Supplemental Materials for the Manuscript Entitled *"One Saddle Point and Two Types of Sensitivities Within the Lorenz 1963 and 1969 Models"*

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Abstract

As Supplemental Materials, the following two sections are provided: Section (A) A Simple Illustration of Ill-conditioning and Section (B) An Illustration of a Stiff Ordinary Differential Equation.

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Section A: A Simple Illustration of Ill-conditioning

To illustrate ill-conditioning, we revise the example from Kreyszig (2011) as follows:

$$0.9999x - 1.0001y = 0, \tag{A1}$$

$$x - y = 0. \tag{A2}$$

The above system can be written as:

$$\mathbf{A}U = 0. \tag{A3}$$

Here, a column vector U contains two components, x and y. A matrix $\mathbf{A}$ consists of the coefficients of variables x and y. A condition number $\kappa(\mathbf{A}) = 20,001$ suggests that the system is ill-conditioned. The system contains a unique solution of $(0,0)$, similar to the trivial critical point in Eq. (4). However, when a tiny perturbation (ϵ) is introduced, as follows:

$$0.9999x - 1.0001y = 0, \tag{A4}$$

$$x - y = \epsilon, \tag{A5}$$

the "perturbed' system produces a solution of $(x, y) = (5000.5\epsilon, 4999.5\epsilon)$. When $\epsilon = 10^{-4}$, the solution is shifted to (0.50005, 0.49995). This indicates numerical sensitivities, a sensitive dependence of the solution on tiny perturbations in the system.

Section B: An Illustration of a Stiff ODE

Here, we discuss the stiffness of ODEs. The simplest stiff ODE can be written as follows (Boyce and Diprima 2012):

$$\frac{dX}{d\tau} = \lambda X, \tag{B1}$$

where $\lambda < 0$ and $|\lambda|$ is large. The solution $e^{\lambda\tau}$ decreases with time. However, it is easy to obtain numerical instability because a truncation error of a p^{th}-order numerical scheme is proportional to the $(p+1)^{th}$ derivatives of the solutions. Using $\lambda = -300$ and the Euler method (with $p = 1$) as an example, the local truncation error is proportional to $\frac{1}{2}\Delta\tau^2 \frac{d^2X}{d\tau^2} \sim \frac{1}{2}\lambda^2\Delta\tau^2 X$, indicating large errors on the order of λ^2 when τ is small (i.e., X is not very small). As long as the magnitude of λ is large, very small step sizes are required to obtain stable numerical solutions.

Within a system of ODEs, stiffness is defined as the magnitude of the ratio between the largest and smallest eigenvalues (i.e., the condition number).

[1] Boyce, W.E. and Diprima, R.C., 2012: Elementary Differential Equations, 10th ed.; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 2012; ISBN1 13: 978-0470458327. ISBN2 10: 0470458321.

[2] Kreyszig, E, 2011: Advanced Engineering Mathematics,, 10th edition. John Wiley & Sons, INC. 1113 pp.