S1: Relationship between the Fredholm integral equation of the first kind and DRSC method

The basis for our analysis in this section is a Fredholm integral equation of the first kind with the kernel function $K(x,y)$

$$f(x)=\int_a^b K(x,y)\varphi(y)dy \quad c\le x\le d \tag{1}$$

where $\varphi(y)$ is the unknown function and $f(x)$ is known. To investigate the relationship between the DSRC method and the integral equation, the authors assume that the kernel, $\varphi(y)$ and $f(x)$ satisfy the following relations

$$K(p,t)=\frac{\partial q_p(\mathbf{X})}{\partial x_t} \tag{2}$$

$$\varphi(t)=\Delta x_t \tag{3}$$

and

$$f(p)=\Delta q_p \tag{4}$$

where $\frac{\partial q_p(\mathbf{X})}{\partial x_t}$ is defined in Eq. (5) of the main manuscript.

In practice, the integral problem is generally discretized to solve it numerically. Obviously, the authors can only get access to the state of the XAJ model and the observation at a discrete set of points. Therefore, the information used for evaluating an approximation to Eq. (1) is restricted to a set of quadrature points $t\in\{t_1,t_2,\cdots,t_m\}$ on the interval $[a,b]$ and p is restricted to $p\in\{p_1,p_2,...,p_n\}$. The main purpose of this section is to investigate the relationship between the integral equation and the DSRC method and thus it is beyond the goal of this section to discuss the criterion for choosing the methods of discretization in detail. Using the Repeated Rectangular rule, the integral is discretized and approximated as

$$f(p)\approx h\sum_{t=1}^{m}K(p,t)\varphi(t) \quad p=1,2,...,n \tag{5}$$

with the uniform interval (h=1). Rewrite Eq. (5) in matrix notation

$$\begin{bmatrix} f(1)\\ f(2)\\ \vdots\\ f(n)\end{bmatrix}=\begin{bmatrix} K(1,1) & K(1,2) & \cdots & K(1,m)\\ K(2,1) & K(2,2) & \cdots & K(2,m)\\ \vdots & \vdots & \ddots & \vdots\\ K(n,1) & K(n,2) & \cdots & K(n,m)\end{bmatrix}\begin{bmatrix}\varphi(1)\\ \varphi(2)\\ \vdots\\ \varphi(m)\end{bmatrix} \tag{6}$$

Inserting Eq. (2), Eq. (3) and Eq. (4) into Eq.(6), the authors obtain

$$\begin{bmatrix}\Delta q_1\\ \Delta q_2\\ \vdots\\ \Delta q_n\end{bmatrix}=\begin{bmatrix}\frac{\partial q_1(\mathbf{X})}{\partial x_1} & \frac{\partial q_1(\mathbf{X})}{\partial x_2} & \cdots & \frac{\partial q_1}{\partial x_m}\\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_2}{\partial x_m}\\ \vdots & \vdots & \cdots & \vdots\\ \frac{\partial q_n}{\partial x_1} & \frac{\partial q_n}{\partial x_2} & \cdots & \frac{\partial q_n}{\partial x_m}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_m\end{bmatrix} \tag{7}$$

In matrix notation, equation (14) can be expressed by

$$\mathbf{F}\approx\mathbf{K\Phi} \tag{8}$$

Thus, it has been demonstrated that under certain assumptions, the basic equitation (Eq. (1) in the main manuscript) of the DSRC method has the same form as the numerical solution of the Fredholm equation of the first kind.