\nabla^2\psi_1+R\frac{\partial}{\partial y}\Pi_1\phi=0 \\ \left(\frac{r}{K_1-K_2}+C_{66}\right)\nabla^2\psi_2-R\frac{\partial}{\partial x}\Pi_2\phi=0 \end{cases} \tag{9}$$

where

$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, $\Pi_1=3\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}$, $\Pi_2=3\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}$, $r=C_{66}(K_1+K_2)-2R^2$.

Introducing a new function $G(x,y)$, one obtains

$$\phi=r_1\nabla^2\nabla^2G,\psi_1=-R\frac{\partial}{\partial y}\Pi_1\nabla^2G,\psi_2=R\frac{\partial}{\partial x}\Pi_2\nabla^2G \tag{10}$$

where $r_1=\frac{r}{K_1-K_2}+C_{66}$.

Equation (9) is automatically satisfied, so G is called the stress function. One has [29]

$$\nabla^2\nabla^2\nabla^2\nabla^2G=0 \tag{11}$$

Therefore, the final control function based on stress potential is a quadruple harmonic equation. Substituting Equation (10) into Equation (6), one obtains

$$\begin{aligned} &\sigma_{xx}=r_1\frac{\partial^2}{\partial y^2}\nabla^2\nabla^2G,\ \sigma_{yy}=r_1\frac{\partial^2}{\partial x^2}\nabla^2\nabla^2G, \\ &\sigma_{xy}=\sigma_{yx}=-r_1\frac{\partial^2}{\partial x\partial y}\nabla^2\nabla^2G, \end{aligned} \tag{12}$$

$$\begin{aligned} &H_{xx}=-R\frac{\partial^2}{\partial y^2}\Pi_1\nabla^2G,\ H_{xy}=R\frac{\partial^2}{\partial x\partial y}\Pi_1\nabla^2G, \\ &H_{yx}=-R\frac{\partial^2}{\partial x\partial y}\Pi_2\nabla^2G,\ H_{yy}=R\frac{\partial^2}{\partial x^2}\Pi_2\nabla^2G \end{aligned} \tag{13}$$

Based on the method of stress potential function, Li and Fan [30] developed the complex variable function solution of the quartic harmonic equation. The fundamental solution of Equation (11) is

$$G=2\mathrm{Re}\left(g_1(z)+\bar{z}g_2(z)+\frac{1}{2}\bar{z}^2g_3(z)+\frac{1}{6}\bar{z}^3g_4(z)\right) \tag{14}$$

in which $g_j(z)(j=1,2,3,4)$ is about four analytic functions of complex variables; $z=x+iy$, and $i=\sqrt{-1}$. Superscript "$-$" indicates complex conjugation, i.e. $z=x-iy$.

Substituting Equation (14) into Equations (12) and (13), the complex expression of stress function is obtained as

$$\begin{aligned} &\sigma_{xx}=-32r_1\mathrm{Re}\left(g_3^{(4)}(z)+\bar{z}g_4^{(4)}(z)-2g_4'''(z)\right) \\ &\sigma_{yy}=32r_1\mathrm{Re}\left(g_3^{(4)}(z)+\bar{z}g_4^{(4)}(z)+2g_4'''(z)\right) \\ &\sigma_{xy}=\sigma_{yx}=32r_1\mathrm{Im}\left(g_3^{(4)}(z)+\bar{z}g_4^{(4)}(z)\right) \end{aligned} \tag{15}$$

$$\begin{aligned} &H_{xx}=32R\mathrm{Re}\left(g_2^{(5)}(z)+\bar{z}g_3^{(5)}(z)+\tfrac{1}{2}\bar{z}^2g_4^{(5)}(z)-g_3^{(4)}(z)-\bar{z}g_4^{(4)}(z)\right) \\ &H_{yy}=-32R\mathrm{Re}\left(g_2^{(5)}(z)+\bar{z}g_3^{(5)}(z)+\tfrac{1}{2}\bar{z}^2g_4^{(5)}(z)+g_3^{(4)}(z)+\bar{z}g_4^{(4)}(z)\right) \\ &H_{xy}=-32R\mathrm{Im}\left(g_2^{(5)}(z)+\bar{z}g_3^{(5)}(z)+\tfrac{1}{2}\bar{z}^2g_4^{(5)}(z)+g_3^{(4)}(z)+\bar{z}g_4^{(4)}(z)\right) \\ &H_{yx}=-32R\mathrm{Im}\left(g_2^{(5)}(z)+\bar{z}g_3^{(5)}(z)+\tfrac{1}{2}\bar{z}^2g_4^{(5)}(z)-g_3^{(4)}(z)-\bar{z}g_4^{(4)}(z)\right) \end{aligned} \tag{16}$$

New analytic functions $f_j(z)(j=2,3,4)$ are introduced to make

$$f_2(z)=g_2^{(4)}(z),\ f_3(z)=g_3'''(z),\ f_4(z)=g_4''(z) \tag{17}$$

Then, Equations (15) and (16) can be rewritten as

$$\begin{aligned}\sigma_{xx}&=-32r_1\mathrm{Re}\big(f_3'(z)+\bar{z}f_4''(z)-2f_4'(z)\big)\\ \sigma_{yy}&=32r_1\mathrm{Re}\big(f_3'(z)+\bar{z}f_4''(z)+2f_4'(z)\big)\\ \sigma_{xy}&=\sigma_{yx}=32r_1\mathrm{Im}\big(f_3'(z)+\bar{z}f_4''(z)\big)\end{aligned} \tag{18}$$

$$\begin{aligned}H_{xx}&=32R\mathrm{Re}\Big(f_2'(z)+\bar{z}f_3''(z)+\tfrac{1}{2}\bar{z}^2f_4'''(z)-f_3'(z)-\bar{z}f_4''(z)\Big)\\ H_{yy}&=-32R\mathrm{Re}\Big(f_2'(z)+\bar{z}f_3''(z)+\tfrac{1}{2}\bar{z}^2f_4'''(z)+f_3'(z)+\bar{z}f_4''(z)\Big)\\ H_{xy}&=-32R\mathrm{Im}\Big(f_2'(z)+\bar{z}f_3''(z)+\tfrac{1}{2}\bar{z}^2f_4'''(z)+f_3'(z)+\bar{z}f_4''(z)\Big)\\ H_{yx}&=-32R\mathrm{Im}\Big(f_2'(z)+\bar{z}f_3''(z)+\tfrac{1}{2}\bar{z}^2f_4'''(z)-f_3'(z)-\bar{z}f_4''(z)\Big)\end{aligned} \tag{19}$$

3. Statement of the Problem

Consider an interface crack with one contact area between two bonded semi-infinite decagonal quasicrystals. σ and τ are uniformly loaded in the phonon field, and $\sigma_{xx}^{(\mathrm{I})\infty}$, $\sigma_{xx}^{(\mathrm{II})\infty}$, $H_{xx}^{(\mathrm{I})\infty}$, $H_{xx}^{(\mathrm{II})\infty}$ are applied at infinity, as shown in Figure 1.

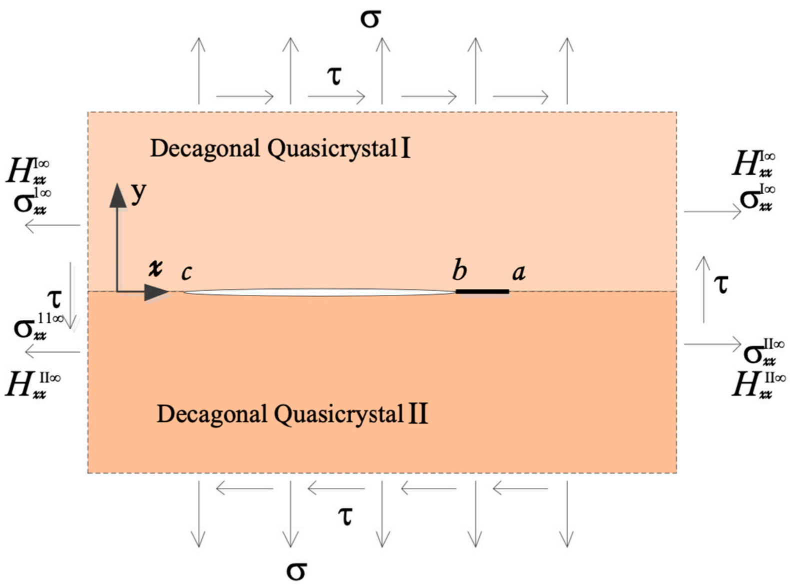

Figure 1. Schematic diagram of an interface crack with one contact area between decagonal quasicrystal bi-materials under infinite load.

The continuity conditions on the interface can be written as follows

$$\begin{cases}\left(\sigma_{yy}^{(\mathrm{I})}(x,0)-\mathrm{i}\sigma_{xy}^{(\mathrm{I})}(x,0)\right)-\left(\sigma_{yy}^{(\mathrm{II})}(x,0)-\mathrm{i}\sigma_{xy}^{(\mathrm{II})}(x,0)\right)=0\\ \left(H_{yy}^{(\mathrm{I})}(x,0)-\mathrm{i}H_{xy}^{(\mathrm{I})}(x,0)\right)-\left(H_{yy}^{(\mathrm{II})}(x,0)-\mathrm{i}H_{xy}^{(\mathrm{II})}(x,0)\right)=0\end{cases}\quad x\in(-\infty,\infty) \tag{20}$$

$$\begin{cases}\left(u_x^{(\mathrm{I})}(x,0)+\mathrm{i}u_y^{(\mathrm{I})}(x,0)\right)-\left(u_x^{(\mathrm{II})}(x,0)+\mathrm{i}u_y^{(\mathrm{II})}(x,0)\right)=0\\ \left(w_x^{(\mathrm{I})}(x,0)+\mathrm{i}w_y^{(\mathrm{I})}(x,0)\right)-\left(w_x^{(\mathrm{II})}(x,0)+\mathrm{i}w_y^{(\mathrm{II})}(x,0)\right)=0\end{cases}\quad x\in(-\infty,c)\cup(a,\infty) \tag{21}$$

The boundary conditions on the crack face can be expressed as follows:

$$\begin{cases} \sigma_{xy}^{(\mathrm{I})}(x,0) = 0 \\ H_{xy}^{(\mathrm{I})}(x,0) = 0 \\ u_y^{(\mathrm{I})}(x,0) - u_y^{(\mathrm{II})}(x,0) = 0 \\ w_y^{(\mathrm{I})}(x,0) - w_y^{(\mathrm{II})}(x,0) = 0 \end{cases} \quad x \in (b,a) \tag{22}$$

$$\begin{cases} \sigma_{yy}^{(\mathrm{I})}(x,0) - \mathrm{i}\sigma_{xy}^{(\mathrm{I})}(x,0) = 0 \\ H_{yy}^{(\mathrm{I})}(x,0) - \mathrm{i}H_{xy}^{(\mathrm{I})}(x,0) = 0 \end{cases} \quad x \in (c,b) \tag{23}$$

where $\sigma_{12}^{(k)}(x,y)$; $\sigma_{22}^{(k)}(x,y)$, $u_1^{(k)}(x,y)$, and $u_2^{(k)}(x,y)$ are the phonon field of shear stresses, normal stresses, and displacements along x- and y-axes, respectively. $H_{12}^{(k)}(x,y)$, $H_{22}^{(k)}(x,y)$, $w_1^{(k)}(x,y)$, and $w_2^{(k)}(x,y)$ are the phason field of shear stresses, normal stresses, and displacements along x- and y-axes, respectively. Subscripts $k = I$ and $k = \mathrm{II}$ mean, respectively, the upper and lower half-planes. Intervals $(-\infty, c) \cup (a, \infty)$, $[c, b]$, and (b, a) denote the bond, open part of the crack, and contact zone, respectively.

4. Theoretical Derivation of Interface Stresses and Displacement Jump

For the plane elastic problems, the stresses and displacements in the decagonal quasicrystal of point groups, 10 can be expressed in terms of the complex potentials as follows:

$$\begin{cases} \sigma_{yy}^{(k)}(x,y) - \mathrm{i}\sigma_{xy}^{(k)}(x,y) = 32r_1^{(k)}\left(f'_{4k}(z) + \overline{f'_{3k}(z)} + z\overline{f''_{4k}(z)} + \overline{f'_{4k}(z)}\right) \\ u_x^{(k)}(x,y) + \mathrm{i}u_y^{(k)}(x,y) = 32r_6^{(k)} f_{4k}(z) - 32r_5^{(k)}\left(\overline{f_{3k}(z)} + z\overline{f'_{4k}(z)}\right) \end{cases} \tag{24}$$

$$\begin{cases} H_{yy}^{(k)}(x,y) - \mathrm{i}H_{xy}^{(k)}(x,y) = -32R^{(k)}\left(\overline{f'_{2k}(z)} + z\overline{f''_{3k}(z)} + \frac{1}{2}z^2\overline{f'''_{4k}(z)} + \overline{f'_{3k}(z)} + z\overline{f''_{4k}(z)}\right) \\ w_x^{(k)}(x,y) + \mathrm{i}w_y^{(k)}(x,y) = 32r_7^{(k)}\left(\overline{f_{2k}(z)} + z\overline{f'_{3k}(z)} + \frac{1}{2}z^2\overline{f''_{4k}(z)}\right) \end{cases} \tag{25}$$

$$\begin{cases} \sigma_{xx}^{(k)}(x,y) + \sigma_{yy}^{(k)}(x,y) = 64r_1^{(k)}\left(f'_{4k}(z) + \overline{f'_{4k}(z)}\right) \\ H_{xx}^{(k)}(x,y) + H_{yy}^{(k)}(x,y) = -32R^{(k)}\left(f'_{3k}(z) + \overline{z}f''_{4k}(z) + \overline{f'_{3k}(z)} + z\overline{f''_{4k}(z)}\right) \end{cases} \tag{26}$$

in which

$r^{(k)} = C_{66}^{(k)}\left(K_1^{(k)} + K_2^{(k)}\right) - 2R^{(k)2}$, $r_1^{(k)} = \frac{r^{(k)}}{K_1^{(k)} - K_2^{(k)}} + C_{66}^{(k)}$,

$r_2^{(k)} = \frac{1}{2\left(C_{12}^{(k)} + C_{66}^{(k)}\right)} + \frac{K_1^{(k)} + K_2^{(k)}}{2r^{(k)}}$, $r_3^{(k)} = \frac{R^{(k)2}}{r^{(k)}}$, $r_4^{(k)} = \frac{K_1^{(k)} + K_2^{(k)}}{r^{(k)}}$,

$r_5^{(k)} = r_1^{(k)} r_4^{(k)} - r_3^{(k)}$, $r_6^{(k)} = 4r_1^{(k)} r_2^{(k)} - r_3^{(k)} - r_1^{(k)} r_4^{(k)}$, $r_7^{(k)} = \frac{R^{(k)}}{K_1^{(k)} - K_2^{(k)}}$

Introducing two new functions

$$s_k(z) = \overline{f}_{3k}(z) + z\overline{f}'_{4k}(z) \tag{27}$$

$$t_k(z) = \overline{f}_{2k}(z) + z\overline{f}'_{3k}(z) + \frac{1}{2}z^2\overline{f}''_{4k}(z) \tag{28}$$

Replacing z with $\overline{z}$ in Equations (27) and (28) to obtain the following expressions

$$\overline{f_{3k}(z)} = s_k(\overline{z}) - \overline{z}\overline{f'_{4k}(z)} \tag{29}$$

$$\overline{f_{2k}(z)} = t_k(\overline{z}) - \overline{z}\overline{f'_{3k}(z)} - \frac{1}{2}\overline{z}^2\overline{f''_{4k}(z)} \tag{30}$$

The differential of the above formula is

$$\overline{f'_{3k}(z)} + \overline{f'_{4k}(z)} = s'_k(\overline{z}) - \overline{z}\overline{f''_{4k}(z)} \tag{31}$$

$$\overline{f''_{3k}(z)} + 2\overline{f''_{4k}(z)} = s''_k(\overline{z}) - \overline{z}\overline{f'''_{4k}(z)} \tag{32}$$

$$\overline{f'_{2k}(z)} + \overline{f'_{3k}(z)} = t'_k(\overline{z}) - \overline{z}\overline{f''_{4k}(z)} - \overline{z}\overline{f''_{3k}(z)} - \frac{1}{2}\overline{z}^2\overline{f'''_{4k}(z)} \tag{33}$$

Substituting Equations (27), (29) and (31) into Equation (24) yields

$$\begin{cases} \sigma_{yy}^{(k)}(x,y) - \mathrm{i}\sigma_{xy}^{(k)}(x,y) = 32r_1^{(k)}\left(\mathrm{I}_k(z) + \Gamma_k(\overline{z}) + (z-\overline{z})\overline{\mathrm{I}'_k(z)}\right) \\ u_x^{(k)}(x,y) + \mathrm{i}u_y^{(k)}(x,y) = 32r_6^{(k)}f_{4k}(z) - 32r_5^{(k)}\left(s_k(\overline{z}) + (z-\overline{z})\overline{\mathrm{I}_k(z)}\right) \end{cases} \tag{34}$$

Substituting Equations (28), (30) and (32) into Equation (25), one obtains

$$\begin{cases} H_{yy}^{(k)}(x,y) - \mathrm{i}H_{xy}^{(k)}(x,y) = -32R^{(k)}\left(H_k(\overline{z}) + (z-\overline{z})\left(\Gamma'_k(\overline{z}) - \overline{I'_k(z)}\right) + \frac{1}{2}(z-\overline{z})^2\overline{\mathrm{I}''_k(z)}\right) \\ w_x^{(k)}(x,y) + \mathrm{i}w_y^{(k)}(x,y) = 32r_7^{(k)}\left(t_k(\overline{z}) + (z-\overline{z})\overline{f'_{3k}(z)} + \frac{1}{2}(z^2-\overline{z}^2)\overline{f''_{4k}(z)}\right) \end{cases} \tag{35}$$

Introduce the following notation

$$\mathrm{I}_k(z) = f'_{4k}(z),\ \Gamma_k(\overline{z}) = s'_k(\overline{z}),\ \mathrm{H}_k(\overline{z}) = t'_k(\overline{z}) \tag{36}$$

Thus,

$$\begin{array}{l} \overline{f'_{3k}(z)} = \Gamma_k(\overline{z}) - \overline{\mathrm{I}_k(z)} - \overline{z}\overline{\mathrm{I}'_k(z)}, \\ f'_{3k}(z) = \overline{\Gamma_k(\overline{z})} - \mathrm{I}_k(z) - z\mathrm{I}'_k(z), \end{array} \tag{37}$$

$\overline{f''_{3k}(z)} = \Gamma'_k(\overline{z}) - 2\overline{\mathrm{I}'_k(z)} - \overline{z}\overline{\mathrm{I}''_k(z)}$

Substituting Equation (37) into Equations (34) and (35), one obtains

$$\begin{cases} \sigma_{yy}^{(k)}(x,y) - \mathrm{i}\sigma_{xy}^{(k)}(x,y) = 32r_1^{(k)}\left(\mathrm{I}_k(z) + \Gamma_k(\overline{z}) + (z-\overline{z})\overline{\mathrm{I}'_k(z)}\right) \\ \frac{\partial\left(u_x^{(k)}(x,y) + \mathrm{i}u_y^{(k)}(x,y)\right)}{\partial x} = 32r_6^{(k)}\mathrm{I}_k(z) - 32r_5^{(k)}\left(\Gamma_k(\overline{z}) + (z-\overline{z})\overline{\mathrm{I}'_k(z)}\right) \end{cases} \tag{38}$$

$$\begin{cases} H_{yy}^{(k)}(x,y) - \mathrm{i}H_{xy}^{(k)}(x,y) = -32R^{(k)}\left(\mathrm{H}_k(\overline{z}) + (z-\overline{z})\left(\Gamma'_k(\overline{z}) - \overline{I'_k(z)}\right) + \frac{1}{2}(z-\overline{z})^2\overline{\mathrm{I}''_k(z)}\right) \\ \frac{\partial\left(w_x^{(k)}(x,y) + \mathrm{i}w_y^{(k)}(x,y)\right)}{\partial x} = 32r_7^{(k)}\left(\mathrm{H}_k(\overline{z}) + (z-\overline{z})\left(\Gamma'_k(\overline{z}) - \overline{\Gamma'_k(z)}\right) + \frac{1}{2}(z-\overline{z})^2\overline{\mathrm{I}''_k(z)}\right) \end{cases} \tag{39}$$

Equation (26) can be represented as

$$\begin{cases} \sigma_{xx}^{(k)}(x,y) + \sigma_{yy}^{(k)}(x,y) = 128r_1^{(k)}\mathrm{Re}\mathrm{I}_k(z) \\ H_{xx}^{(k)}(x,y) + H_{yy}^{(k)}(x,y) = -64R^{(k)}\mathrm{Re}\left(\overline{\Gamma_k(\overline{z})} - \mathrm{I}_k(z) + (\overline{z}-z)\mathrm{I}'_k(z)\right) \end{cases} \tag{40}$$

Combined with continuity conditions Equations (20) and (21), the following equations are given:

$$\begin{cases} r_1^{(\mathrm{I})}\mathrm{I}_{\mathrm{I}}^{+}(x) - r_1^{(\mathrm{II})}\Gamma_{\mathrm{II}}^{+}(x) = r_1^{(\mathrm{II})}\mathrm{I}_{\mathrm{II}}^{-}(x) - r_1^{(\mathrm{I})}\Gamma_{\mathrm{I}}^{-}(x) & x \in (-\infty,\infty) \\ r_6^{(\mathrm{I})}\mathrm{I}_{\mathrm{I}}^{+}(x) + r_5^{(\mathrm{II})}\Gamma_{\mathrm{II}}^{+}(x) = r_6^{(\mathrm{II})}\mathrm{I}_{\mathrm{II}}^{-}(x) + r_5^{(\mathrm{I})}\Gamma_{\mathrm{I}}^{-}(x) & x \in (-\infty,c)\cup(a,\infty) \end{cases} \tag{41}$$

$$\begin{cases} R^{(\mathrm{II})}\mathrm{H}_{\mathrm{II}}^{+}(x) = R^{(\mathrm{I})}\mathrm{H}_{\mathrm{I}}^{-}(x) & x \in (-\infty,\infty) \\ r_7^{(\mathrm{II})}\mathrm{H}_{\mathrm{II}}^{+}(x) = r_7^{(\mathrm{I})}\mathrm{H}_{\mathrm{I}}^{-}(x) & x \in (-\infty,c)\cup(a,\infty) \end{cases} \tag{42}$$

where superscripts "+" and "−" denote the limit values of the analytical functions, when $z \to x + i0$ and $z \to x - i0$, respectively.

Both sides of Equation (41) represent the boundary values of two analytical functions in their respective half-planes, and the two functions can be analytically extended into the whole plane. For the phonon field, we can introduce the following functions:

$$A_1(z) = \begin{cases} r_1^{(\mathrm{I})}\mathrm{I}_\mathrm{I}(z) - r_1^{(\mathrm{II})}\Gamma_\mathrm{II}(z) & y > 0 \\ r_1^{(\mathrm{II})}\mathrm{I}_\mathrm{II}(z) - r_1^{(\mathrm{I})}\Gamma_\mathrm{I}(z) & y < 0 \end{cases} \tag{43}$$

and

$$A_2(z) = \begin{cases} r_6^{(\mathrm{I})}\mathrm{I}_\mathrm{I}(z) + r_5^{(\mathrm{II})}\Gamma_\mathrm{II}(z) & y > 0 \\ r_6^{(\mathrm{II})}\mathrm{I}_\mathrm{II}(z) + r_5^{(\mathrm{I})}\Gamma_\mathrm{I}(z) & y < 0 \end{cases} \tag{44}$$

Corresponding, for the phason field, introducing new functions $B_1(z)$ and $B_2(z)$

$$B_1(z) = \begin{cases} R^{(\mathrm{II})}\mathrm{H}_\mathrm{II}(z) & y > 0 \\ R^{(\mathrm{I})}\mathrm{H}_\mathrm{I}(z) & y < 0 \end{cases} \tag{45}$$

and

$$B_2(z) = \begin{cases} r_7^{(\mathrm{II})}\mathrm{H}_\mathrm{II}(z) & y > 0 \\ r_7^{(\mathrm{I})}\mathrm{H}_\mathrm{I}(z) & y < 0 \end{cases} \tag{46}$$

$A_1(z)$ and $B_1(z)$ are two analytical functions in the whole plane; $A_2(z)$ and $B_2(z)$ are analytical in the region $(-\infty, c) \cup (a, \infty)$, and the functions $\mathrm{I}_k(z)$, $\Gamma_k(z)$ and $\mathrm{H}_k(z)$ are bounded at infinity. Based on Liouvill's theorem, we can conclude that $A_1(z)$ and $B_1(z)$ are constants in the whole plane. That is

$$A_1(z) \equiv A \tag{47}$$

$$B_1(z) \equiv B \tag{48}$$

After algebraic manipulations, Equations (43) and (44) can be represented in the form

$$\begin{cases} r_1^{(\mathrm{I})}\mathrm{I}_\mathrm{I}(z) = g\left(A_2(z) + r_8^{(\mathrm{II})}A\right) & y > 0 \\ r_1^{(\mathrm{II})}\mathrm{I}_\mathrm{II}(z) = g\gamma\left(A_2(z) + r_8^{(\mathrm{I})}A\right) & y < 0 \end{cases} \tag{49}$$

and

$$\begin{cases} r_1^{(\mathrm{II})}\Gamma_\mathrm{II}(z) = g\left(A_2(z) + r_8^{(\mathrm{II})}A\right) - A & y > 0 \\ r_1^{(\mathrm{I})}\Gamma_\mathrm{I}(z) = g\gamma\left(A_2(z) + r_8^{(\mathrm{I})}A\right) - A & y < 0 \end{cases} \tag{50}$$

$$\begin{cases} \mathrm{H}_\mathrm{II}(z) = \frac{B}{R^{(\mathrm{II})}} & y > 0 \\ \mathrm{H}_\mathrm{I}(z) = \frac{B}{R^{(\mathrm{I})}} & y < 0 \end{cases} \tag{51}$$

where

$g = \frac{r_1^{(\mathrm{I})}r_1^{(\mathrm{II})}}{r_1^{(\mathrm{I})}r_5^{(\mathrm{II})}+r_1^{(\mathrm{II})}r_6^{(\mathrm{I})}}$, $\gamma = \frac{r_1^{(\mathrm{I})}r_5^{(\mathrm{II})}+r_1^{(\mathrm{II})}r_6^{(\mathrm{I})}}{r_1^{(\mathrm{II})}r_5^{(\mathrm{I})}+r_1^{(\mathrm{I})}r_6^{(\mathrm{II})}}$, $r_8^{(k)} = \frac{r_5^{(k)}}{r_1^{(k)}}$.

The expressions for stresses in the phonon field can be rewritten as

$$\sigma_{yy}(x,y) - \mathrm{i}\sigma_{xy}(x,y) = \begin{cases} 32\left(g\left(A_2(z) + r_8^{(\mathrm{II})}A\right) + g\gamma\left(A_2(z) + r_8^{(\mathrm{I})}A\right) - A + g(z-\bar{z})\overline{A_2'(z)}\right) & y > 0 \\ 32\left(g\gamma\left(A_2(z) + r_8^{(\mathrm{I})}A\right) + g\left(A_2(z) + r_8^{(\mathrm{II})}A\right) - A + g\gamma(z-\bar{z})\overline{A_2'(z)}\right) & y < 0 \end{cases} \tag{52}$$

and

$$\sigma_{xx}(x,y) + \sigma_{yy}(x,y) = 128\begin{cases} g\mathrm{Re}\left(A_2(z) + r_8^{(\mathrm{II})}A\right), & y > 0 \\ g\gamma\mathrm{Re}\left(A_2(z) + r_8^{(\mathrm{I})}A\right), & y < 0 \end{cases} \tag{53}$$

The expressions for stresses in the phason field can be rewritten as

$$H_{yy}(x,y) - \mathrm{i}H_{xy}(x,y) = -32\begin{cases} B + R^{(\mathrm{I})}(z-\bar{z})\frac{g}{r_1^{(\mathrm{I})}}\left(\gamma A_2'(\bar{z}) - \overline{A_2'(z)}\right) + \frac{1}{2}(z-\bar{z})^2\overline{A_2''(z)}, & y>0 \\ B + R^{(\mathrm{II})}(z-\bar{z})\frac{g}{r_1^{(\mathrm{II})}}\left(A_2'(\bar{z}) - \gamma\overline{A_2'(z)}\right) + \frac{1}{2}\gamma(z-\bar{z})^2\overline{A_2''(z)}, & y<0 \end{cases} \quad (54)$$

and

$$H_{xx}(x,y) + H_{yy}(x,y) = -64\begin{cases} \frac{R^{(\mathrm{I})}}{r_1^{(\mathrm{I})}}\mathrm{Re}\left(g\left(\gamma\overline{A}_2(z) - A_2(z) + (\bar{z}-z)A_2'(z)\right) + \left(g\gamma r_8^{(\mathrm{I})} - g r_8^{(\mathrm{II})} - 1\right)A\right), & y>0 \\ \frac{R^{(\mathrm{II})}}{r_1^{(\mathrm{II})}}\mathrm{Re}\left(g\left(\overline{A}_2(z) - \gamma A_2(z) + (\bar{z}-z)\gamma A_2'(z)\right) + \left(g r_8^{(\mathrm{II})} - g\gamma r_8^{(\mathrm{I})} - 1\right)A\right), & y<0 \end{cases} \quad (55)$$

The expressions for displacements in the phonon and phason fields can be rewritten in the form

$$\frac{\partial\left(u_x(x,y) + \mathrm{i}u_y(x,y)\right)}{\partial x} = 32\begin{cases} r_6^{(\mathrm{I})} g\left(A_2(z) + r_8^{(\mathrm{II})}A\right) \\ -r_5^{(\mathrm{I})}\left(g\gamma\left(A_2(z) + r_8^{(\mathrm{I})}A\right) - A + g(z-\bar{z})\overline{A_2'(z)}\right), & y>0 \\ r_6^{(\mathrm{II})} g\gamma\left(A_2(z) + r_8^{(\mathrm{I})}A\right) \\ -r_5^{(\mathrm{II})}\left(g\left(A_2(z) + r_8^{(\mathrm{II})}A\right) - A + g\gamma(z-\bar{z})\overline{A_2'(z)}\right), & y<0 \end{cases} \quad (56)$$

and

$$\frac{\partial\left(w_x(x,y) + iw_y(x,y)\right)}{\partial x} = \begin{cases} 32 r_7^{(\mathrm{I})}\left(\frac{B}{R^{(\mathrm{I})}} + \frac{g}{r_1^{(\mathrm{I})}}(z-\bar{z})\left(\gamma A_2'(\bar{z}) - \overline{A'_2(z)} + \frac{1}{2}(z-\bar{z})\overline{A_2''(z)}\right)\right), & y>0 \\ 32 r_7^{(\mathrm{II})}\left(\frac{B}{R^{(\mathrm{II})}} + \frac{g}{r_1^{(\mathrm{II})}}(z-\bar{z})\left(A_2'(\bar{z}) - \gamma\overline{A'_2(z)} + \frac{1}{2}\gamma(z-\bar{z})\overline{A_2''(z)}\right)\right), & y<0 \end{cases} \quad (57)$$

Using Equations (49)–(54), we can introduce the following functions:

$$F(z) = A_2(z) + pA \quad (58)$$

$$H(z) = B_2(z) \quad (59)$$

Substituting Equations (58) and (59) into Equations (54) and (55), one obtains

$$\frac{\sigma_{yy}(x,y) - \mathrm{i}\sigma_{xy}(x,y)}{32g} = \begin{cases} F(z) + \gamma F(\bar{z}) + (z-\bar{z})\overline{F}'(\bar{z}) & y>0 \\ \gamma F(z) + F(\bar{z}) + \gamma(z-\bar{z})\overline{F}'(\bar{z}) & y<0 \end{cases} \quad (60)$$

and

$$\sigma_{xx}(x,y) + \sigma_{yy}(x,y) = 128\begin{cases} g\mathrm{Re}\left(F(z) + \left(r_8^{(\mathrm{II})} - p\right)A\right), & y>0 \\ g\mathrm{Re}\left(\gamma F(z) + \gamma\left(r_8^{(\mathrm{I})} - p\right)A\right), & y<0 \end{cases} \quad (61)$$

$$\frac{H_{yy}(x,y) - \mathrm{i}H_{xy}(x,y)}{-32g} = \begin{cases} R^{(\mathrm{I})}\left(\mathrm{H_I}(\bar{z}) + \frac{(z-\bar{z})}{r_1^{(\mathrm{I})}}\left(\left(\gamma F'(\bar{z}) - \overline{F}'(\bar{z})\right) + \frac{1}{2}(z-\bar{z})\overline{F}''(\bar{z})\right)\right), & y>0 \\ R^{(\mathrm{II})}\left(\mathrm{H_{II}}(\bar{z}) + \frac{(z-\bar{z})}{r_1^{(\mathrm{II})}}\left(F'(\bar{z}) - \gamma\overline{F}'(\bar{z})\right) + \frac{\gamma}{2}(z-\bar{z})\overline{F}''(\bar{z})\right), & y<0 \end{cases} \quad (62)$$

and

$$H_{xx}(x,y) + H_{yy}(x,y) = -64\begin{cases} \frac{R^{(\mathrm{I})}}{r_1^{(\mathrm{I})}}\mathrm{Re}\left(g\left(\gamma\overline{F}(z) - F(z) + (\bar{z}-z)F'(z)\right) + gp(1-\gamma)A + \left(g\gamma r_8^{(\mathrm{I})} - g r_8^{(\mathrm{II})} - 1\right)A\right), & y>0 \\ \frac{R^{(\mathrm{II})}}{r_1^{(\mathrm{II})}}\mathrm{Re}\left(g\left(\overline{F}(z) - \gamma F(z) + (\bar{z}-z)\gamma F'(z)\right) + gp(\gamma-1)A + \left(g r_8^{(\mathrm{II})} - g\gamma r_8^{(\mathrm{I})} - 1\right)A\right), & y<0 \end{cases} \quad (63)$$

$$\frac{\partial(u_x+\mathrm{i}u_y)}{\partial x}=32g\begin{cases}\frac{r_6^{(\mathrm{I})}}{r_1^{(\mathrm{I})}}\Big(F(z)+r_8^{(\mathrm{II})}A-pA\Big)\\-\frac{r_5^{(\mathrm{I})}}{r_1^{(\mathrm{I})}}\Big(\gamma\Big(F(\overline{z})-pA+r_8^{(\mathrm{I})}A\Big)-\frac{A}{g}+(z-\overline{z})\overline{F}'(\overline{z})\Big) & y>0\\\frac{r_6^{(\mathrm{II})}}{r_1^{(\mathrm{II})}}\gamma\Big(F(z)+r_8^{(\mathrm{I})}A-pA\Big)\\-\frac{r_5^{(\mathrm{II})}}{r_1^{(\mathrm{II})}}\Big(\Big(F(\overline{z})+r_8^{(\mathrm{II})}A-pA\Big)-\frac{A}{g}+(z-\overline{z})\gamma\overline{F}'(\overline{z})\Big) & y<0\end{cases}\tag{64}$$

$$\frac{\partial}{\partial x}\left(w_x(x,y)+\mathrm{i}w_y(x,y)\right)=\begin{cases}32r_7^{(\mathrm{I})}\Big(\mathrm{H_I}(\overline{z})+\frac{g}{r_1^{(\mathrm{I})}}(z-\overline{z})\Big(\Big(\gamma F'(\overline{z})-\overline{F}'(\overline{z})\Big)+\frac{1}{2}(z-\overline{z})\overline{F}''(\overline{z})\Big)\Big), & y>0\\32r_7^{(\mathrm{II})}\Big(\mathrm{H_{II}}(\overline{z})+\frac{g}{r_1^{(\mathrm{II})}}(z-\overline{z})\Big(\Big(F'(\overline{z})-\gamma\overline{F}'(\overline{z})\Big)+\frac{\gamma}{2}(z-\overline{z})\overline{F}''(\overline{z})\Big)\Big), & y<0\end{cases}\tag{65}$$

Thus, the complex function expressions of the stresses and displacement jump derivatives on the interface are written as

$$\begin{cases}\sigma_{yy}^{(\mathrm{I})}(x,0)-\mathrm{i}\sigma_{xy}^{(\mathrm{I})}(x,0)=32g(F^+(x)+\gamma F^-(x))\\\frac{\partial}{\partial x}\Big(\Big(u_x^{(\mathrm{I})}(x)+\mathrm{i}u_y^{(\mathrm{I})}(x)\Big)-\Big(u_x^{(\mathrm{II})}(x)+\mathrm{i}u_y^{(\mathrm{II})}(x)\Big)\Big)=32(F^+(x)-F^-(x))\end{cases}\tag{66}$$

$$\begin{cases}H_{yy}^{(\mathrm{I})}(x,0)-\mathrm{i}H_{xy}^{(\mathrm{I})}(x,0)=-32\Big(K_1^{(\mathrm{I})}-K_2^{(\mathrm{I})}\Big)\mathrm{H}^-(x)\\\frac{\partial}{\partial x}\Big(\Big(w_x^{(\mathrm{I})}(x,0)+\mathrm{i}w_y^{(\mathrm{I})}(x,0)\Big)-\Big(w_x^{(\mathrm{II})}(x,0)+\mathrm{i}w_y^{(\mathrm{II})}(x,0)\Big)\Big)=32\Big(r_7^{(\mathrm{I})}\mathrm{H}^-(x)-r_7^{(\mathrm{II})}\mathrm{H}^+(x)\Big)\end{cases}\tag{67}$$

5. Complex Potential Solution of the Problem

From the derivation in the previous section, the problem is transformed into the homogeneous Dirichlet–Riemann boundary value problem

$$\begin{cases}F^+(x)+\gamma F^-(x)=0, & x\in(c,b)\\\mathrm{Im}F^{\pm}(x)=0, & x\in(b,a)\end{cases}\tag{68}$$

$$\begin{cases}H^-(x)=0, & x\in(c,b)\\\mathrm{Im}H^{\mp}(x)=0, & x\in(b,a)\end{cases}\tag{69}$$

From the second equation of the above equations, one obtains

$$\mathrm{H}(z)=0\tag{70}$$

Thus, the stresses and displacements of the phason field can be expressed as

$$\frac{H_{yy}(x,y)-\mathrm{i}H_{xy}(x,y)}{-32g}=\begin{cases}\frac{R^{(\mathrm{I})}}{r_1^{(\mathrm{I})}}(z-\overline{z})\Big(\Big(\gamma F'(\overline{z})-\overline{F}'(\overline{z})\Big)+\frac{1}{2}(z-\overline{z})\overline{F}''(\overline{z})\Big), & y>0\\\frac{R^{(\mathrm{II})}}{r_1^{(\mathrm{II})}}(z-\overline{z})\Big(\Big(F'(\overline{z})-\gamma\overline{F}'(\overline{z})\Big)+\frac{\gamma}{2}(z-\overline{z})\overline{F}''(\overline{z})\Big), & y<0\end{cases}\tag{71}$$

$$\frac{\partial}{\partial x}\left(w_x(x,y)+\mathrm{i}w_y(x,y)\right)=\begin{cases}\frac{32gr_7^{(1)}}{r_1^{(\mathrm{I})}}(z-\overline{z})\Big(\Big(\gamma F'(\overline{z})-\overline{F}'(\overline{z})\Big)+\frac{1}{2}(z-\overline{z})\overline{F}''(\overline{z})\Big), & y>0\\\frac{32gr_7^{(1I)}}{r_1^{(\mathrm{II})}}(z-\overline{z})\Big(\Big(F'(\overline{z})-\gamma\overline{F}'(\overline{z})\Big)+\frac{\gamma}{2}(z-\overline{z})\overline{F}''(\overline{z})\Big), & y<0\end{cases}\tag{72}$$

when $x\in(-\infty,c)\cup(a,\infty)$, $F^+(x)=F^-(x)=F(x)$ is valid.

Using the conditions at infinity, one obtains

$$32g(1+\gamma)F(x)=\sigma-\mathrm{i}\tau,\ x\in(-\infty,c)\cup(a,\infty)\tag{73}$$

Function $F(z)$ is analytic at infinity; one obtains

$$F(z)|_{z\to\infty} = \frac{\sigma - \mathrm{i}\tau}{32g(1+\gamma)} \tag{74}$$

The general solution of Equation (68) of the combined Dirichlet–Riemann boundary value problem from [30] is unbounded at all points a, b, c and can be written as

$$F(z) = \mathrm{i}P(z)X_1(z) + Q(z)X_2(z) \tag{75}$$

where

$P(z) = C_1 z + C_2,\ Q(z) = D_1 z + D_2,$

$X_1(z) = \frac{e^{\mathrm{i}\varphi(z)}}{\sqrt{(z-a)}\sqrt{(z-c)}},\ X_2(z) = \frac{e^{\mathrm{i}\varphi(z)}}{\sqrt{(z-b)}\sqrt{(z-c)}}$

$\varphi(z) = 2\varepsilon \ln \frac{\sqrt{(a-b)(z-c)}}{\sqrt{(a-c)(z-b)}+\sqrt{(b-c)(z-a)}}$

$\varepsilon = \frac{1}{2\pi}\ln\gamma$

Replacing z with $\overline{z}$ in Equations (31)–(33), one has

$$F(\overline{z}) = \mathrm{i}P(\overline{z})X_1(\overline{z}) + Q(\overline{z})X_2(\overline{z}) \tag{76}$$

derived by differentiation

$$\begin{aligned}
F'(z) &= \mathrm{i}C_1X_1(z) + \mathrm{i}P(z)X_1'(z) + D_1X_2(z) + Q(z)X_2'(z) \\
F''(z) &= 2\mathrm{i}C_1X_1'(z) + \mathrm{i}P(z)X_1''(z) + 2D_1X_2'(z) + Q(z)X_2''(z) \\
\overline{F'(z)} &= \mathrm{i}C_1\overline{X_1(z)} + \mathrm{i}\overline{P(z)X_1'(z)} + D_1\overline{X_2(z)} + \overline{Q(z)X_2'(z)} \\
\overline{F''(z)} &= 2\mathrm{i}C_1\overline{X_1'(z)} + \mathrm{i}\overline{P(z)}\overline{X_1''(z)} + 2D_1\overline{X_2'(z)} + \overline{Q(z)}\overline{X_2''(z)}
\end{aligned} \tag{77}$$

$X_1(z)$ and $X_2(z)$ have an oscillating singularity at the point $z = c + \mathrm{i}0$ and square-root singularities at the points $z = a + \mathrm{i}0$ and $z = b + \mathrm{i}0$, respectively. $X_1(z)$ and $X_2(z)$ at infinity can be written as

$$\begin{aligned}
X_1(z) &= z^{-2}e^{\mathrm{i}\beta}\left(z + \mathrm{i}\beta_1 + \tfrac{c+a}{2}\right) + O\left(z^{-3}\right) \\
X_2(z) &= z^{-2}e^{\mathrm{i}\beta}\left(z + \mathrm{i}\beta_1 + \tfrac{c+b}{2}\right) + O\left(z^{-3}\right)
\end{aligned} \tag{78}$$

where

$\beta = \varepsilon \ln \frac{\sqrt{a-c}-\sqrt{b-c}}{\sqrt{a-c}+\sqrt{b-c}}, \beta_1 = \varepsilon\sqrt{(a-c)(b-c)},$

The arbitrary constants C_1, C_2, D_1, D_2 have the following forms

$$\begin{aligned}
&C_1 = -\tau\cos\beta - \sigma\sin\beta, C_2 = -\tfrac{c+a}{2}C_1 - \beta_1 D_1, \\
&D_1 = \sigma\cos\beta - \tau\sin\beta, D_2 = \beta_1 C_1 - \tfrac{c+b}{2}D_1,
\end{aligned} \tag{79}$$

The stresses and the derivatives of the displacement jumps for $z = x + \mathrm{i}0$ can be expressed as follows:

for $x > a$:

$$\begin{aligned}
&\sigma_{yy}^{(\mathrm{I})}(x,0) - \mathrm{i}\sigma_{xy}^{(\mathrm{I})}(x,0) = 32g(1+\gamma)\left(\tfrac{Q(x)}{\sqrt{x-b}} + \mathrm{i}\tfrac{P(x)}{\sqrt{x-a}}\right)\tfrac{\exp[\mathrm{i}\varphi(x)]}{\sqrt{x-c}} \\
&H_{yy}^{(\mathrm{I})}(x,0) - \mathrm{i}H_{xy}^{(\mathrm{I})}(x,0) = 0
\end{aligned} \tag{80}$$

for $x \in (b,a)$:

$$\begin{aligned}
\sigma_{yy}^{(1)} = &\tfrac{32gP(x)}{\sqrt{(x-c)(a-x)}}((1-\gamma)ch\varphi_0(x) + (1+\gamma)sh\varphi_0(x)) \\
&+\tfrac{32gQ(x)}{\sqrt{(x-c)(x-b)}}((1+\gamma)ch\varphi_0(x) + (1-\gamma)sh\varphi_0(x))
\end{aligned} \tag{81}$$

$$\left(u_x^{(\mathrm{I})} - u_x^{(\mathrm{II})}\right)' = \frac{2}{\sqrt{x-c}}\left(\frac{P(x)}{\sqrt{a-x}} ch\varphi_0(x) + \frac{Q(x)}{\sqrt{x-b}} sh\varphi_0(x)\right) \tag{82}$$

for $x \in (c, b)$:

$$\begin{aligned} &\left(u_x^{(\mathrm{I})}(x) + \mathrm{i}u_y^{(\mathrm{I})}(x)\right)' - \left(u_x^{(\mathrm{II})}(x) + \mathrm{i}u_y^{(\mathrm{II})}(x)\right)' = \tfrac{32(1+\gamma)}{\sqrt{\gamma}}\left(\tfrac{P(x)}{\sqrt{a-x}} - \mathrm{i}\tfrac{Q(x)}{\sqrt{b-x}}\right)\tfrac{\exp(\mathrm{i}\varphi^*(x))}{\sqrt{x-c}} \\ &\left(w_x^{(\mathrm{I})}(x) + \mathrm{i}w_y^{(\mathrm{I})}(x)\right)' - \left(w_x^{(\mathrm{II})}(x) + \mathrm{i}w_y^{(\mathrm{II})}(x)\right)' = 0 \end{aligned} \tag{83}$$

where

$$\varphi^*(x) = 2\varepsilon \ln \frac{\sqrt{(a-b)(x-c)}}{\sqrt{(a-c)(b-x)} + \sqrt{(b-c)(a-x)}},$$

$$\varphi_0(x) = 2\varepsilon \mathrm{arctan}\sqrt{\frac{(b-c)(a-x)}{(a-c)(x-b)}}$$

From Equations (80) and (82), it can be seen that the normal stress in the phonon field has a square-root singularity for $x \to b+0$, and the shear stress has the same singularity for $x \to a+0$. The relevant stress intensity factors in the phonon field can be defined as

$$K_{S1} = \lim_{x \to b+0} \sqrt{2(x-b)}\sigma_{yy}(x,0),\ K_{S2} = \lim_{x \to a+0} \sqrt{2(x-a)}\sigma_{xy}(x,0), \tag{84}$$

and can be written as

$$K_{S1} = 64g\sqrt{\pi\gamma}\frac{Q(b)}{\sqrt{b-c}},\ K_{S2} = -32g(1+\gamma)\sqrt{\frac{2\pi}{a-c}}P(a) \tag{85}$$

Further, the stress intensity factors can be expressed in the form

$$\begin{aligned} K_{S1} &= \tfrac{\sqrt{\pi\gamma}}{1+\gamma}\left(\sqrt{b-c}(\sigma\cos\beta - \tau\sin\beta) - 2\varepsilon\sqrt{a-c}(\sigma\sin\beta + \tau\cos\beta)\right) \\ K_{S2} &= \sqrt{\tfrac{\pi}{2}}\left(\sqrt{a-c}(\sigma\sin\beta + \tau\cos\beta) + 2\varepsilon\sqrt{b-c}(\sigma\cos\beta - \tau\sin\beta)\right) \end{aligned} \tag{86}$$

The asymptotic behavior of the stresses and the displacement jumps in the phonon field at the points a and b can be written as

$$\begin{aligned} &\sigma_{yy}^{(\mathrm{I})}(x,0)\Big|_{x\to b+0} = \tfrac{K_{S1}}{\sqrt{2\pi(x-b)}},\ \sigma_{xy}^{(\mathrm{I})}(x,0)\Big|_{x\to a+0} = \tfrac{K_{S2}}{\sqrt{2\pi(x-a)}}, \\ &\left(u_y^{(\mathrm{I})}(x,0) - u_y^{(\mathrm{II})}(x,0)\right)\Big|_{x\to b-0} = \tfrac{K_{S1}}{16g\sqrt{2\pi\gamma}}\sqrt{b-x}, \\ &\left(u_x^{(\mathrm{I})}(x,0) - u_x^{(\mathrm{II})}(x,0)\right)\Big|_{x\to a-0} = \tfrac{K_{S2}}{8g(1+\gamma)\sqrt{2\pi}}\sqrt{a-x} \end{aligned} \tag{87}$$

6. Numerical Results and Discussion

The elastic constants of Al Ni Co quasicrystal alloy are taken as the elastic constants of the phonon field, phason field, and coupling constants of the phonon–phason field [31], as shown in Tables 1 and 2. In order to avoid matrix ill condition caused by material parameters in different orders of magnitude, the material constants are dimensionless, and the dimensionless stress intensity factors in the phonon field are obtained.

Table 1. Material I constants of decagonal quasicrystal.

Elastic Constants	The Value of Elastic Constant
Phonon field elastic constant/GPa	$C_{11} = 234.33,\ C_{66} = 88.46$
Phason field elastic constant/GPa	$K_1 = 122,\ K_2 = 24$
Coupling constant/GPa	$R = 0.8846$

Table 2. Material II constants of decagonal quasicrystal.

Elastic Constants	The Value of Elastic Constant
Phonon field elastic constant/GPa	$C_{11} = 214.33,\ C_{66} = 68.46$
Phason field elastic constant/GPa	$K_1 = 102,\ K_2 = 22$
Coupling constant/GPa	$R = 0.8646$

It is clear from Equations (84)–(86) that the length of the contact zone depends only on point b. Figure 2 shows the change in the stress intensity factor with the crack contact zone, where the relative contact zone length of a crack with a right contact zone is given. It can be found that for any point, the model framework that only considers the contact area can clearly define the area with a large contact area. The smaller the contact length, the greater the normal stress intensity factor. This is consistent with the trend of classical elasticity.

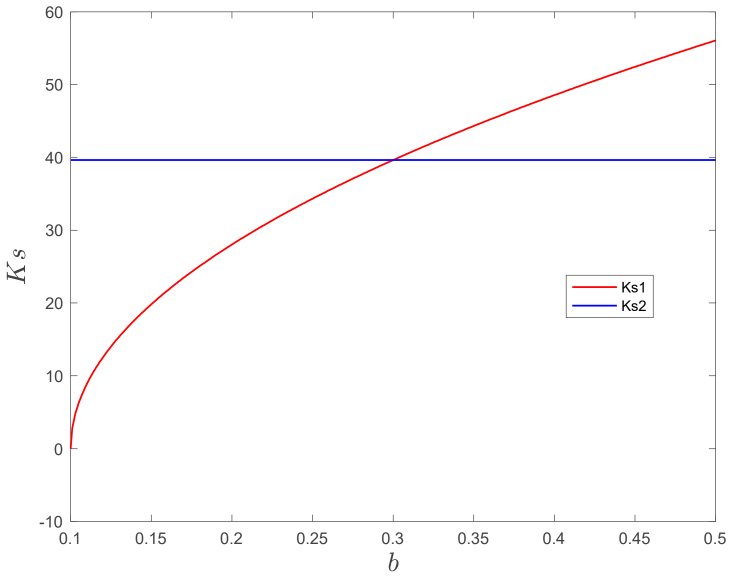

Figure 2. Variation in stress intensity factors with the contact zone length.

7. Conclusions

The interface crack contact zone of decagonal quasicrystal bi-materials under far-field mixed loading is studied. Based on the theory of complex variable function, the problem is transformed into a Dirichlet–Riemann problem for analytical solution. The expressions of stress, stress intensity factor, and displacement jump along the material interface are obtained by using the closed analytical formula of the interface crack in the single contact zone of the decagonal quasicrystal bi-materials, and the relationship between the fracture mechanics parameters of the interface crack is given. The analytical expression obtained can be used to verify some numerical analysis and can also accurately show the physical nature of the crack problem in the contact zone of decagonal quasicrystal materials.

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