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Review

The Decay of Energy and Scalar Variance in Axisymmetric Turbulence

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Abstract: We review the recent progress in our understanding of the large scales in homogeneous (but anisotropic) turbulence. We focus on turbulence which emerges from Saffman-like initial conditions, in which the vortices possess a finite linear impulse. Such turbulence supports long-range velocity correlations of the form $\left\langle u_i u'_j \right\rangle = O(r^{-3})$, where $\mathbf{u}$ and $\mathbf{u}'$ are separated by a distance r, and these long-range interactions dominate the dynamics of large eddies. We show that, for axisymmetric turbulence, the energy and integral scales evolve as $u_\perp^2 \sim u_{//}^2 \sim t^{-6/5}$ and $\ell_\perp \sim \ell_{//} \sim t^{2/5}$, where $\perp$ and $//$ indicate directions that are perpendicular and parallel to the symmetry axis, respectively. These predictions are consistent with the evidence of direct numerical simulations. Similar results are obtained for the passive scalar variance, where we find that $\langle \theta^2 \rangle \sim t^{-6/5}$. The primary point of novelty in our discussion of passive scalar decay is that it is based in real (rather than spectral) space, making use of an integral invariant which is a generalization of the isotropic Corrsin integral.

Keywords: scalar variance; Saffman turbulence; axisymmetric turbulence

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Citation: Davidson, P.A. The Decay of Energy and Scalar Variance in Axisymmetric Turbulence. *Atmosphere* **2023**, *14*, 1019. https://doi.org/10.3390/atmos14061019

Academic Editors: Boris Galperin, Annick Pouquet, Peter Sullivan and Doris Folini

Received: 16 March 2023
Revised: 31 May 2023
Accepted: 5 June 2023
Published: 13 June 2023

1. Introduction

1.1. The Summer of 2002

In the summer of 2002, the author visited NCAR to discuss the behavior of the large scales in freely decaying, homogeneous turbulence with Jack Herring. It was to be the first of many memorable visits to NCAR. The primary focus at that time was on the behavior of Loitsyansky's integral in isotropic turbulence, but it left the author with a longstanding interest in the dynamics of large scales in all types of homogeneous turbulence. Initially, that interest centered on large scales in so-called Batchelor turbulence, whose energy spectrum satisfies $E(k \to 0) \sim k^4$, and where Loitsyansky's integral was shown to be conserved in fully developed turbulence (see Jackson et al. [1] and Ishida, Davidson, and Kaneda [2]). However, interest soon spread to include Saffman ($E(k \to 0) \sim k^2$) turbulence (Davidson, Okamoto, and Kaneda [3]; Krogstad and Davidson [4]), as well as the evolution of the large scales in rotating, stratified, and MHD turbulence (Davidson [5,6]). Finally, as part of an international network, the author began a study on the large scales in passive scalar mixing (Yoshimatsu and Kaneda [7]). Many of these studies have their roots in those discussions at NCAR in the summer of 2002.

This paper reviews some of the more recent developments that have been made in the large-scale behavior of freely decaying Saffman turbulence, with particular emphasis on energy decay and the kinematics of passive scalars. It particularly builds on the work of Davidson [5,6] and Davidson, Okamoto, and Kaneda [3], and on the fruitful discussions the author had with Yukio Kaneda and Katsunori Yoshimatsu in 2016 and 2017 as part of an international network.

1.2. Some Important Questions about the Decay of Passive Scalar Fluctuations

Let us start with some background information on the decay of passive scalar fluctuations in homogeneous turbulence, particularly the decay of scalar variance. Perhaps the first point to make is that, although there are points of contact with Richardson's celebrated theory of relative dispersion (Richardson [8]), passive scalar decay and Richardson dispersion are distinct phenomena, with the former focusing on the integral scales and the latter focusing on the inertial sub-range (Monin and Yoglam [9]).

Perhaps the most important paper in passive scalar decay is that written by Corrsin [10]. Crucially, Corrsin noted that, under certain conditions, a passive scalar, θ, evolving in freely decaying *isotropic* turbulence, possesses an integral invariant of the two-point correlation function $\langle\theta\theta'\rangle = \left\langle\theta(\mathbf{x})\theta(\mathbf{x}')\right\rangle$. Specifically, he noted that

$$L_\theta = \int_{V_\infty} \langle\theta\theta'\rangle d\mathbf{r},\ \mathbf{r} = \mathbf{x}' - \mathbf{x}, \tag{1}$$

is conserved (when it exists), provided that the triple correlation $\langle\mathbf{u}\theta\theta'\rangle$ falls off faster than $O(r^{-2})$ at large separations, where $r = |\mathbf{r}| = \left|\mathbf{x}' - \mathbf{x}\right|$. The possible existence of such an invariant immediately raises a host of questions, including the following.

(i) Under what conditions is this integral convergent? In other words, when does L_θ exist?
(ii) If the integral exists, under what conditions would we expect to find $\langle\mathbf{u}\theta\theta'\rangle \le O(r^{-3})$, so that L_θ is an invariant?
(iii) If the invariant exists in isotropic turbulence, how does this generalize to anisotropic turbulence?
(iv) Can we use the existence of the invariant to predict the decay rate of passive scalar variance?

We shall address all of these questions in this review. In particular, we generalize Corrsin's integral to anisotropic and axisymmetric turbulence and use it to predict the decay rate of passive scalar variance.

Of course, the statistics of θ depend on those of the velocity field, and so the answers to the questions above must depend on the nature of the turbulence in which the scalar finds itself. The strongest long-range correlations are to be found in anisotropic Saffman turbulence, where the two-point velocity correlations fall off slowly as $O(r^{-3})$, as discussed in Saffman [11,12] and Davidson, Okamoto, and Kaneda [3]. Consequently, the most stringent test for the existence of Corrsin's integral invariant is in anisotropic Saffman turbulence, and so that is the case that we consider here. To focus our thoughts, we follow Davidson, Okamoto, and Kaneda [3] and restrict the discussion to statistically axisymmetric (but anisotropic) turbulence.

We shall see that, in axisymmetric Saffman turbulence, L_θ is indeed convergent, though only conditionally so. It is also conserved. Moreover, the fact that L_θ is conserved in Saffman turbulence strongly suggests that it is an invariant in all other forms of homogeneous turbulence. Finally, we combine the conservation of L_θ with the self-similarity of large scales to show that $\langle\theta^2\rangle\ell^\theta_{//}\left(\ell^\theta_\perp\right)^2$ is an invariant, where $\ell^\theta_{//}$ and $\ell^\theta_\perp$ are integral scales parallel and perpendicular to the axis of symmetry, respectively. This in turn demands that the passive scalar variance decays as $\langle\theta^2\rangle \sim t^{-6/5}$, at least in Saffman turbulence.

1.3. The Structure of the Paper

The structure of the paper is as follows. We start, in Section 2, by reminding readers about the properties of certain integral invariants (Saffman invariants) associated with the two-point velocity correlations. Such invariants exist in so-called Saffman turbulence (turbulence in which the eddies possess a statistically significant amount of linear impulse) and are a statistical manifestation of linear momentum conservation. These invariants are important as their conservation constrains the way in which the integral scales can evolve, and this in turn has consequences for the rate of energy decay and the rate of growth of integral length scales.

Next, in Sections 3 and 4, we generalize Corrsin's passive scalar integral to anisotropic axisymmetric turbulence and note that its invariance reflects the conservation of the total amount of passive scalar within a large control volume embedded in the turbulence. We also note the analogy between our generalized Corrsin integral and the Saffman invariants.

The results of Sections 2–4 are then combined in Section 5, where we use these various integral invariants to determine the behavior of integral scales in axisymmetric turbulence. In particular, we deduce the rate of energy decay and passive scalar variance, as well as the evolution of various integral scales. We close, in Sections 6 and 7, with the discussion and conclusions, respectively. As noted earlier, the primary point of novelty in the paper lies in the use of our generalized Corrsin integral to deduce the integral properties of a passive scalar.

2. Invariants of Freely Decaying, Axisymmetric Saffman Turbulence

As a prelude to our discussion of invariants in passive scalar mixing, it seems appropriate to review the large-scale dynamics of freely decaying Saffman turbulence. The review is based on the results of Saffman [11] for the most general anisotropic case, and on the findings of Davidson [5] and Davidson, Okamoto, and Kaneda [3], hereafter denoted as DOK12, for the case where the fields are statistically axisymmetric.

2.1. Anisotropic Saffman Turbulence

Saffman [11] considered homogeneous turbulence which emerges from an initial condition in which there are no long-range algebraic correlations in the vorticity field. Thus, he assumed that $\langle \omega_i \omega'_j \rangle_{r\to\infty}$ and $\langle \omega_i \omega_j \omega'_k \rangle_{r\to\infty}$, as well as all higher-order long-range vorticity cumulants, are exponentially small at $t = 0$. (From now on, we shall abbreviate expressions such as $\langle \omega_i(\mathbf{x}) \omega_j(\mathbf{x}') \rangle_{r\to\infty}$ and $\langle \omega_i \omega'_j \rangle_\infty$). This leads to a low-wavenumber energy spectrum of the form $E(k \to 0) \sim k^2$ and so provides an alternative to the more traditional analysis of Batchelor and Proudman [13], who found $E(k \to 0) \sim k^4$. Interestingly, recent measurements taken by Krogstad and Davidson [4] support the idea that, at least for certain types of grid turbulence, the dynamics of freely decaying turbulence are indeed consistent with $E(k \to 0) \sim k^2$, rather than the more traditional assumption of $E(k \to 0) \sim k^4$.

The spectral tensor,

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int \langle u_i u'_j \rangle(\mathbf{r}) e^{-j\mathbf{k}\cdot\mathbf{r}} d\mathbf{r},$$

in Saffman turbulence takes the low-k form

$$\Phi_{ij}(\mathbf{k} \to 0) = \left\{ \delta_{i\alpha} - \frac{k_i k_\alpha}{k^2} \right\} \left\{ \delta_{j\beta} - \frac{k_j k_\beta}{k^2} \right\} M_{\alpha\beta} + O(k), \tag{2}$$

where $M_{\alpha\beta}$ is symmetric and independent of $\mathbf{k}$. Moreover, the coefficients $M_{\alpha\beta}$ is related to the second moment of the two-point vorticity correlation, $\langle \omega_i \omega'_j \rangle$, by

$$(2\pi)^3 M_{ij} = \frac{1}{2} \Omega_{ij} - \frac{1}{4} \delta_{ij} \Omega_{kk},\ \Omega_{ij} = \int r^2 \langle \omega_i \omega'_j \rangle d\mathbf{r}, \tag{3}$$

(Saffman [11]). Evidently, $\Phi_{ij}(\mathbf{k})$ is non-analytic at $\mathbf{k} = 0$ as the value depends on the way in which we approach the origin. Expansion (2) yields

$$\Phi_{ii}(\mathbf{k}) = \left\{ \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right\} M_{\alpha\beta} + O(k)\,,\ E(k \to 0) = \frac{4}{3} \pi M_{\alpha\alpha} k^2\,, \tag{4}$$

and the corresponding asymptotic form of $\langle u_i u'_j \rangle_\infty$ is

$$\langle u_i u'_j \rangle_\infty = -M_{\alpha\beta} \pi^2 \left\{ \delta_{i\alpha} \nabla^2 - \frac{\partial^2}{\partial r_i \partial r_\alpha} \right\} \left\{ \delta_{j\beta} \nabla^2 - \frac{\partial^2}{\partial r_j \partial r_\beta} \right\} r. \tag{5}$$

Note that $\langle u_i u'_j \rangle_\infty = O(r^{-3})$.

The origin of the r^{-3} tails in $\langle u_i u'_j \rangle_\infty$ reverts back to Biot–Savart law, in which an eddy with a finite linear impulse induces irrotational far-field velocity fluctuations of the form

$$\mathbf{u}_\infty(\mathbf{x}) = \frac{1}{4\pi}\left(\mathbf{L}_{\text{eddy}} \cdot \nabla\right)\nabla(1/r) + O(r^{-4}), \tag{6}$$

where $\mathbf{L}_{\text{eddy}}$ is the linear impulse of the eddy, $\mathbf{L}_{\text{eddy}} = \frac{1}{2}\int_{V_{\text{eddy}}} \mathbf{x} \times \boldsymbol{\omega} dV$. Thus, individual eddies cast a long shadow which falls off slowly as $|\mathbf{u}_\infty| = O(r^{-3})$, leading to the long-range algebraic correlations $\langle u_i u'_j \rangle_\infty = O(r^{-3})$ and $\langle u_i u_j u'_k \rangle_\infty = O(r^{-3})$. Since these far-field fluctuations are irrotational, they do not contribute to the leading-order terms in $\langle \omega_i \omega'_j \rangle_\infty$, which are of order $\langle \omega_i \omega'_j \rangle_\infty \leq O(r^{-6})$, rather than $\langle \omega_i \omega'_j \rangle_\infty \leq O(r^{-5})$, for $t > 0$ (Saffman [11]).

Since $\langle \omega_i \omega'_j \rangle_\infty \leq O(r^{-6})$, the integral Ω_{ij} defined by (3) is absolutely convergent, and so M_{ij} is well defined. On the other hand, the integrals

$$L_{ij} = \underset{V\to\infty}{\text{Lim}} \int_V \langle u_i u'_j \rangle d\mathbf{r} \tag{7}$$

are only conditionally convergent. That is to say, they are convergent in the sense that they are independent of the size of V as $V \to \infty$. They are also uniquely determined by M_{ij}, since L_{ij} may be written as a surface integral whose integrand is the far-field Expansion (5):

$$L_{ij} = \underset{V\to\infty}{\text{Lim}} \int_V \langle u_i u'_j \rangle d\mathbf{r} = \underset{S\to\infty}{\text{Lim}} \oint_S \langle u_i u'_k \rangle_\infty r_j dS_k. \tag{8}$$

However, the numerical value of L_{ij} depends on the shape of the surface S as it recedes to infinity. This conditional convergence is a direct consequence of the non-analytic form of $\Phi_{ij}(\mathbf{k})$ at $\mathbf{k} = 0$. For the particular case of a large spherical volume of radius R and volume V_R, Saffman [11] established that

$$L_{ij} = \underset{V_R\to\infty}{\text{Lim}} \int_{V_R} \langle u_i u'_j \rangle d\mathbf{r} = (2\pi)^3 \left[\frac{7}{15} M_{ij} + \frac{1}{15}\delta_{ij} M_{kk}\right]. \tag{9}$$

The integral L_{ij} is particularly important because, as Saffman [11] suggested, L_{ij}, and hence M_{ij}, are invariants:

$$L_{ij} = \text{constant}, \;\; M_{ij} = \text{constant}. \tag{10}$$

This idea is as follows. Homogeneity tells us that

$$L_{ij} = \underset{V_R\to\infty}{\text{Lim}} \left\langle \left\{ \frac{1}{V_R^{1/2}} \int_{V_R} u_i d\mathbf{x} \right\} \left\{ \frac{1}{V_R^{1/2}} \int_{V_R} u'_j d\mathbf{x}' \right\} \right\rangle \tag{11}$$

(see, for example, Davidson [6]), where V_R is a large spherical control volume with surface S_R and radius R. So, the dynamical behavior of L_{ij} depends on that of the linear momentum $\mathbf{P} = \int_{V_R} \mathbf{u}\, dV$. However, $\mathbf{P}$ can only change as a result of pressure forces acting on the surface S_R, or because momentum is convected across S_R. Moreover, these are both random events spread over a large surface, and so the central limit theorem tentatively suggests that their cumulative effects are scaled as $O\left(S_R^{1/2}\right) \sim R$, i.e., the momentum change in any finite time is $O\left(V_R^{1/3} t\right)$. This in turn suggests that the pressure forces and momentum fluxes across S_R are too weak to change $\mathbf{P}/V_R^{1/2}$ in the limit of $V_R \to \infty$, and that consequently L_{ij} is an invariant in accordance with (11). This is essentially the argument of Saffman [11].

Unfortunately, it is rare that one satisfies the strict requirements for the central limit theorem to hold, especially in turbulence where the long-range interactions mean that remote events are rarely statistically independent. Consequently, we must regard the argument above as merely suggestive. However, formal proof of the invariance of L_{ij} is given in the study conducted by Davidson [5,6].

2.2. Statistically Axisymmetric Saffman Turbulence

We now turn to the case of statistically axisymmetric Saffman turbulence, with or without reflectional symmetry. In particular, we need to summarize the findings of Davidson [5,6] and DOK12, as these directly lead to invariants of passive scalar mixing in axisymmetric turbulence. As usual, we use subscripts // and $\perp$ to indicate quantities that are parallel and perpendicular to the symmetry axis, respectively, which we take to be the z axis. Here, it may be shown that $M_{ij} = 0$ if $i \neq j$, and so the only non-zero components of M_{ij} are $M_{//}$ and $M_{xx} = M_{yy} = \frac{1}{2}M_{\perp}$. This supports the idea that $M_{//}$ and $M_{\perp}$ are the only invariants of axisymmetric Saffman turbulence. Similarly, (3) tells us that Ω_{ij} is diagonal, and that

$$(2\pi)^3 M_{//} = \frac{1}{4}\Omega_{//} - \frac{1}{4}\Omega_{\perp},\ (2\pi)^3 M_{\perp} = -\frac{1}{2}\Omega_{//}. \tag{12}$$

Therefore, the most general form of $\Phi_{ij}(\mathbf{k} \to 0)$ for Saffman turbulence is:

$$\Phi_{//} = \Phi_{zz} = M_{//}\frac{k_{\perp}^4}{k^4} + \frac{1}{2}M_{\perp}\frac{k_z^2 k_{\perp}^2}{k^4} + O(k), \tag{13}$$

$$\Phi_{\perp} = \Phi_{xx} + \Phi_{yy} = M_{//}\frac{k_z^2 k_{\perp}^2}{k^4} + \frac{1}{2}M_{\perp}\left[1 + \frac{k_z^4}{k^4}\right] + O(k), \tag{14}$$

(Davidson [5]), where $k_{\perp}^2 = k^2 - k_z^2$, and we omit the off-diagonal terms. Thus, with regards to the spectral tensor, our two invariants can be written as follows:

$$M_{//} = \Phi_{//}(k_z = 0, k_{\perp} \to 0) = \frac{1}{(2\pi)^3}\lim_{k_{\perp}\to 0}\int e^{-j\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}}\langle u_{//}u'_{//}\rangle d\mathbf{r} = \text{constant}, \tag{15}$$

$$\frac{1}{2}M_{\perp} = \Phi_{\perp}(k_z = 0, k_{\perp} \to 0) = \frac{1}{(2\pi)^3}\lim_{k_{\perp}\to 0}\int e^{-j\mathbf{k}_{\perp}\cdot\mathbf{r}_{\perp}}\left\langle \mathbf{u}_{\perp}\cdot\mathbf{u}'_{\perp}\right\rangle d\mathbf{r} = \text{constant}, \tag{16}$$

$$M_{\perp} = \Phi_{\perp}(k_{\perp} = 0, k_z \to 0) = \frac{1}{(2\pi)^3}\lim_{k_z\to 0}\int e^{-jk_z r_z}\left\langle \mathbf{u}_{\perp}\cdot\mathbf{u}'_{\perp}\right\rangle d\mathbf{r} = \text{constant}, \tag{17}$$

while

$$\Phi_{//}(k_{\perp} = 0, k_z \to 0) = 0.$$

Similarly, the only non-zero components of L_{ij} are $L_{//}$ and $L_{\perp}$. As for the general anisotropic case, $L_{//}$ and $L_{\perp}$ are uniquely determined by M_{ij}, although the relationship between $L_{//}$, $L_{\perp}$, $M_{//}$, and $M_{\perp}$ depends on the shape of V in (7). For example, in the particular case of a large cylindrical control volume of radius R and height $2H$, the following can be deduced:

$$L_{//} = (2\pi)^3 M_{//}\frac{1 + \frac{1}{2}(R/H)^2}{\left[1 + (R/H)^2\right]^{3/2}} + (2\pi)^3 M_{\perp}\frac{\frac{1}{4}(R/H)^2}{\left[1 + (R/H)^2\right]^{3/2}}, \tag{18}$$

$$L_{\perp} = (2\pi)^3 M_{\perp}\left\{1 - \frac{\frac{1}{2} + \frac{3}{4}(R/H)^2}{\left[1 + (R/H)^2\right]^{3/2}}\right\} + (2\pi)^3 M_{//}\frac{\frac{1}{2}(R/H)^2}{\left[1 + (R/H)^2\right]^{3/2}}, \tag{19}$$

(see DOK12), from which

$$L_{//}(H/R \to \infty) = (2\pi)^3 M_{//} = \text{constant, (long cylinder)}, \tag{20}$$

$$L_{\perp}(H/R \to \infty) = (2\pi)^3 \frac{1}{2} M_{\perp} = \text{constant, (long cylinder)}, \tag{21}$$

$$L_{//}(R/H \to \infty) = 0, \text{(thin disc)}, \tag{22}$$

$$L_{\perp}(R/H \to \infty) = (2\pi)^3 M_{\perp} = \text{constant, (thin disc)}. \tag{23}$$

These are equivalent to the spectral results (15)→(17), where $H/R \to \infty$ corresponds to $k_z = 0$ and $k_{\perp} \to 0$, while $R/H \to \infty$ is equivalent to $k_{\perp} = 0$ and $k_z \to 0$. The constraint that L_{ij} = constant is now reduced to

$$L_{//} = \int \langle u_{//} u'_{//} \rangle d\mathbf{r} = \text{constant}, \tag{24}$$

$$L_{\perp} = \int \left\langle \mathbf{u}_{\perp} \cdot \mathbf{u}'_{\perp} \right\rangle d\mathbf{r} = \text{constant}, \tag{25}$$

and this is true, irrespective of the shape of V. The invariance of $L_{//}$ and $L_{\perp}$, or equivalently $M_{//}$ and $M_{\perp}$, is confirmed in the direct numerical simulations of DOK12.

3. Passive Scalar Statistics in Axisymmetric Saffman Turbulence

Let us now turn to passive scalar mixing in Saffman turbulence. Let θ be the passive scalar which has a mean of zero, $\langle \theta \rangle = 0$, and is governed by

$$\frac{D\theta}{Dt} = \alpha \nabla^2 \theta. \tag{26}$$

We shall assume that, like the velocity field, its initial energy spectrum takes the form $E_{\theta}(k \to 0) \sim k^2$. (As we shall see, such an initial condition ensures that the spectrum $E_{\theta}(k \to 0) \sim k^2$ then persists for all time.) In such a case, the spectral function

$$\Phi_{\theta}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int \langle \theta \theta' \rangle (\mathbf{r}) e^{-j\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}, \tag{27}$$

takes the low-k form

$$\Phi_{\theta}(\mathbf{k} \to 0) = \left\{ \delta_{\alpha\beta} - \frac{k_{\alpha} k_{\beta}}{k^2} \right\} N_{\alpha\beta} + O(k), \tag{28}$$

and

$$E_{\theta}(k \to 0) = \frac{4}{3} \pi N_{\alpha\alpha} k^2,$$

where $N_{\alpha\beta}$ is symmetric and independent of $\mathbf{k}$. This has the same structural form as $\Phi_{ii}(\mathbf{k} \to 0)$ in (4), provided that we substitute $N_{\alpha\beta}$ for $M_{\alpha\beta}$, and so we conclude that $\langle \theta \theta' \rangle_{\infty} = O(r^{-3})$.

In statistically axisymmetric turbulence, $N_{ij} = 0$ if $i \neq j$, and so the only non-zero components of N_{ij} are $N_{//}$ and $N_{xx} = N_{yy} = \frac{1}{2} N_{\perp}$. The low-$k$ form of the spectral function $\Phi_{\theta}(\mathbf{k})$ is then simplified to

$$\Phi_{\theta}(\mathbf{k} \to 0) = \left[N_{//} + \frac{1}{2} N_{\perp} \right] - \left[N_{//} - \frac{1}{2} N_{\perp} \right] \frac{k_{//}^2}{k^2} + O(k), \tag{29}$$

which is analytic at $\mathbf{k} = 0$ only for the isotropic case of $N_{//} = \frac{1}{2}N_{\perp}$. Note that $N_{//}$ and $N_{\perp}$ can be written in terms of the spectral function as

$$N_{//} + \frac{1}{2}N_{\perp} = \Phi_{\theta}(k_{//} = 0, k_{\perp} \to 0) = \frac{1}{(2\pi)^3} \operatorname*{Lim}_{k_{\perp} \to 0} \int e^{-j\mathbf{k}_{\perp} \cdot \mathbf{r}_{\perp}} \langle \theta\theta' \rangle d\mathbf{r}, \tag{30}$$

and

$$N_{\perp} = \Phi_{\theta}(k_{\perp} = 0, k_{//} \to 0) = \frac{1}{(2\pi)^3} \operatorname*{Lim}_{k_{//} \to 0} \int e^{-jk_{//}r_{//}} \langle \theta\theta' \rangle d\mathbf{r}. \tag{31}$$

Thus, we see that $N_{//}$ and $N_{\perp}$ are closely related to the integral $\operatorname*{Lim}_{V \to \infty} \int_V \langle \theta\theta' \rangle d\mathbf{r}$, which we shall shortly see is an invariant of scalar mixing.

Let us now turn to the Corrsin integral

$$L_{\theta} = \operatorname*{Lim}_{V \to \infty} \int_V \langle \theta\theta' \rangle d\mathbf{r}, \tag{32}$$

which is central to scalar mixing analysis. Since $\langle \theta\theta' \rangle_{\infty} = O(r^{-3})$, L_{θ} is, in general, only conditionally convergent, and so it is important to determine the precise form of $\langle \theta\theta' \rangle$ in the far-field. Here, the analogy to $\Phi_{ii}(\mathbf{k})$ is again helpful. Based on (5), we have

$$\left\langle \mathbf{u} \cdot \mathbf{u}' \right\rangle_{\infty} = -\pi^2 [2M_{//} - M_{\perp}] \frac{r^2 - 3r_{//}^2}{r^5}, \tag{33}$$

and so the structural similarity between (4) and (28) tells us that

$$\langle \theta\theta' \rangle_{\infty} = -\pi^2 [2N_{//} - N_{\perp}] \frac{r^2 - 3r_{//}^2}{r^5}. \tag{34}$$

Note that the long-range correlation, $\langle \theta\theta' \rangle_{\infty} = O(r^{-3})$, vanishes only in the isotropic case.

The key point about (34) is that its right-hand side may be written as a divergence of a vector of magnitude r^{-2},

$$\langle \theta\theta' \rangle_{\infty} = \pi^2 [2N_{//} - N_{\perp}] \, \nabla \cdot \left[\left(\frac{\partial}{\partial r_{//}} \frac{1}{r} \right) \hat{\mathbf{e}}_z \right]. \tag{35}$$

Thus, if we integrate $\langle \theta\theta' \rangle_{\infty}(\mathbf{r})$ over the volume bounded by two large concentric surfaces of the same shape but different sizes, the net result is zero. Moreover, it remains zero as the outer surface recedes to infinity. This is important because it means that L_{θ} is convergent, if only conditionally convergent (in the sense defined above). For example, in the case of a large cylindrical control volume of radius R and height $2H$, (30) and (31) suggest the following:

$$\int \langle \theta\theta' \rangle d\mathbf{r} = (2\pi)^3 \left[N_{//} + \frac{1}{2}N_{\perp} \right], \qquad (R/H \to 0, \text{ long cylinder}), \tag{36}$$

$$\int \langle \theta\theta' \rangle d\mathbf{r} = (2\pi)^3 N_{\perp}, \qquad (R/H \to \infty, \text{ thin disc}). \tag{37}$$

In summary, in the isotropic case of $N_{//} = \frac{1}{2}N_{\perp}$, there are no long-range correlations of the form $\langle \theta\theta' \rangle_{\infty} = O(r^{-3})$, and so L_{θ} is absolutely convergent and given by either (30) or (31):

$$L_{\theta} = \int \langle \theta\theta' \rangle d\mathbf{r} = (2\pi)^3 \left[N_{//} + \frac{1}{2}N_{\perp} \right] = (2\pi)^3 N_{\perp}. \tag{38}$$

However, in the anisotropic case of $N_{//} \neq \frac{1}{2}N_{\perp}$, we have $\langle \theta\theta' \rangle_{\infty} = O(r^{-3})$ and L_{θ} is only conditionally convergent, and its value depends on the shape of V.

4. An Invariant for Passive Scalar Mixing in Axisymmetric Saffman Turbulence

Crucially, integral L_θ is an invariant of freely decaying Saffman turbulence. This may be shown as follows. From (26), we have the two-point evolution equation:

$$\frac{\partial}{\partial t}\langle\theta\theta'\rangle = -\frac{\partial}{\partial r_i}\langle(\delta u_i)\theta\theta'\rangle + 2\alpha\nabla^2\langle\theta\theta'\rangle, \tag{39}$$

where $\delta\mathbf{u} = \mathbf{u}' - \mathbf{u}$ is a velocity increment. By integrating over a volume V in $\mathbf{r}$-space, and noting that $\langle\theta\theta'\rangle_\infty \le O(r^{-3})$, we obtain

$$\frac{d}{dt}L_\theta = -\oint_{S_\infty}\langle\delta\mathbf{u}\theta\theta'\rangle \cdot d\mathbf{S}, \tag{40}$$

where S_∞ is the surface of V as that surface recedes to infinity. However, in Saffman turbulence, the long-range correlations are no stronger than order r^{-3}, and so $\langle\delta\mathbf{u}\theta\theta'\rangle_\infty \le O(r^{-3})$. The surface integral in (40) now vanishes, yielding

$$L_\theta = \operatorname*{Lim}_{V\to\infty}\int_V \langle\theta\theta'\rangle d\mathbf{r} = \text{constant}. \tag{41}$$

The physical interpretation of (40) and (41) is clear. Homogeneity allows us to rewrite L_θ as

$$L_\theta = \operatorname*{Lim}_{V\to\infty}\int_V \langle\theta\theta'\rangle d\mathbf{r} = \operatorname*{Lim}_{V\to\infty}\frac{1}{V}\left\langle\left[\int_V \theta dV\right]^2\right\rangle, \tag{42}$$

and so L_θ is conserved if the scalar flux across the bounding surface of V, say S, is less than order $V^{1/2}$ (or equivalently $S^{3/4}$) as $V \to \infty$. However, the central limit theorem suggests that this flux should be no larger that $S^{1/2}$, consistent with L_θ being an invariant. In fact, (40) is nothing more than a concise form of the scalar budget equation

$$\frac{d}{dt}L_\theta = \frac{d}{dt}\frac{1}{V}\left\langle\left[\int_V \theta dV\right]^2\right\rangle = -\frac{2}{V}\left\langle\int_V \theta' dV' \oint_S \theta\,\mathbf{u}\cdot d\mathbf{S}\right\rangle = -\frac{2}{V}\int_V\left\langle\oint_S \theta'\theta\,\mathbf{u}\cdot d\mathbf{S}\right\rangle dV', \tag{43}$$

and the vanishing of the surface integral in (40) amounts to the statement that there is negligible convection of the passive scalar in or out of V in the limit of $V \to \infty$.

Perhaps some comments are in order at this point. First, since $N_{//}$ and $N_\perp$ are related to L_θ through (30) and (31), we would expect the invariance of L_θ to translate to the statement that $N_{//}$ and $N_\perp$ are both invariants. This is indeed the case (Yoshimatsu and Kaneda, 2018 [14]). Second, if $\ell^\theta_{//}$ and $\ell^\theta_\perp$ are integral scales for the passive scalar, say

$$\ell^\theta_{//} = \frac{1}{\langle\theta^2\rangle}\int\left\langle\theta(\mathbf{x})\cdot\theta(\mathbf{x}+r\hat{\mathbf{e}}_z)\right\rangle dr, \tag{44}$$

$$\ell^\theta_\perp = \frac{1}{\langle\theta^2\rangle}\int\left\langle\theta(\mathbf{x})\cdot\theta(\mathbf{x}+r\hat{\mathbf{e}}_x)\right\rangle dr, \tag{45}$$

then, since L_θ, $N_{//}$, and $N_\perp$ are all dominated by large scales, we have

$$L_\theta \sim N_{//} \sim N_\perp \sim \left\langle\theta^2\right\rangle\ell^\theta_{//}\left(\ell^\theta_\perp\right)^2. \tag{46}$$

Moreover, when the *large scales* in the passive scalar field evolve in a self-similar way, i.e., when $\langle\theta\theta'\rangle/\langle\theta^2\rangle$ is a function of $r_{//}/\ell^\theta_{//}$ and $\mathbf{r}_\perp/\ell^\theta_\perp$ only, then the invariance of $N_{//}$ and $N_\perp$ demands

$$\left\langle\theta^2\right\rangle\ell^\theta_{//}\left(\ell^\theta_\perp\right)^2 = \text{constant}, \tag{47}$$

which places a strong constraint on the evolution of the scalar variance $\langle\theta^2\rangle$.

Third, the constraint (47) is similar to that imposed on the velocity field as a result of (15)→(17). That is to say, $M_{//}$ and $M_{\perp}$ are dominated by large scales and so

$$M_{//} \sim \left\langle \mathbf{u}_{//}^2 \right\rangle \ell_{//}^u (\ell_{\perp}^u)^2, \; M_{\perp} \sim \left\langle \mathbf{u}_{\perp}^2 \right\rangle \ell_{//}^u (\ell_{\perp}^u)^2, \tag{48}$$

where $\ell_{//}^u$ and $\ell_{\perp}^u$ are suitable integral scales for the velocity field (see Section 5). Moreover, when the large-scale velocity correlations are self-similar, as is typical of fully developed turbulence, (15) →(17) demand

$$\left\langle \mathbf{u}_{//}^2 \right\rangle \ell_{//}^u (\ell_{\perp}^u)^2 = \text{constant}, \; \left\langle \mathbf{u}_{\perp}^2 \right\rangle \ell_{//}^u (\ell_{\perp}^u)^2 = \text{constant}. \tag{49}$$

The existence of invariants $M_{//}$ and $M_{\perp}$ (or, equivalently, $L_{//}$ and $L_{\perp}$), as well as the associated conservation relationships (49), is confirmed in DOK12.

Finally, we combine $\langle \delta \mathbf{u} \theta \theta' \rangle_{\infty} \leq O(r^{-3})$ with (39). This tells us that, if $\langle \theta \theta' \rangle_{\infty} = O(r^{-3})$ at $t = 0$, then we have $\langle \theta \theta' \rangle_{\infty} = O(r^{-3})$ for all $t > 0$. Thus, the initial Saffman-like spectrum $E_{\theta}(k \to 0) \sim k^2$ is preserved for all time, as noted above. Indeed, this has to be the case since

$$E_{\theta}(k \to 0) = \frac{4}{3} \pi N_{\alpha\alpha} k^2, \tag{50}$$

and $N_{ii} = N_{//} + N_{\perp}$ is an invariant. Conversely, if the initial spectrum takes an alternative form, say $E_{\theta}(k \to 0) \sim k^4$, then the Saffman spectrum $E_{\theta}(k \to 0) \sim k^2$ remains kinematically inaccessible to the passive scalar.

5. The Predicted Decay Rate of Energy and Scalar Variance

We shall now use the invariants $M_{//}$, $M_{\perp}$, $N_{//}$, and $N_{\perp}$, or equivalently $L_{//}$, $L_{\perp}$, and L_{θ}, combined with the assumption of self-similarity of large scales, to predict the evolution of kinetic energy and scalar variance in freely evolving turbulence. The approach is the same as that used by Saffman [12] and DOK12, where the invariance of $M_{//}$ and $M_{\perp}$ is used to predict the decay rate of $\left\langle \mathbf{u}_{//}^2 \right\rangle$ and $\langle \mathbf{u}_{\perp}^2 \rangle$.

5.1. The Rate of Energy Decay

Before discussing the decay of scalar variance, we should note how the kinetic energy and integral length scales $\ell_{//}^u$ and $\ell_{\perp}^u$ evolve, as the passive scalar is slave to the velocity field.

Let us start by introducing some notation. We define the velocity integral scales as follows:

$$\ell_{//}^u = \frac{1}{u_{//}^2} \int \left\langle u_{//}(\mathbf{x}) u_{//}(\mathbf{x} + r\hat{\mathbf{e}}_z) \right\rangle dr, \; u_{//} = \left\langle \mathbf{u}_{//}^2 \right\rangle^{1/2}, \tag{51}$$

$$\ell_{\perp}^u = \frac{1}{2u_{\perp}^2} \left\{ \int \left\langle u_x(\mathbf{x}) u_x(\mathbf{x} + r\hat{\mathbf{e}}_x) \right\rangle dr + \int \left\langle u_y(\mathbf{x}) u_y(\mathbf{x} + r\hat{\mathbf{e}}_y) \right\rangle dr \right\}, \; u_{\perp} = \left\langle \frac{1}{2} \mathbf{u}_{\perp}^2 \right\rangle^{1/2}. \tag{52}$$

Next, we assume that, once the turbulence is fully developed, the large scales in the velocity field (scales of order $\ell_{//}^u$ and $\ell_{\perp}^u$) are self-similar, i.e., $\langle u_{//} u'_{//} \rangle / u_{//}^2$ and $\langle \mathbf{u}_{\perp} \cdot \mathbf{u}'_{\perp} \rangle / u_{\perp}^2$ are functions of $r_{//}/\ell_{//}^u$ and $\mathbf{r}_{\perp}/\ell_{\perp}^u$ only. This is certainly observed in the numerical experiments of DOK12. We shall refer to this as *partial self-similarity*, as it does not extend down to Kolmogorov scales, nor perhaps to the far-field correlations $\langle u_i u'_j \rangle_{\infty}$ given by (5). Now, $M_{//}$ and $M_{\perp}$ are absolutely convergent and so are independent of the far-field contribution $\langle u_i u'_j \rangle_{\infty}$. (The long-wavelength oscillation associated with $e^{-j\mathbf{k}_{\perp} \cdot \mathbf{r}_{\perp}}$ and $e^{-j\mathbf{k}_{//} \cdot \mathbf{r}_{//}}$ kills off the far-field contributions to these integrals, leaving (15)→(17) well behaved.) It follows that partial self-similarity may be applied to (15)→(17), which then demands that, in fully developed turbulence,

$$M_{\perp} = c_{\perp} u_{\perp}^2 (\ell_{\perp}^u)^2 \ell_{//}^u = \text{constant}, \; M_{//} = c_{//} u_{//}^2 (\ell_{\perp}^u)^2 \ell_{//}^u = \text{constant}, \tag{53}$$

where $c_\perp$ and $c_{//}$ are dimensionless constants. In such a situation, we have

$$u_\perp^2/u_{//}^2 = \text{constant}. \tag{54}$$

This suggests that, rather surprisingly, any velocity anisotropy which is present at the beginning of the fully developed stage will be locked into the turbulence thereafter. The persistence of anisotropy in the form of (54) is confirmed in the numerical simulations of DOK12.

We now estimate the decay rate of energy under the assumption that partial self-similarity holds and that, as usual, the flux of energy to small scales is controlled by large scales. Since $u_{//}^2/u_\perp^2 = \text{constant}$, we only need to consider one component of energy, say $u_\perp^2$, and dimensional analysis then gives us

$$\frac{du_\perp^2}{dt} = -A\frac{u_\perp^3}{\ell_\perp^u}, \qquad A = A\left(\ell_{//}^u/\ell_\perp^u\right), \quad \text{[fully developed turbulence only]} \tag{55}$$

where A is a dimensionless function of $\ell_{//}^u/\ell_\perp^u$. Now, certain types of axisymmetric turbulence satisfy $u_\perp/\ell_\perp^u \sim u_{//}/\ell_{//}^u$ as a result of continuity. In such cases, the constraint $u_{//}^2/u_\perp^2 = \text{constant}$ demands that $\ell_{//}^u/\ell_\perp^u = \text{constant}$, and this is certainly observed in the fully developed state in the numerical experiments of DOK12. Equations (53) and (55) then simplify to

$$u_\perp^2(\ell_\perp^u)^3 = \text{constant}, \; \frac{du_\perp^2}{dt} = -A\frac{u_\perp^3}{\ell_\perp^u}, \tag{56}$$

which results in the following decay laws:

$$u_\perp^2 \sim u_{//}^2 \sim t^{-6/5}, \; \ell_\perp^u \sim \ell_{//}^u \sim t^{2/5}, \quad \text{[fully developed turbulence only]} \tag{57}$$

as suggested by Saffman [12], and as observed in the direct numerical simulations of DOK12.

5.2. The Decay of Scalar Variance

Given (57), we can now predict the decay rate of scalar variance in fully developed turbulence. The analysis differs somewhat depending on whether $\ell_{//}^u$ and $\ell_\perp^u$ are greater or smaller than the scalar integral scales $\ell_{//}^\theta$ and $\ell_\perp^\theta$. Consider first the case where the initial conditions have $\ell_{\min}^\theta >> \ell_{\max}^u$. (Here, the subscripts indicate whether the minimum or maximum of the two possible length scales is taken.) Then, mixing occurs on the scale of $\ell_{\max}^u$, and dimensional analysis, combined with $u_{//}^2/u_\perp^2 = \text{constant}$ and $\ell_{//}^u/\ell_\perp^u = \text{constant}$, gives us

$$\frac{d\langle\theta^2\rangle}{dt} = -C\frac{u_\perp}{\ell_\perp^u}\left\langle\theta^2\right\rangle, \tag{58}$$

where C is a constant. Given (57), and the fact that mixing occurs on a scale much smaller than $\ell_{\min}^\theta$, we expect $\langle\theta^2\rangle$ to decay as a power law, while $\ell_{//}^\theta$ and $\ell_\perp^\theta$ will be more or less constant, and this will continue for as long as $\ell_{\min}^\theta >> \ell_{\max}^u$. Note that the scalar field is not self-similar during this stage, so that (47) does not apply. Eventually, $\ell_{\max}^u$ will grow to be of the same order as $\ell_{\min}^\theta$, and thereafter $\ell_{//}^\theta$ and $\ell_\perp^\theta$ will grow along with $\ell_{//}^u$ and $\ell_\perp^u$, as $t^{2/5}$. If the scalar spectrum exhibits partial self-similarity during this growth, then invariant (47) combined with (57) yields

$$\left\langle\theta^2\right\rangle \sim t^{-6/5}, \; \ell_\perp^\theta \sim \ell_{//}^\theta \sim t^{2/5}. \quad \text{[fully developed turbulence, } \ell^\theta \sim \ell^u\text{]}. \tag{59}$$

Now consider the initial conditions in which $\ell^{\theta}_{\max} << \ell^{u}_{\min}$. In this case, mixing is no longer accomplished by the integral-scale eddies, but rather by inertial-range eddies of scale $\ell^{\theta}_{\max}$. In such a situation, we have

$$\frac{d\langle\theta^2\rangle}{dt} \sim -C\frac{\langle\theta^2\rangle}{\tau_\theta}, \quad C = C\left(\ell^{\theta}_{//}/\ell^{\theta}_{\perp}\right), \tag{60}$$

where C is a dimensionless function of $\ell^{\theta}_{//}/\ell^{\theta}_{\perp}$ and τ_θ is the eddy turnover time of scales of order $\ell^{\theta}_{\max}$. Now, τ_θ can be estimated from Kolmogorov's 2/3 law as $\left(\ell^{\theta}_{\max}\right)^{2/3}/\varepsilon^{1/3}$. (Here, ε is the rate of dissipation of kinetic energy.) Since the anisotropy of the velocity integral scales is fixed in fully developed turbulence, i.e., $u^2_{//}/u^2_{\perp} =$ constant and $\ell^{u}_{//}/\ell^{u}_{\perp} =$ constant, we would expect the velocity anisotropy at scale $\ell^{\theta}_{\max}$ to also be constant. This in turn suggests that $\ell^{\theta}_{//}/\ell^{\theta}_{\perp}$ will be constant, or equivalently $\ell^{\theta}_{//} \sim \ell^{\theta}_{\perp} \sim \ell^{\theta}$, provided that the velocity anisotropy is not too large at scale $\ell^{\theta}_{\max}$. In such cases, (47) and (60) simplify to

$$\frac{d\langle\theta^2\rangle}{dt} \sim -C\frac{\varepsilon^{1/3}\langle\theta^2\rangle}{\left(\ell^{\theta}\right)^{2/3}}, \quad \left\langle\theta^2\right\rangle\left(\ell^{\theta}\right)^3 = \text{constant}, \tag{61}$$

from which

$$\frac{d\langle\theta^2\rangle}{dt} \sim -\varepsilon^{1/3}\left\langle\theta^2\right\rangle^{11/9}, \tag{62}$$

where (57) has $\varepsilon \sim t^{-11/5}$. Once again, we conclude that

$$\left\langle\theta^2\right\rangle \sim t^{-6/5},\ \ell^{\theta}_{\perp} \sim \ell^{\theta}_{//} \sim t^{2/5}. \quad [\text{fully developed turbulence, } \ell^{\theta} << \ell^{u}] \tag{63}$$

In summary, we would expect the scalar variance to decay as $\left\langle\theta^2\right\rangle \sim t^{-6/5}$, and the scalar integral scales to grow as $\ell^{\theta}_{\perp} \sim \ell^{\theta}_{//} \sim t^{2/5}$, whenever the turbulence is fully developed and $\ell^{\theta} \le O(\ell^{u})$. These various predictions are confirmed in the direct numerical simulations of Yoshimatsu and Kaneda [7].

6. Discussion

Our understanding of the large-scale dynamics of homogeneous turbulence has come a long way since those early discussions at NCAR in 2002, and this author has greatly enjoyed the journey, particularly those collaborations with Jack Herring, Yukio Kaneda, and Per-Age Krogstad. We now have a reasonably complete understanding of large-scale dynamics, both in terms of kinetic energy and passive scalar mixing, and this is true of both Batchelor $E(k \to 0) \sim k^4$ and Saffman $E(k \to 0) \sim k^2$ turbulence.

As noted by Kolmogorov, in Batchelor turbulence, the kinetic energy should decay as $u^2 \sim t^{-10/7}$, provided that Loitsyansky's integral is conserved. This decay law was confirmed in the direct numerical simulations of Ishida, Davidson, and Kaneda [2], but only for fully developed turbulence. (Loitsyansky's integral is not conserved during an initial transient, before the turbulence becomes fully developed.) Similarly, for isotropic Saffman turbulence, the prediction of $u^2 \sim t^{-6/5}$ is observed in the wind tunnel experiments, as stipulated by Krogstad and Davidson [4]. Finally, in anisotropic (but axisymmetric) Saffman turbulence, the theoretical predictions of

$$u^2_{\perp} \sim u^2_{//} \sim t^{-6/5},\ \ell^{u}_{\perp} \sim \ell^{u}_{//} \sim t^{2/5},$$

are well supported by the direct numerical simulations of DOK12, while the estimates

$$\left\langle\theta^2\right\rangle \sim t^{-6/5},\ \ell^{\theta}_{\perp} \sim \ell^{\theta}_{//} \sim t^{2/5},$$

are consistent with the direct numerical simulations of Yoshimatsu and Kaneda [7], as discussed above.

Ironically, the one question which has proved to be most resistant to theoretical attack turns out to be the very subject of those early discussions with Jack Herring back in 2002, i.e., the observation that Loitsyansky's integral is effectively constant in fully developed Batchelor turbulence. The point is this: Batchelor and Proudman [13] noted that the long-range velocity correlations in $E(k \to 0) \sim k^4$ turbulence fall off sufficiently slowly with r so as to invalidate the usual proof that Loitsyansky's integral is an invariant. However, the direct numerical simulations show that, once the turbulence is fully developed, but not before, Loitsyansky's integral is indeed constant (Herring et al. [1]; Ishida, Davidson, and Kaneda, [2]). It seems that there is something particular about the structure of fully developed turbulence that suppresses the long-range interactions of Batchelor and Proudman [13], and we still do not understand what that might be. Evidently, there is still much to accomplish, and so this is a story without an ending.

7. Conclusions

We reviewed the integral invariant theory of Saffman turbulence, showing that the various statistical invariants, both the Saffman and Corrsin integrals, have a simple physical interpretation. We have also shown that, under a wide range of conditions, the integral scales of a passive scalar evolve as

$$\left\langle \theta^2 \right\rangle \sim t^{-6/5},\ \ell_\perp^\theta \sim \ell_{//}^\theta \sim t^{2/5}.$$

The primary point of novelty of our approach lies in its use of various integral invariants to deduce the evolution of integral scales.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The author would like to thank the Leverhulme Trust for its generous support through the grant RPG-2015-195/RG77943, as well as Yukio Kaneda for their fruitful discussions on passive scalar mixing.

Conflicts of Interest: The author declares no conflict of interest.

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