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On Selection of the Efficient Water Purification Strategy at Commercial Scale Using Complex Intuitionistic Fuzzy Dombi Environment

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Abstract: The primary objective of the water purification process is to remove harmful chemical compounds and microorganisms from water sources in order to produce water suitable for human consumption. Water purification satisfies the demand for drinkable water, which is a requirement for many industries, including the medical, pharmaceutical, and chemical industries, among others. Access to purified water is the single most important factor determining a nation's strength and prosperity. As a consequence, researchers all over the world are investigating a wide variety of potential strategies for improving and preserving the water supply. In this paper, we present the findings of our research into a possible water strategy for purifying water and improving accessibility to drinkable water in areas prone to drought. This article presents the concepts of the complex intuitionistic fuzzy Dombi weighted averaging (CIFDWA) operator, the complex intuitionistic fuzzy Dombi ordered weighted averaging (CIFDOWA) operator, the complex intuitionistic fuzzy Dombi weighted geometric (CIFDWG) operator, and the complex intuitionistic fuzzy Dombi ordered weighted geometric (CIFDOWG) operator in complex intuitionistic fuzzy (CIF) settings. In addition, we investigate several important key features of these operators. Moreover, we introduce an improved score function to overcome the deficiencies of the existing score function under CIF knowledge. Furthermore, we effectively apply the proposed score function and newly defined operators to select the best technique for purifying water at a commercial scale. Additionally, we establish a comparative analysis to show the validity and feasibility of the proposed techniques compared to existing methods.

Keywords: CIFDWA operator; CIFDOWA operator; CIFDWG operator; CIFDOWG operator; multi-criteria decision-making problem

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1. Introduction

Multiple-attribute decision-making (MADM) problems manifest in a diverse array of scenarios, necessitating the selection of a number of alternatives, actions, or candidates according to a predefined set of criteria. MADM through aggregation operators is becoming popular as it is easy to handle real-life problems in almost every field, such as science, engineering, environmental and social sciences, and many others. Aggregation operators combine multiple values into a unified value within a specific set, thereby enabling the final aggregation outcome to encompass all individual values. Before the inception of aggregation operators, crisp sets were used to make decisions. However, the fact is that in real-world situations, membership in a set is not always so crisp, where typical mathematical tools are ineffective, particularly in the social and biological sciences, psychology, economics, linguistics, and soft sciences. To overcome this problem, Zadeh [1] introduced the concept of the partial belongingness of a set and named it the fuzzy set (FS) in 1965. In 1975, Kahne [2] introduced a decision-making approach for situations in which each alternative must be evaluated based on multiple attributes that possess distinct degrees of significance. In 1977, Jain [3] presented a method for decision-making that computed a fuzzy optimal alternative. In 1978, Dubois and Prade [4] discussed some operations on fuzzy sets. Yager [5] introduced some aggregation operators on fuzzy sets.

Atanassov [6] introduced intuitionistic fuzzy sets (IFS) in 1986, which extend the concept of fuzzy sets by characterizing both membership and non-membership degrees. Specifically, the sum of these degrees is constrained to be less than or equal to one. Chen and Tan [7] proposed score functions for IFS in 1994 to handle MADM problems. In 1996, Szmidt and Kaeprzyk [8] presented a method for determining solutions in group decision-making within the IFS framework. Li [9] suggested several linear programming models and methods for MADM under IFS settings in 2005, while Xu and Yager [10] defined geometric aggregation operators on IFS in 2006. Xu [11] subsequently developed arithmetic aggregation operators for IFS in 2007. In 2010, Zhao et al. [12] suggested generalized aggregation operators on IFS and used them to solve the MADM problem. Xu and Wang [13] presented the induced generalized aggregation operators for IFS and utilized them in group decision-making problems. Huang [14] designed a decision-making approach by investigating Hamacher aggregation operators on IFS in 2014. In 2015, Verma [15] presented Bonferroni mean operators on IFS. In addition, many useful strategies were invented to address the issue of different MCDM problems in [16–20].

Dombi aggregation operators are one of the most effective methods to solve MADM problems for researchers. Akram et al. [21] introduced Pythagorean fuzzy Dombi aggregation operators in 2019. Later on, Q-rung orthopair fuzzy Dombi aggregation operators [22], bipolar fuzzy Dombi aggregation operators [23], and picture fuzzy Dombi aggregation operators [24] were defined. Ashraf et al. [25] proposed spherical fuzzy Dombi aggregation operators in 2020. Liu et al. [26] presented interval-valued fuzzy Dombi aggregation operators for information security risk assessment in 2020. Moreover, in 2021, Seikh and Mandal [27] introduced Dombi operators on IFS. Karaaslan and Husseinawi [28] presented hesitant T-spherical fuzzy Dombi aggregation operators and their applications in MADM in 2022.

The theories of FS and IFS focus on solving one-dimensional issues. The fascinating scenario only becomes apparent when two-dimensional difficulties are brought up. Complex fuzzy sets (CFS) are a remarkable, innovative ideology developed by Ramot et al. [29] in 2002 that handles the challenges involving two-dimensional difficulties. As a consequence, the theory of CFS changes the fundamental idea of fuzzy membership by adding a second dimension to the statement of membership. Many physical issues, including complex amplitude and impedance in electrical engineering, wave function in quantum physics, and complex amplitude, have been successfully resolved by using the concept of CFS. The present theory bears significant importance in diverse applications, specifically in advanced control and forecasting of periodic events, in which multiple fuzzy variables are intricately interconnected in a manner that cannot be suitably identified through conventional fuzzy operations. The current investigations of complex fuzzy sets utilize the degrees of membership, which are a subset of complex numbers, to cope with the uncertainties inherent in the data. Nevertheless, this methodology may result in the loss of crucial information, ultimately compromising the decision-making process. In 2012, Alkouri and Salleh [30] added a complex degree of non-membership function to the concept of CFS to create a complex intuitionistic fuzzy set (CIFS) in order to solve this issue. Therefore, a CIFS is a generalization of many theories, such as FS, IFS, and CFS. A comparative analysis of CIFS and other sets based on their characteristic features is provided in Table 1. In the following table, the symbol ✓ indicates the ability to tackle a situation under a certain fuzzy environment. It is important to note that CIFS modeling is very capable of handling key factors such as uncertainty, falsity, hesitation, periodicity, and 2-D information of a physical phenomenon. In contrast, the other environments mentioned in the table do not possess the complete ability to address all of the listed factors.

Table 1. A Comparative Analysis of CIFS Model against Other Established Models.

Models	Uncertainity	Falsity	Hesitation	Periodicity	Ability to Represent 2-D Information	Have Characteristics of Generalization
FS	✓	✘	✘	✘	✘	✘
IFS	✓	✓	✓	✘	✘	✘
CFS	✓	✘	✘	✓	✓	✘
CIFS	✓	✓	✓	✓	✓	✓

Moreover, CF geometric aggregation operators [31] and CF arithmetic aggregation operators [32] were designed. Garg and Rani [33] proposed arithmetic and geometric aggregation operators in a CIF environment. Akram et al. [34] discussed Hamacher aggregation operators on CIFS. Mahmood et al. [35] initiated the principle of CIF Aczel-Alsina aggregation operators in 2022. Furthermore, one can study the recent developments in these theories in [36–42].

The hydrosphere of Earth and the bodily fluids of all recognized living entities are primarily composed of water. However, due to increasing industries and technology, chemicals and other harmful elements contaminate water supplies, increasing the risk of some types of cancer. Some toxins do more than just endanger human health. They sometimes affect the flavor of drinking water, imparting a metallic or other unpleasant taste. Chlorine in drinking water can lead to a variety of major health problems. Eliminating them through water purification may help reduce the risk of getting diseases caused by exposure to these materials [43–45]. Water purification is a crucial process aimed at eliminating undesirable organic and inorganic chemical compounds as well as biological contaminants from water, with a primary focus on providing potable water. Moreover, water purification fulfills the necessity for clean and portable water in various industries, including the medical, pharmaceutical, and chemical sectors. The purification process entails reducing the concentration of impurities, such as suspended particles, parasites, bacteria, algae, viruses, and fungi, to a significant extent. Some of the most notable health benefits of drinking pure water include improved absorption, increased hydration, improved metabolism, reduced skin and scalp irritation, healthier hair, and toxin elimination. Hence, water purification is unquestionably worthwhile.

The primary objective of this study is to develop effective techniques for handling various MCDM problems within a CIFS environment in a straightforward manner. Our approach stands out from others because it can manage input dependencies, making it more adaptable to different contexts. Additionally, implementing MADM solutions can be challenging in practice because current techniques cannot dynamically adjust parameters to account for the decision-makers' risk aversion. However, the methods outlined in this article are more than capable of addressing this issue. It is worth noting that the technique presented in this article is innovative in its own right, as there is currently no viable solution to the challenge of addressing water purification issues in a CIF setting.

The objective of this manuscript is to present several aggregation operators for aggregating complex intuitionistic fuzzy sets (CIFSs). These methods are intended to consider the dependency that exists between pairs of membership degrees. Existing studies on fuzzy and its extensions use the degree of membership, a subset of real numbers, to deal with data uncertainties. This results in the loss of valuable information, which can affect the decisions made. CIFSs are a subtype of these that can handle two-dimensional information within a single set. This is accomplished by managing uncertainties through the use of degrees whose ranges are enlarged from the real subset to the complex subset with the unit disk. Inspired by this, we have designed novel operators for aggregation in the context of complex intuitionistic fuzzy Dombi settings.

The multi-purpose Dombi operators have outstanding adaptability for computing imprecise information, owing to their aggregation features, decision-making abilities, and operational characteristics. These operators help convert information into a single value. Dombi operators are highly adaptable to operational conditions and effectively resolve decision-making problems. It is noteworthy that the strategies proposed in this article are more generalized than existing techniques because the best preference changes when information is lost in the framework of existing IF operators. In contrast, CIF Dombi aggregation operators effectively handle this situation. Furthermore, the newly defined operators are more flexible as they involve a parametric value. The above discussion clearly indicates that CIF Dombi aggregation operators, when used with other powerful mathematical tools, offer a groundbreaking approach to resolving MCDM problems. They can consider all data within the aggregation process and increase the accuracy and certainty of optimal results when applied to real-world MCDM issues.

The fundamental goal of this study is to provide a solution to the research question of selecting an appropriate technique for purifying water at a commercial scale using novel strategies within the CIF Dombi environment. Our study focuses on several key goals in the theoretical framework, which are as follows:

1. Create a new score function that improves upon the CIF system without the drawbacks of the old score function. To achieve this objective, we will combine sophisticated mathematical and statistical methods for a more reliable and precise grading system.
2. Develop a Dombi operation framework that is fundamental for use with CIFSs. It will be necessary to develop mathematical models that describe the relationships between the different components of CIFSs to improve both the analysis and forecasting of the results of using these systems.
3. Explore different ways in which CIFD operators can be combined to make the process of aggregating CIFS data more efficient.
4. Demonstrate that the newly specified operators meet critical requirements by examining their behavior. To demonstrate the practicality and usefulness of the proposed operators, we will utilize mathematical analysis and rigorous proofs.
5. Establish a method for resolving problems relating to MADM by utilizing CIFS aggregation operators. To accomplish this goal, we will establish a methodical procedure for applying novel operators to the analysis and evaluation of challenging decision-making problems.
6. Apply the recently suggested strategy to select the best technique to purify water at a commercial scale. Specifically, this will involve putting the recommended algorithm to use in different situations.
7. Compare the suggested technique with other approaches that are similar to those that have already been tried. To accomplish this, we will evaluate the efficiency of the proposed algorithm in relation to the performance of well-established methods using data and scenarios drawn from the real world.

This manuscript is structured as follows: Section 2 provides an overview of fundamental definitions. Section 3 identifies a limitation in the existing score function and proposes an alternative score function to address this issue within the context of the CIF environment. Section 4 introduces Dombi aggregation operators for CIFS and examines their basic properties. Section 5 employs the newly defined operators to select the optimal approach for large-scale water purification. In Section 6, a comparative analysis is presented to demonstrate the effectiveness and feasibility of this novel approach relative to established techniques. Lastly, the paper concludes by summarizing the key findings and implications.

2. Preliminaries

This section comprises some basic definitions that are helpful to understand the work presented in this article.

Definition 1. ([6]). *An intuitionistic fuzzy set (IFS) A of universal set X is defined as*

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\},$$

where $\mu_A, \nu_A : X \to [0,\ 1]$ *represents the membership degree and non-membership degree functions, respectively, that satisfy the condition* $0 \le \mu_A(x) + \nu_A(x) \le 1$. *The hesitancy margin of the IFS is defined as* $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 2. ([29]). *A complex fuzzy set (CFS) A defined on a universe of discourse X is characterized by a membership function* $\mu_A(x)$ *that allocates each element of X to a closed-unit circle in a complex plane and is written as* $\hat{r}_A(x)e^{i2\pi\hat{\theta}_A(x)}$, *where* $\hat{r}_A(x)$ *denotes the real-valued function from X to the closed-unit interval and* $e^{i2\pi\hat{\theta}_A(x)}$ *is a periodic function whose periodic law and principal period are* 2π *and* $0 \le \hat{\theta}_A(x) \le 1$, *respectively.*

Definition 3. ([30]). *A CIF A of universal set X is defined as* $A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$, *where* μ_A *and* ν_A *are the complex-valued membership degree and non-membership degree functions, respectively, defined from X to the unit closed circle, defined as* $\mu_A(x) = \hat{r}_A(x)e^{i2\pi\hat{\theta}_A(x)}$, $\nu_A(x) = \hat{k}_A(x)e^{i2\pi\hat{\phi}_A(x)}$, *along with* $0 \le \hat{r}_A(x),\ \hat{k}_A(x), \hat{\theta}_A(x), \hat{\phi}_A(x), \hat{r}_A(x) + \hat{k}_A(x), \hat{\theta}_A(x) + \hat{\phi}_A(x) \le 1$.

In the rest of the article, we write membership and non-membership degrees of $x \in X$ as $x = \left((\hat{r}, \hat{\theta}), \left(\hat{k}, \hat{\phi}\right)\right)$ and we call this representation of the element x as a CIF number where $0 \le \hat{r}, \hat{k}, \hat{r} + \hat{k} \le 1$ and $0 \le \hat{\theta}, \hat{\phi}, \hat{\theta} + \hat{\phi} \le 1$.

Definition 4. ([30]). *Consider two CIFNs:* $a_1 = \left((\hat{r}_1, \hat{\theta}_1), \left(\hat{k}_1, \hat{\phi}_1\right)\right)$ *and* $a_2 = \left((\hat{r}_2, \hat{\theta}_2), \left(\hat{k}_2, \hat{\phi}_2\right)\right)$. *The basic operations on these numbers are defined in the subsequent ways:*

1. $a_1 \prec a_2$, *if* $\hat{r}_1 \left\langle \hat{r}_2, \hat{k}_1 \right\rangle \hat{k}_2, \hat{\theta}_1 \langle \hat{\theta}_2$ *and* $\hat{\phi}_1 \rangle \hat{\phi}_2$,
2. $a_1 = a_2$, *if* $\hat{r}_1 = \hat{r}_2, \hat{k}_1 = \hat{k}_2, \hat{\theta}_1 = \hat{\theta}_2$ *and* $\hat{\phi}_1 = \hat{\phi}_2$,
3. $a_1^c = \left(\left(\hat{k}_1, \hat{\phi}_1\right), (\hat{r}_1, \hat{\theta}_1)\right)$.

Definition 5. ([33]). *Let* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right) (\gamma = 1, 2, 3, \ldots, m)$ *represent the number of CIFNs. In light of the CIF weighted averaging (CIFWA) operator, the aggregated value of these CIFNs is interpreted as follows:*

$$CIFWA(a_1, a_2, \ldots, a_m) = \begin{pmatrix} \left(1 - \prod_{\gamma=1}^{m} (1 - \hat{r}_\gamma)^{\xi_\gamma}, 1 - \prod_{\gamma=1}^{m} \left(1 - \hat{\theta}_\gamma\right)^{\xi_\gamma}\right), \\ \left(\prod_{\gamma=1}^{m} \left(\hat{k}_\gamma\right)^{\xi_\gamma}, \prod_{\gamma=1}^{m} \left(\hat{\phi}_\gamma\right)^{\xi_\gamma}\right) \end{pmatrix}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_\gamma(\gamma = 1, 2, 3, \ldots, m)$, *such that* $0 \le \xi_\gamma \le 1$ *for* $\gamma = 1, 2, 3, \ldots, m$ *and* $\sum_{\gamma=1}^{m} \xi_\gamma = 1$.

Definition 6. ([33]). *Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma\right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1,2,3,\ldots,m)$ *represent the number of CIFNs. In light of the CIF ordered weighted averaging (CIFOWA) operator, the aggregated value of these CIFNs is interpreted as follows:*

$$CIFOWA(a_1, a_2, \ldots, a_m) = \left(\begin{array}{c}\left(1 - \prod_{\gamma=1}^{m}\left(1 - \hat{r}_{\sigma(\gamma)}\right)^{\xi_\gamma}, 1 - \prod_{\gamma=1}^{m}\left(1 - \hat{\theta}_{\sigma(\gamma)}\right)^{\xi_\gamma}\right), \\ \left(\prod_{\gamma=1}^{m}\left(\hat{k}_{\sigma(\gamma)}\right)^{\xi_\gamma}, \prod_{\gamma=1}^{m}\left(\hat{\phi}_{\sigma(\gamma)}\right)^{\xi_\gamma}\right)\end{array}\right)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a permutation of* $(1,2,\ldots,m)$, *such that* $a_{\sigma(\gamma-1)} \geq a_{\sigma(\gamma)} \forall \gamma = 1,2,\ldots m$; $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is an associated weight vector of* $a_\gamma(\gamma = 1,2,3,\ldots,m)$, *such that* $0 \leq \xi_\gamma \leq 1$ *for* $\gamma = 1,2,3,\ldots,m$ *and* $\sum_{\gamma=1}^{\mathrm{m}} \xi_\gamma = 1$.

Definition 7. ([33]). *Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma\right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1,2,3,\ldots,m)$ *represent the number of CIFNs. In light of the CIF weighted geometric (CIFWG) operator, the aggregated value of these CIFNs is interpreted as follows:*

$$CIFWG(a_1, a_2, \ldots, a_m) = \left(\begin{array}{c}\left(\prod_{\gamma=1}^{m}(\hat{r}_\gamma)^{\xi_\gamma}, \prod_{\gamma=1}^{m}(\hat{\theta}_\gamma)^{\xi_\gamma}\right), \\ \left(1 - \prod_{\gamma=1}^{m}\left(1 - \hat{k}_\gamma\right)^{\xi_\gamma}, 1 - \prod_{\gamma=1}^{m}(1 - \hat{\phi}_\gamma)^{\xi_\gamma}\right)\end{array}\right)$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_\gamma(\gamma = 1,2,3,\ldots,m)$, *such that* $0 \leq \xi_\gamma \leq 1$ *for* $\gamma = 1,2,3,\ldots,m$ *and* $\sum_{\gamma=1}^{\mathrm{m}} \xi_\gamma = 1$.

Definition 8. ([33]). *Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma\right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1,2,3,\ldots,m)$ *represent the number of CIFNs. In light of the CIF ordered weighted geometric (CIFOWG) operator, the aggregated value of these CIFNs is interpreted as follows:*

$$CIFOWG(a_1, a_2, \ldots, a_m) = \left(\begin{array}{c}\left(\prod_{\gamma=1}^{m}\left(\hat{r}_{\sigma(\gamma)}\right)^{\xi_\gamma}, \prod_{\gamma=1}^{m}\left(\hat{\theta}_{\sigma(\gamma)}\right)^{\xi_\gamma}\right), \\ \left(1 - \prod_{\gamma=1}^{m}\left(1 - \hat{k}_{\sigma(\gamma)}\right)^{\xi_\gamma}, \prod_{\gamma=1}^{m}\left(1 - \hat{\phi}_{\sigma(\gamma)}\right)^{\xi_\gamma}\right)\end{array}\right)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a permutation of* $(1,2,\ldots,m)$, *such that* $a_{\sigma(\gamma-1)} \geq a_{\sigma(\gamma)} \forall \gamma = 1,2,\ldots m$; $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is an associated weight vector of* $a_\gamma(\gamma = 1,2,3,\ldots,m)$, *such that* $0 \leq \xi_\gamma \leq 1$ for $\gamma = 1,2,3,\ldots,m$ *and* $\sum_{\gamma=1}^{m} \xi_\gamma = 1$.
Some certain types of triangular norms and conorms are discussed in the following way.

Definition 9. ([46]). *For a,b* $\in R$, *the Dombi t-norm and t-conorm are denoted by* $Dom(a,b)$ *and* $Dom'(a,b)$, *respectively, such that* $Dom(a,b) = \frac{1}{1+\left\{\left(\frac{1-a}{a}\right)^{\varrho}+\left(\frac{1-b}{b}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}$ *and* $Dom'(a,b) = \frac{1}{1+\left\{\left(\frac{a}{1-a}\right)^{\varrho}+\left(\frac{b}{1-b}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}$, *where* $\varrho \geq 1$ *and* $(a,b) \in [0,1] \times [0,1]$. *Here,* $Dom(a,b) = \frac{1}{1+\left\{\left(\frac{1-a}{a}\right)^{\varrho}+\left(\frac{1-b}{b}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}$ *and* $Dom'(a,b) = \frac{1}{1+\left\{\left(\frac{a}{1-a}\right)^{\varrho}+\left(\frac{b}{1-b}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}$ *represent the Dombi product Dombi sum, respectively.*

Definition 10. ([26]). *The CIFNs are ranked in the framework of the following score and accuracy functions:*
For any CIFN $a_0 = \left(\left(\hat{r}_0, \hat{\theta}_0\right), \left(\hat{k}_0, \hat{\phi}_0\right)\right)$, *the score function*

and accuracy function

$$Ç(a_0) = \hat{r}_0 + \hat{\theta}_0 - \hat{k}_0 - \hat{\phi}_0$$

$$Ħ(a_0) = \hat{r}_0 + \hat{\theta}_0 + \hat{k}_0 + \hat{\phi}_0$$

where $Ç(a_0) \in [-2, 2]$ *and* $Ħ(a_0) \in [0, 2]$. *Moreover, any two CIFNs* a_1 *and* a_2 *satisfy the following comparison laws:*

1. *If* $Ç(a_1) > Ç(a_2)$, *then* $a_1 \succ a_2$,
2. *If* $Ç(a_1) < Ç(a_2)$, *then* $a_1 \prec a_2$,
3. *If* $Ç(a_1) = Ç(a_2)$, *then* $Ħ(a_1) > Ħ(a_2) \Rightarrow a_1 \succ a_2$, $Ħ(a_1) < Ħ(a_2) \Rightarrow a_1 \prec a_2$ *and* $Ħ(a_1) = Ħ(a_2) \Rightarrow a_1 \sim a_2$.

3. An Improvement of the Existing Score Function of CIFS

In this section, we present an example that indicates deficiencies of the score function of CIFNs developed in [26] and improve it in the subsequent discussion.

Example 1. *Suppose* $a_1 = ((0.3, 0.6), (0.35, 0.55))$ *and* $a_2 = ((0.4, 0.5), (0.2, 0.7))$ *are any two CIFNs. The application of Definition 10 on CIFNs* a_1 *and* a_2 *gives that* $Ç(a_1) = Ç(a_2) = 0$ *and* $Ħ(a_1) = Ħ(a_2) = 1.8$. *In view of property 3(c) of Definition 10, one can easily observe the incomparability of CIFNs* a_1 *and* a_2.

This indicates the deficiency of the score function under consideration. The above discussion leads us to improve this score function in the following definition.

Definition 11. *Let* $a_0 = \left((\hat{r}_0, \hat{\theta}_0), \left(\hat{k}_0, \hat{\phi}_0\right) \right)$ *be a CIFN. The improved score function* $Ҡ(a_0)$ *of CIFN is defined as*

$$Ҡ(a_0) = \frac{\hat{k}_0 + \hat{\phi}_0 - \hat{r}_0 - \hat{\theta}_0}{2} + \frac{\hat{r}_0 + \hat{\theta}_0 + 2\left(\hat{r}_0\hat{\theta}_0 - \hat{k}_0\hat{\phi}_0\right)}{\hat{r}_0 + \hat{\theta}_0 + \hat{k}_0 + \hat{\phi}_0}$$

where $g(a_0)$ *is the range of the score function and* $0 \leq g(a_0) \leq 2$.

Moreover, the above-proposed score function satisfies the comparison law for any two CIFNs a_1 and a_2, that is, $Ҡ(a_1) > Ҡ(a_2) \Rightarrow a_1 \succ a_2$, $Ҡ(a_1) < Ҡ(a_2) \Rightarrow a_1 \prec a_2$, and $Ҡ(a_1) = Ҡ(a_2) \Rightarrow a_1 \sim a_2$.

To illustrate the accuracy of the proposed score function for CIFN, consider the following example.

Example 2. *Suppose* $a_1 = ((0.3, 0.6), (0.35, 0.55))$ *and* $a_2 = ((0.4, 0.5), (0.2, 0.7))$ *are two CIFNs. The application of Definition 11 on the two CIFNs* a_1 *and* a_2 *gives that* $Ҡ(a_1) = 0.486$ and $Ҡ(a_2) = 0.567$. *Thus, in view of property 2 of Definition 11, we have* $a_1 \prec a_2$. *This fact suggests that* a_2 *is better than* a_1.

The above discussion shows that the proposed score function is more suitable and gives more accurate results for decision analysis.

4. Dombi Operations on Complex Intuitionistic Fuzzy Numbers

In this section, we develop the Dombi operations in the framework of the CIF environment.

Definition 12. *Let* $a_1 = \left((\hat{r}_1, \hat{\theta}_1), \left(\hat{k}_1, \hat{\phi}_1\right) \right)$ *and* $a_2 = \left((\hat{r}_2, \hat{\theta}_2), \left(\hat{k}_2, \hat{\phi}_2\right) \right)$ *be any two CIFNs. Some of the basic operations on* a_1 *and* a_2 *for* $\varrho \geq 1$ *and* $\psi > 0$ *are defined as*

1. $a_1 \bigoplus a_2 = \left(\begin{array}{c} \left(1 - \frac{1}{1+\left\{ \left(\frac{\hat{r}_1}{1-\hat{r}_1} \right)^{\varrho} + \left(\frac{\hat{r}_2}{1-\hat{r}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \left(\frac{\hat{\theta}_1}{1-\hat{\theta}_1} \right)^{\varrho} + \left(\frac{\hat{\theta}_2}{1-\hat{\theta}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{ \left(\frac{1-\hat{k}_1}{\hat{k}_1} \right)^{\varrho} + \left(\frac{1-\hat{k}_2}{\hat{k}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \left(\frac{1-\hat{\phi}_1}{\hat{\phi}_1} \right)^{\varrho} + \left(\frac{1-\hat{\phi}_2}{\hat{\phi}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right),$

2. $a_1 \bigotimes a_2 = \left(\begin{array}{c} \left(\frac{1}{1+\left\{ \left(\frac{1-\hat{r}_1}{\hat{r}_1} \right)^{\varrho} + \left(\frac{1-\hat{r}_2}{\hat{r}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \left(\frac{1-\hat{\theta}_1}{\hat{\theta}_1} \right)^{\varrho} + \left(\frac{1-\hat{\theta}_2}{\hat{\theta}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left\{ \left(\frac{\hat{k}_1}{1-\hat{k}_1} \right)^{\varrho} + \left(\frac{\hat{k}_2}{1-\hat{k}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \left(\frac{\hat{\phi}_1}{1-\hat{\phi}_1} \right)^{\varrho} + \left(\frac{\hat{\phi}_2}{1-\hat{\phi}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right),$

3. $\psi a_1 = \left(\left(1 - \frac{1}{1+\left\{ \psi \left(\frac{\hat{r}_1}{1-\hat{r}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \psi \left(\frac{\hat{\theta}_1}{1-\hat{\theta}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \left(\frac{1}{1+\left\{ \psi \left(\frac{1-\hat{k}_1}{\hat{k}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \psi \left(\frac{1-\hat{\phi}_1}{\hat{\phi}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \right),$

4. $a_1^{\psi} = \left(\left(\frac{1}{1+\left\{ \psi \left(\frac{1-\hat{r}_1}{\hat{r}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \psi \left(\frac{1-\hat{\theta}_1}{\hat{\theta}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \left(1 - \frac{1}{1+\left\{ \psi \left(\frac{\hat{k}_1}{1-\hat{k}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \psi \left(\frac{\hat{\phi}_1}{1-\hat{\phi}_1} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \right).$

In the following definition, we propose a Dombi arithmetic aggregation operator for CIFS, namely, a CIF Dombi weighted averaging (CIFDWA) operator.

Definition 13. *Let* $a_{\gamma} = \left((\hat{r}_{\gamma}, \hat{\theta}_{\gamma}), \left(\hat{k}_{\gamma}, \hat{\phi}_{\gamma} \right) \right) (\gamma = 1, 2, 3 \ldots m)$ *represent the number of CIFNs. The CIFDWA operator is a mapping* $CIFDWA : a^m \rightarrow a$ *defined by*

$$CIFDWA(a_1, a_2, \ldots, a_m) = \bigoplus_{\gamma=1}^{m} (\xi_{\gamma} a_{\gamma}),$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_{\gamma}(\gamma = 1, 2, 3, \ldots, m)$*, such that* $0 \leq \xi_{\gamma} \leq 1$ for $\gamma = 1, 2, 3, \ldots, m$ *and* $\sum_{\gamma=1}^{m} \xi_{\gamma} = 1$.

Theorem 1. *Let* $a_{\gamma} = \left((\hat{r}_{\gamma}, \hat{\theta}_{\gamma}), \left(\hat{k}_{\gamma}, \hat{\phi}_{\gamma} \right) \right) (\gamma = 1, 2, 3, \ldots, m)$ *represents the number of CIFNs. Then, the aggregated value of these CIFNs in the framework of the CIFDWA operator is also a CIFN and is determined in the following way:*

$$CIFDWA(a_1, a_2, \ldots, a_m) = \bigoplus_{\gamma=1}^{m} (\xi_{\gamma} a_{\gamma}),$$

$$= \left(\begin{array}{c} \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_{\gamma} \left(\frac{\hat{r}_{\gamma}}{1-\hat{r}_{\gamma}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_{\gamma} \left(\frac{\hat{\theta}_{\gamma}}{1-\hat{\theta}_{\gamma}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_{\gamma} \left(\frac{1-\hat{k}_{\gamma}}{\hat{k}_{\gamma}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_{\gamma} \left(\frac{1-\hat{\phi}_{\gamma}}{\hat{\phi}_{\gamma}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

where $\varrho > 0$, $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_{\gamma}(\gamma = 1, 2, 3, \ldots, m)$*, such that* $0 \leq \xi_{\gamma} \leq 1$ *for* $\gamma = 1, 2, 3, \ldots, m$ *and* $\sum_{\gamma=1}^{m} \xi_{\gamma} = 1$.

Proof. This theorem is demonstrated through the use of the mathematical induction technique.

When $\gamma = 1$, then clearly $\xi_1 = 1$.

$$\begin{aligned} CIFDWA(a_1) &= \left(\left(1 - \frac{1}{1+\frac{\hat{r}_1}{1-\hat{r}_1}}, 1 - \frac{1}{1+\frac{\hat{\theta}_1}{1-\hat{\theta}_1}} \right), \left(\frac{1}{1+\frac{1-\hat{k}_1}{\hat{k}_1}}, \frac{1}{1+\frac{1-\hat{\phi}_1}{\hat{\phi}_1}} \right) \right) \\ &= \left(\left(1 - \frac{1-\hat{r}_1}{1-\hat{r}_1+\hat{r}_1}, 1 - \frac{1-\hat{\theta}_1}{1-\hat{\theta}_1+\hat{\theta}_1} \right), \left(\frac{\hat{k}_1}{\hat{k}_1+1-\hat{k}_1}, \frac{\hat{\phi}_1}{\hat{\phi}_1+1-\hat{\phi}_1} \right) \right) \\ &= \left(\left(\hat{r}_1, \hat{\theta}_1 \right), \left(\hat{k}_1, \hat{\phi}_1 \right) \right) \end{aligned}$$

This means that

$$CIFDWA(a_1) = a_1$$

Therefore, the equation holds for $\gamma = 1$.

Moreover, the application of Definition 13 for $\gamma = 2$ gives the following outcome:

$$CIFDWA(a_1, a_2) = \xi_1 a_1 \bigoplus \xi_2 a_2$$

$$= \begin{pmatrix} \left(1 - \frac{1}{1+\left\{ \xi_1 \left(\frac{\hat{r}_1}{1-\hat{r}_1} \right)^{\varrho} + \xi_2 \left(\frac{\hat{r}_2}{1-\hat{r}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \xi_1 \left(\frac{\hat{\theta}_1}{1-\hat{\theta}_1} \right)^{\varrho} + \xi_2 \left(\frac{\hat{\theta}_2}{1-\hat{\theta}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{ \xi_1 \left(\frac{1-\hat{k}_1}{\hat{k}_1} \right)^{\varrho} + \xi_2 \left(\frac{1-\hat{k}_2}{\hat{k}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \xi_1 \left(\frac{1-\hat{\phi}_1}{\hat{\phi}_1} \right)^{\varrho} + \xi_2 \left(\frac{1-\hat{\phi}_2}{\hat{\phi}_2} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{pmatrix}$$

This means that

$$CIFDWA(a_1, a_2)$$

$$= \begin{pmatrix} \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{\hat{r}_\gamma}{1-\hat{r}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{\hat{\theta}_\gamma}{1-\hat{\theta}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{1-\hat{k}_\gamma}{\hat{k}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{1-\hat{\phi}_\gamma}{\hat{\phi}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{pmatrix}$$

Hence, the result is valid for $\gamma = 2$.

Assuming that the statement is true for $\gamma = s$, we have:

$$CIFDWA(a_1, a_2, \ldots, a_s) = (\xi_1 a_1) \bigoplus (\xi_2 a_2) \bigoplus \ldots \bigoplus (\xi_s a_s) = \bigoplus_{\gamma=1}^{s} (\xi_\gamma a_\gamma)$$

$$= \begin{pmatrix} \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{\hat{r}_\gamma}{1-\hat{r}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{\theta}_\gamma}{1-\hat{\theta}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{k}_\gamma}{\hat{k}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{\phi}_\gamma}{\hat{\phi}_\gamma} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{pmatrix}$$

Moreover, for $\gamma = s+1$, we have

$$CIFDWA(a_1, a_2, \ldots, a_s, a_{s+1}) = (\xi_1 a_1) \bigoplus (\xi_2 a_2) \bigoplus \ldots \bigoplus (\xi_s a_s) \bigoplus (\xi_{s+1} a_{s+1})$$

$$= \bigoplus_{\gamma=1}^{m} (\xi_\gamma a_\gamma) \bigoplus (\xi_{s+1} a_{s+1})$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\gamma)}}{\hat{r}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\gamma)}}{\hat{\theta}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\gamma)}}{1-\hat{k}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\gamma)}}{1-\hat{\phi}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

$$\bigotimes \left(\begin{array}{c} \left(\frac{1}{1+\left\{\xi_{s+1} \left(\frac{1-\hat{r}_{\sigma(s+1)}}{\hat{r}_{\sigma(s+1)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\xi_{s+1} \left(\frac{1-\hat{\theta}_{\sigma(s+1)}}{\hat{\theta}_{\sigma(s+1)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left\{\xi_{s+1} \left(\frac{\hat{k}_{\sigma(s+1)}}{1-\hat{k}_{\sigma(s+1)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\xi_{s+1} \left(\frac{\hat{\phi}_{\sigma(s+1)}}{1-\hat{\phi}_{\sigma(s+1)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

This means that

$$CIFDWA(a_1, a_2, \cdots, a_{s+1})$$

$$= \left(\begin{array}{c} \left(1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{\hat{r}_\gamma}{1-\hat{r}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{\hat{\theta}_\gamma}{1-\hat{\theta}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{1-\hat{k}_\gamma}{\hat{k}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{1-\hat{\phi}_\gamma}{\hat{\phi}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

Consequently, we conclude that the statement is true for all positive integral values of γ. □

The following example describes the above-stated fact.

Example 3. *Consider the CIFNs* $a_1 = ((0.56, 0.45), (0.44, 0.4))$, $a_2 = ((0.35, 0.45), (0.4, 0.5))$, $a_3 = ((0.45, 0.6), (0.45, 0.15))$, $a_4 = ((0.45, 0.15), (0.2, 0.5))$ *and the associated weight vector* $\xi = (0.2, 0.3, 0.4, 0.1)^T$. *Then, for* $\varrho = 3$, *we have*

$$\left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{\hat{r}_\gamma}{1-\hat{r}_\gamma}\right)^3\right\}^{\frac{1}{3}} = 0.9016, \left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{\hat{\theta}_\gamma}{1-\hat{\theta}_\gamma}\right)^3\right\}^{\frac{1}{3}} = 1.1755,$$

$$\left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{1-\hat{k}_\gamma}{\hat{k}_\gamma}\right)^3\right\}^{\frac{1}{3}} = 2.0452, \left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{1-\hat{\phi}_\gamma}{\hat{\phi}_\gamma}\right)^3\right\}^{\frac{1}{3}} = 4.1957.$$

This implies that

$$CIFDWA(a_1, a_2, a_3, a_4) = \bigoplus_{\gamma=1}^{4} (\xi_\gamma a_\gamma) = ((0.4741, 0.5403), (0.3284, 0.1925))$$

Thus, we conclude that the outcome of the above discussion is also a CIFN.

Theorem 2. *Idempotency property) If* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1, 2, 3, \ldots, m)$ *are CIFNs such that* $a_\gamma = a_0 \forall\gamma$, *where* $a_0 = \left((\hat{r}_0, \hat{\theta}_0), \left(\hat{k}_0, \hat{\phi}_0\right)\right)$ *is a CIFN. Then* $CIFDWA(a_1, a_2, \ldots, a_m) = a_0$.

Proof. Since $a_\gamma = a_0 \forall\gamma$. Then, by Definition 4, $\hat{r}_\gamma = \hat{r}_0, \hat{\theta}_\gamma = \hat{\theta}_0, \hat{k}_\gamma = \hat{k}_0$ and $\hat{\phi}_\gamma = \hat{\phi}_0 \ \forall\gamma$. By substituting the above relations in Theorem 1, we obtain

$$CIFDOWG(a_1, a_2, \ldots, a_m)$$

$$= \left(\begin{array}{c} \left(1 - \frac{1}{1+\left(\frac{1-\hat{r}_0}{\hat{r}_0}\right)\left\{\sum_{\gamma=1}^{m} \xi_\gamma\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left(\frac{\hat{\theta}_0}{\hat{\theta}_0}\right)\left\{\sum_{\gamma=1}^{m} \xi_\gamma\right\}^{\frac{1}{\varrho}}}\right), \\ \left(\frac{1}{1+\left(\frac{1-\hat{k}_0}{\hat{k}_0}\right)\left\{\sum_{\gamma=1}^{m} \xi_\gamma\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left(\frac{1-\hat{\phi}_0}{\hat{\phi}_0}\right)\left\{\sum_{\gamma=1}^{m} \xi_\gamma\right\}^{\frac{1}{\varrho}}}\right) \end{array} \right)$$

$$= \left(\left(1 - \frac{1}{1+\left(\frac{\hat{r}_0}{1-\hat{r}_0}\right)}, 1 - \frac{1}{1+\left(\frac{\hat{\theta}_0}{1-\hat{\theta}_0}\right)}\right), \left(\frac{1}{1+\left(\frac{1-\hat{k}_0}{\hat{k}_0}\right)}, \frac{1}{1+\left(\frac{1-\hat{\phi}_0}{\hat{\phi}_0}\right)}\right) \right)$$

$$= \left((\hat{r}_0, \hat{\theta}_0), \left(\hat{k}_0, \hat{\phi}_0\right)\right)$$

This shows that

$$CIFDWA(a_1, a_2, \ldots, a_m) = a_0$$

□

Theorem 3. *(Boundedness property) Let* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1, 2, 3, \ldots, m)$ *be CIFNs. Let* $a^- = \min_\gamma\{a_\gamma\} = \left((\hat{r}^-, \hat{\theta}^-), \left(\hat{k}^-, \hat{\phi}^-\right)\right)$ *and* $a^+ = \max_\gamma\{a_\gamma\} = \left((\hat{r}^+, \hat{\theta}^+), \left(\hat{k}^+, \hat{\phi}^+\right)\right)$ *where* $\hat{r}^- = \min_\gamma\{\hat{r}_\gamma\}$, $\hat{\theta}^- = \min_\gamma\{\hat{\theta}_\gamma\}$, $\hat{k}^- = \max_\gamma\left\{\hat{k}_\gamma\right\}$, $\hat{\phi}^- = \max_\gamma\{\hat{\phi}_\gamma\}$, $\hat{r}^+ = \max_\gamma\{\hat{r}_\gamma\}$, $\hat{\theta}^+ = \max_\gamma\{\hat{\theta}_\gamma\}, \hat{k}^+ = \min_\gamma\left\{\hat{k}_\gamma\right\}, \hat{\phi}^+ = \min_\gamma\{\hat{\phi}_\gamma\}$. *Then,* $a^- \ CIFDWA(a_1, a_2, \ldots, a_m) \leq \ a^+$.

Proof. In view of the given conditions, we have

$$1-\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{\hat{r}^-}{1-\hat{r}^-}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\leq 1-\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{\hat{r}_\gamma}{1-\hat{r}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}$$

$$\leq 1-\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{\hat{r}^+}{1-\hat{r}^+}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},$$

$$1-\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{\hat{\theta}^-}{1-\hat{\theta}^-}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\leq 1-\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{\hat{\theta}_\gamma}{1-\hat{\theta}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}$$

$$\leq 1-\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{\hat{\theta}^+}{1-\hat{\theta}^+}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},$$

$$\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{1-\hat{k}^-}{\hat{k}^-}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\geq\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{1-\hat{k}_\gamma}{\hat{k}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\geq\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{1-\hat{k}^+}{\hat{k}^+}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},$$

$$\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{1-\hat{\phi}^-}{\hat{\phi}^-}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\geq\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{1-\hat{\phi}_\gamma}{\hat{\phi}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\geq\frac{1}{1+\left\{\sum_{\gamma=1}^{m}\xi_\gamma\left(\frac{1-\hat{\phi}^+}{\hat{\phi}^+}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}.$$

In light of Definition 4 and the above relations, we get
$a^- \leq CIFDWA(a_1, a_2, \ldots, a_m) \leq a^+$. □

Theorem 4. *(Monotonicity property) Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma\right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)$ and $a'_\gamma = \left(\left(\hat{r}'_\gamma, \hat{\theta}'_\gamma\right), \left(\hat{k}'_\gamma, \hat{\phi}'_\gamma\right)\right)$ *for* $\gamma = 1, 2, 3, \ldots, m$ *be two collections of CIFNs. If* $\hat{r}_\gamma \leq \hat{r}'_\gamma$, $\hat{k}_\gamma \geq \hat{k}'_\gamma$, $\hat{\theta}_\gamma \leq \hat{\theta}'_\gamma$ *and* $\hat{\phi}_\gamma \geq \hat{\phi}'_\gamma\ \forall\gamma$. *Then* $CIFDWA(a_1, a_2, \ldots, a_m) \leq CIFDWA\left(a'_1, a'_2, \ldots, a'_m\right)$.

Proof. The proof is a straightforward implementation of Definition 4. □

In the following definition, we propose a Dombi arithmetic aggregation operator for CIFS, namely, a CIF Dombi ordered weighted averaging (CIFDOWA) operator.

Definition 14. *Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma\right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1, 2, 3, \ldots, m)$ *represent the number of CIFNs. The CIFDOWA operator is a mapping* $CIFDOWA : a^m \to a$ *defined by*

$$CIFDOWA(a_1, a_2, \ldots, a_m) = \bigoplus_{\gamma=1}^{m}\left(\xi_\gamma a_{\sigma(\gamma)}\right),$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_\gamma(\gamma = 1, 2, 3, \ldots, m)$, *such that* $0 \leq \xi_\gamma \leq 1$ *for* $\gamma = 1, 2, 3, \ldots, m$ *and* $\sum_{\gamma=1}^{m}\xi_\gamma = 1$; $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a permutation of* $(1, 2, \ldots, m)$ *such that* $a_{\sigma(\gamma-1)} \geq a_{\sigma(\gamma)} \forall\gamma = 1, 2, \ldots m$.

Theorem 5. *Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma\right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1, 2, 3, \ldots, m)$ *represent the number of CIFNs. Then, the aggregated value of these CIFNs in the framework of the CIFDOWA operator is also a CIFN and is determined in the following way:*

$$CIFDOWA(a_1, a_2, \ldots, a_m) = \bigoplus_{\gamma=1}^{m}\left(\xi_\gamma a_{\sigma(\gamma)}\right)$$

$$= \left(\begin{array}{c} \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_\gamma \left(\frac{\hat{r}_{\sigma(\varsigma\gamma)}}{1-\hat{r}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_\gamma \left(\frac{\hat{\theta}_{\sigma(\varsigma\gamma)}}{1-\hat{\theta}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_\gamma \left(\frac{1-\hat{k}_{\sigma(\varsigma\gamma)}}{\hat{k}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{m} \xi_\gamma \left(\frac{1-\hat{\phi}_{\sigma(\varsigma\gamma)}}{\hat{\phi}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a permutation of* $(1, 2, \ldots, m)$ *such that* $a_{\sigma(\gamma-1)} \geq a_{\sigma(\gamma)} \forall \gamma = 1, 2, \ldots, m$; $\varrho > 0$, $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_\gamma (\gamma = 1, 2, 3, \ldots, m)$ *and* $0 \leq \xi_\gamma \leq 1$ *for* $\gamma = 1, 2, 3, \ldots, m$ *such that* $\sum_{\gamma=1}^{m} \xi_\gamma = 1$.

Proof. The proof of this theorem is similar to that of Theorem 1. □

The following example illustrates the aforementioned fact.

Example 4. *Consider the CIFNs* $a_1 = ((0.4, 0.3), (0.5, 0.55))$, $a_2 = ((0.47, 0.5), (0.49, 0.25))$, $a_3 = ((0.63, 0.7), (0.1, 0.25))$, $a_4 = ((0.3, 0.5), (0.15, 0.3))$, $a_5 = ((0.55, 0.63), (0.3, 0.2))$ *and the associated weight vector* $\xi = (0.25, 0.15, 0.13, 0.27, 0.2)^T$. *To aggregate these values by* $CIFDOWA$ *operator, we first permute these numbers using Definition 11 to acquire the subsequent data.*

$$\text{Ҡ}(a_1) = 0.398,\ \text{Ҡ}(a_2) = 0.584,\ \text{Ҡ}(a_3) = 0.797,\ \text{Ҡ}(a_4) = 0.633,\ \text{Ҡ}(a_5) = 0.703$$

By applying Definition 14, the permuted values of the CIFNs are calculated as follows: $a_{\sigma(1)} = ((0.63, 0.7), (0.1, 0.25)), a_{\sigma(2)} = ((0.55, 0.63), (0.3, 0.2)), a_{\sigma(3)} = ((0.4, 0.3), (0.5, 0.55)), a_{\sigma(4)} = ((0.47, 0.5), (0.49, 0.25))$ *and* $a_{\sigma(5)} = ((0.4, 0.3), (0.5, 0.55))$. *Then, for* $\varrho = 3$, *we have*

$$\left\{ \sum_{\gamma=1}^{5} \xi_\gamma \left(\frac{\hat{r}_{\sigma(\gamma)}}{1-\hat{r}_{\sigma(\gamma)}} \right)^3 \right\}^{\frac{1}{3}} = 1.209, \left\{ \sum_{\gamma=1}^{5} \xi_\gamma \left(\frac{\hat{\theta}_{\sigma(\gamma)}}{1-\hat{\theta}_{\sigma(\gamma)}} \right)^3 \right\}^{\frac{1}{3}} = 1.630,$$

$$\left\{ \sum_{\gamma=1}^{5} \xi_\gamma \left(\frac{1-\hat{k}_{\sigma(\gamma)}}{\hat{k}_{\sigma(\gamma)}} \right)^3 \right\}^{\frac{1}{3}} = 5.928, \left\{ \sum_{\gamma=1}^{5} \xi_\gamma \left(\frac{1-\hat{\phi}_{\sigma(\gamma)}}{\hat{\phi}_{\sigma(\gamma)}} \right)^3 \right\}^{\frac{1}{3}} = 2.940.$$

This implies that

$$\begin{aligned} CIFDOWA(a_1, a_2, a_3, a_4, a_5) &= \bigoplus_{\gamma=1}^{5} \left(\xi_\gamma a_{\sigma(\gamma)} \right) \\ &= ((0.547, 0.620), (0.144, 0.254) \end{aligned}$$

Consequently, we conclude that the result of the preceding discussion is also a CIFN.

Theorem 6. *(Idempotency property) If* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma \right) \right) (\gamma = 1, 2, 3, \ldots, m)$ *are CIFNs such that* $a_\gamma = a_0 \forall \gamma$, *where* $a_0 = \left((\hat{r}_0, \hat{\theta}_0), \left(\hat{k}_0, \hat{\phi}_0 \right) \right)$ *is a CIFN. Then* $CIFDOWA(a_1, a_2, \ldots, a_m) = a_0$.

Proof. The proof of this theorem is similar to that of Theorem 2. □

Theorem 7. *(Boundedness property) Let* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma \right) \right) (\gamma = 1, 2, 3, \ldots, m)$ *be CIFNs. Let* $a^- = \min_\gamma \{a_\gamma\} = \left((\hat{r}^-, \hat{\theta}^-), \left(\hat{k}^-, \hat{\phi}^- \right) \right)$ *and* $a^+ = \max_\gamma \{a_\gamma\}$

$= \left(\left(\hat{r}^{+}, \hat{\theta}^{+}\right), \left(\hat{k}^{+}, \hat{\phi}^{+}\right)\right)$ *where* $\hat{r}^{-} = \min_{\gamma}\{\hat{r}_{\gamma}\}$, $\hat{\theta}^{-} = \min_{\gamma}\{\hat{\theta}_{\gamma}\}$, $\hat{k}^{-} = \max_{\gamma}\left\{\hat{k}_{\gamma}\right\}$, $\hat{\phi}^{-} = \max_{\gamma}\{\hat{\phi}_{\gamma}\}$, $\hat{r}^{+} = \max_{\gamma}\{\hat{r}_{\gamma}\}, \hat{\theta}^{+} = \max_{\gamma}\{\hat{\theta}_{\gamma}\}, \hat{k}^{+} = \min_{\gamma}\left\{\hat{k}_{\gamma}\right\}, \hat{\phi}^{+} = \min_{\gamma}\{\hat{\phi}_{\gamma}\}$. *Then,* $a^{-} \leq CIFDOWA(a_1, a_2, \ldots, a_m) \leq a^{+}$.

Proof. The proof of this theorem is similar to that of Theorem 3. □

Theorem 8. *(Monotonicity property) Let* $a_{\gamma} = \left(\left(\hat{r}_{\gamma}, \hat{\theta}_{\gamma}\right), \left(\hat{k}_{\gamma}, \hat{\phi}_{\gamma}\right)\right)$ *and* $a'_{\gamma} = \left(\left(\hat{r}'_{\gamma}, \hat{\theta}'_{\gamma}\right), \left(\hat{k}'_{\gamma}, \hat{\phi}'_{\gamma}\right)\right)$ *for* $\gamma = 1, 2, 3, \ldots, m$ *be two collections of CIFNs. If* $\hat{r}_{\sigma(\gamma)} \leq \hat{r}'_{\sigma(\gamma)}$, $\hat{k}_{\sigma(\gamma)} \geq \hat{k}'_{\sigma(\gamma)}$, $\hat{\theta}_{\sigma(\gamma)} \leq \hat{\theta}'_{\sigma(\gamma)}$ *and* $\hat{\phi}_{\sigma(\gamma)} \geq \hat{\phi}'_{\sigma(\gamma)}$ $\forall\gamma$. *Then* $CIFDOWA(a_1, a_2, \ldots, a_m) \leq CIFDOWA(a'_1, a'_2, \ldots, a'_m)$.

Proof. The proof is a simple application of Definition 4. □

In the subsequent definition, we propose the CIF Dombi weighted geometric (CIFDWG) operator.

Definition 15. *Let* $a_{\gamma} = \left(\left(\hat{r}_{\gamma}, \hat{\theta}_{\gamma}\right), \left(\hat{k}_{\gamma}, \hat{\phi}_{\gamma}\right)\right)(\gamma = 1, 2, 3, \ldots, m)$ *represent the number of CIFNs. The CIFDWG operator is a mapping* $CIFDWG : a^m \to a$ *defined by*

$$CIFDWG(a_1, a_2, \ldots, a_m) = \otimes_{\gamma=1}^{m}(a_{\gamma})^{\xi_{\gamma}}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_{\gamma}(\gamma = 1, 2, 3, \ldots, m)$ *and* $0 \leq \xi_{\gamma} \leq 1$ for $\gamma = 1, 2, 3, \ldots, m$ *such that* $\sum_{\gamma=1}^{m} \xi_{\gamma} = 1$.

Theorem 9. *Let* $a_{\gamma} = \left(\left(\hat{r}_{\gamma}, \hat{\theta}_{\gamma}\right), \left(\hat{k}_{\gamma}, \hat{\phi}_{\gamma}\right)\right)(\gamma = 1, 2, 3 \ldots m)$ *represent the number of CIFNs. Then, the aggregate value of these CIFNs within the context of the CIFDWG operator is also a CIFN and is calculated as follows:*

$$CIFDWG(a_1, a_2, \ldots, a_m) = \otimes_{\gamma=1}^{m}(a_{\gamma})^{\xi_{\gamma}}$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{m} \xi_{\gamma}\left(\frac{\hat{r}_{\gamma}}{1-\hat{r}_{\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{m} \xi_{\gamma}\left(\frac{\hat{\theta}_{\gamma}}{1-\hat{\theta}_{\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{m} \xi_{\gamma}\left(\frac{1-\hat{k}_{\gamma}}{\hat{k}_{\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{m} \xi_{\gamma}\left(\frac{1-\hat{\phi}_{\gamma}}{\hat{\phi}_{\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

where $\varrho > 0$, $\xi = (\xi_1, \xi_1, \ldots, \xi_m)^T$ *is a weighted vector of* $a_{\gamma}(\gamma = 1, 2, 3 \ldots m)$, *such that* $0 \leq \xi_{\gamma} \leq 1$ *for* $\gamma = 1, 2, 3, \ldots, m$ *and* $\sum_{\gamma=1}^{m} \xi_{\gamma} = 1$.

Proof. Proof of this theorem is analogous to Theorem 1. □

The following example describes the above-stated fact.

Example 5. *Consider the CIFNs* $a_1 = ((0.3, 0.5), (0.4, 0.4))$, $a_2 = ((0.7, 0.5), (0.2, 0.5))$, $a_3 = ((0.45, 0.53), (0.55, 0.25))$ *and the associated weight vector* $\xi = (0.12, 0.55, 0.31)^T$. *Then, for* $\varrho = 5$, *we have*

$$\left\{\sum_{\gamma=1}^{3}\xi_\gamma\left(\frac{1-\hat{r}_\gamma}{\hat{r}_\gamma}\right)^5\right\}^{\frac{1}{5}} = 1.557, \left\{\sum_{\gamma=1}^{3}\xi_\gamma\left(\frac{1-\hat{\theta}_\gamma}{\hat{\theta}_\gamma}\right)^5\right\}^{\frac{1}{5}} = 0.966,$$

$$\left\{\sum_{\gamma=1}^{3}\xi_\gamma\left(\frac{\hat{k}_\gamma}{1-\hat{k}_\gamma}\right)^5\right\}^{\frac{1}{5}} = 0.971, \left\{\sum_{\gamma=1}^{3}\xi_\gamma\left(\frac{\hat{\phi}_\gamma}{1-\hat{\phi}_\gamma}\right)^5\right\}^{\frac{1}{5}} = 0.893.$$

This implies that

$$CIFDWG(a_1, a_2, a_3) = \otimes_{\gamma=1}^{3}(a_\gamma)^{\xi_\gamma} = ((0.391, 0.509), (0.493, 0.472))$$

Thus, we conclude that the outcome of the above discussion is also a CIFN.

Theorem 10. *(Idempotency property) If* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1, 2, 3 \ldots m)$ *are CIFNs such that* $a_\gamma = a_0 \forall \gamma$, *where* $a_0 = \left((\hat{r}_0, \hat{\theta}_0), \left(\hat{k}_0, \hat{\phi}_0\right)\right)$ *is a CIFN. Then* $CIFDWG(a_1, a_2, \ldots, a_m) = a_0$.

Proof. Proof of this theorem is analogous to Theorem 2. □

Theorem 11. *(Boundedness property) Let* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1, 2, 3 \ldots m)$ *be CIFNs. Let* $a^- = \min_\gamma\{a_\gamma\} = \left((\hat{r}^-, \hat{\theta}^-), \left(\hat{k}^-, \hat{\phi}^-\right)\right)$ *and* $a^+ = \max_\gamma\{a_\gamma\} = \left((\hat{r}^+, \hat{\theta}^+), \left(\hat{k}^+, \hat{\phi}^+\right)\right)$ *where* $\hat{r}^- = \min_\gamma\{\hat{r}_\gamma\}$, $\hat{\theta}^- = \min_\gamma\{\hat{\theta}_\gamma\}$, $\hat{k}^- = \max_\gamma\left\{\hat{k}_\gamma\right\}$, $\hat{\phi}^- = \max_\gamma\{\hat{\phi}_\gamma\}$, $\hat{r}^+ = \max_\gamma\{\hat{r}_\gamma\}, \hat{\theta}^+ = \max_\gamma\{\hat{\theta}_\gamma\}, \hat{k}^+ = \min_\gamma\left\{\hat{k}_\gamma\right\}, \hat{\phi}^+ = \min_\gamma\{\hat{\phi}_\gamma\}$. *Then,* $a^- \leq CIFDWG(a_1, a_2, \ldots, a_m) \leq a^+$.

Proof. Proof of this theorem is analogous to Theorem 3. □

Theorem 12. *(Monotonicity property) Let* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)$ *and* $a'_\gamma = \left(\left(\hat{r}'_\gamma, \hat{\theta}'_\gamma\right), \left(\hat{k}'_\gamma, \hat{\phi}'_\gamma\right)\right)$ *for* $\gamma = 1, 2, 3, \ldots, m$ *be two collections of CIFNs. If* $\hat{r}_\gamma \leq \hat{r}'_\gamma$, $\hat{k}_\gamma \geq \hat{k}'_\gamma$, $\hat{\theta}_\gamma \leq \hat{\theta}'_\gamma$ *and* $\hat{\phi}_\gamma \geq \hat{\phi}'_\gamma\ \forall\gamma$. *Then* $CIFDWG(a_1, a_2, \ldots, a_m) \leq CIFDWG\left(a'_1, a'_2, \ldots, a'_m\right)$.

Proof. The proof is a straightforward implementation of Definition 4. □

In the following definition, we propose a Dombi geometric aggregation operator for CIFS, namely, a CIF Dombi ordered weighted geometric (CIFDOWG) operator.

Definition 16. *Let* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)(\gamma = 1, 2, 3 \ldots m)$ *represent the number of CIFNs. The CIFDOWG operator is a mapping* $CIFDOWG : a^m \to a$ *defined by*

$$CIFDOWG(a_1, a_2, \ldots, a_m) = \bigotimes_{\gamma=1}^{m}\left(a_{\sigma(\gamma)}\right)^{\xi_\gamma}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_\gamma(\gamma = 1, 2, 3, \ldots, m)$, *such that* $0 \leq \xi_\gamma \leq 1$ for $\gamma = 1, 2, 3, \ldots, m$ *and* $\sum_{\gamma=1}^{m}\xi_\gamma = 1$; $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a permutation of* $(1, 2, \ldots, m)$ *such that* $a_{\sigma(\gamma-1)} \geq a_{\sigma(\gamma)} \forall\gamma = 1, 2, \ldots m$.

Theorem 13. *Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma \right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma \right) \right) (\gamma = 1, 2, 3, \ldots, m)$ *represent the number of CIFNs. Then, the aggregate value of these CIFNs within the framework of the CIFDOWG operator is also a CIFN and is calculated as follows:*

$$CIFDOWG(a_1, a_2, \ldots, a_m) = \bigotimes_{\gamma=1}^{m} \left(a_{\sigma(\gamma)} \right)^{\xi_\gamma}$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\gamma)}}{\hat{r}_{\sigma(\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\gamma)}}{\hat{\theta}_{\sigma(\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\gamma)}}{1-\hat{k}_{\sigma(\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\gamma)}}{1-\hat{\phi}_{\sigma(\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a permutation of* $(1, 2, \ldots, m)$ *such that* $a_{\sigma(\gamma-1)} \geq a_{\sigma(\gamma)} \forall \gamma = 1, 2, \ldots m$; $\varrho > 0$, $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ *is a weighted vector of* $a_\gamma (\gamma = 1, 2, 3, \ldots, m)$ *and* $0 \leq \xi_\gamma \leq 1$ *for* $\gamma = 1, 2, 3, \ldots, m$ *such that* $\sum_{\gamma=1}^{m} \xi_\gamma = 1$.

Proof. This theorem is proven through mathematical induction.

When $\gamma = 1$, then clearly $\xi_1 = 1$ and $a_{\sigma(1)} = a_1$.

$$CIFDOWG(a_1) = \left(\left(\frac{1}{1+\frac{1-\hat{r}_1}{\hat{r}_1}}, \frac{1}{1+\frac{1-\hat{\theta}_1}{\hat{\theta}}} \right), \left(1 - \frac{1}{1+\frac{\hat{k}_1}{1-\hat{k}_1}}, 1 - \frac{1}{1+\frac{\hat{\phi}_1}{1-\hat{\phi}_1}} \right) \right)$$

$$= \left(\left(\frac{\hat{r}_1}{\hat{r}_1+1-\hat{r}_1}, \frac{\hat{\theta}_1}{\hat{\theta}_1+1-\hat{\theta}_1} \right), \left(1 - \frac{1-\hat{k}_1}{1-\hat{k}_1+\hat{k}_1}, 1 - \frac{1-\hat{\phi}_1}{1-\hat{\phi}_1+\hat{\phi}_1} \right) \right)$$

$$= \left(\left(\hat{r}_1, \hat{\theta}_1 \right), \left(\hat{k}_1, \hat{\phi}_1 \right) \right)$$

This means that

$$CIFDOWG(a_1) = a_1$$

Therefore, the equation holds for $\gamma = 1$.

Moreover, the application of Definition 16 for $\gamma = 2$ gives the following outcome:

$$CIFDOWG(a_1, a_2) = \left(a_{\sigma(1)} \right)^{\xi_1} \bigoplus \left(a_{\sigma(2)} \right)^{\xi_2}$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left\{ \xi_1 \left(\frac{1-\hat{r}_{\sigma(1)}}{\hat{r}_{\sigma(1)}} \right)^{\varrho} + \xi_2 \left(\frac{1-\hat{r}_{\sigma(2)}}{\hat{r}_{\sigma(2)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \xi_1 \left(\frac{1-\hat{\theta}_{\sigma(1)}}{\hat{\theta}_{\sigma(1)}} \right)^{\varrho} + \xi_2 \left(\frac{1-\hat{\theta}_{\sigma(2)}}{\hat{\theta}_{\sigma(2)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\begin{array}{c} 1 - \frac{1}{1+\left\{ \xi_1 \left(\frac{\hat{k}_{\sigma(1)}}{1-\hat{k}_{\sigma(1)}} \right)^{\varrho} + \xi_2 \left(\frac{\hat{k}_{\sigma(2)}}{1-\hat{k}_{\sigma(2)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \\ 1 - \frac{1}{1+\left\{ \xi_1 \left(\frac{\hat{\phi}_{\sigma(1)}}{1-\hat{\phi}_{\sigma(1)}} \right)^{\varrho} + \xi_2 \left(\frac{\hat{\phi}_{\sigma(2)}}{1-\hat{\phi}_{\sigma(2)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \end{array} \right) \end{array} \right)$$

This means that

$$CIFDOWG(a_1, a_2)$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\gamma)}}{\hat{r}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\gamma)}}{\hat{\theta}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \\ \left(1-\frac{1}{1+\left\{\sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\gamma)}}{1-\hat{k}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{\gamma=1}^{2} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\gamma)}}{1-\hat{\phi}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

Hence, the result is valid for $\gamma = 2$.

Assuming that the statement is true for $\gamma = s$, we have:

$$CIFDOWG(a_1, a_2, \ldots, a_s) = (a_{\sigma(1)})^{\xi_1} \bigotimes (a_{\sigma(2)})^{\xi_2} \bigotimes \ldots \bigotimes (a_{\sigma(s)})^{\xi_s} = \bigotimes_{\gamma=1}^{s} \left(a_{\sigma(\gamma)}\right)^{\xi_\gamma}$$

$$= \left(\begin{array}{c} \left(1-\frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\gamma)}}{1-\hat{k}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\gamma)}}{1-\hat{\phi}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \\ \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\gamma)}}{\hat{r}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\gamma)}}{\hat{\theta}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \end{array} \right)$$

Moreover, for $\gamma = s+1$, we have

$$CIFDOWG(a_1, a_2, \ldots, a_s) = (a_{\sigma(1)})^{\xi_1} \otimes (a_{\sigma(2)})^{\xi_2} \otimes \ldots \otimes (a_{\sigma(s+1)})^{\xi_{s+1}}$$

$$= \otimes_{\gamma=1}^{s} \left(a_{\sigma(\gamma)}\right)^{\xi_\gamma} \otimes (a_{\sigma(s+1)})^{\xi_{s+1}}$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{k}_\gamma}{\hat{k}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{1-\hat{\phi}_\gamma}{\hat{\phi}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \\ \left(1-\frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{r}_\gamma}{1-\hat{r}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{\gamma=1}^{s} \xi_\gamma \left(\frac{\hat{\theta}_\gamma}{1-\hat{\theta}_\gamma}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \left(\frac{1}{1+\left\{\xi_{s+1} \left(\frac{1-\hat{k}_{s+1}}{\hat{k}_{s+1}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\xi_{s+1} \left(\frac{1-\hat{\phi}_{s+1}}{\hat{\phi}_{s+1}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \\ \left(1-\frac{1}{1+\left\{\xi_{s+1} \left(\frac{\hat{r}_{s+1}}{1-\hat{r}_{s+1}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\xi_{s+1} \left(\frac{\hat{\theta}_{s+1}}{1-\hat{\theta}_{s+1}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \end{array} \right)$$

This means that

$$CIFDOWG(a_1, a_2, \cdots, a_{s+1})$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\gamma)}}{\hat{r}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\gamma)}}{\hat{\theta}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\gamma)}}{1-\hat{k}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{s+1} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\gamma)}}{1-\hat{\phi}_{\sigma(\gamma)}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

This leads us to the conclusion that for every positive integral value of γ, the assertion is correct.
□

The following examns the aforementioned fact.

Example 6. *Consider the CIFNs* $a_1 = ((0.25, 0.3), (0.6, 0.4)), a_2 = ((0.3, 0.67), (0.5, 0.3))$, $a_3 = ((0.4, 0.5), (0.5, 0.4))$, $a_4 = ((0.6, 0.4), (0.4, 0.3))$ *and the associated weight vector* $\xi = (0.15, 0.25, 0.35, 0.25)^T$. *To aggregate these values by CIFDOWG operator, we first permute these numbers using Definition 11 to acquire the subsequent data.*

$$\text{Ꝁ}(a_1) = 0.367\ \text{Ꝁ}(a_2) = 0.521,\ \text{Ꝁ}(a_3) = 0.500,\ \text{Ꝁ}(a_4) = 0.579$$

By applying Definition 16, the permuted values of the CIFNs are calculated as follows: $a_{\sigma(1)} = ((0.6, 0.4), (0.4, 0.3)), a_{\sigma(2)} = ((0.3, 0.67), (0.5, 0.3)), a_{\sigma(3)} = ((0.4, 0.5), (0.5, 0.4))$ *and* $a_{\sigma(4)} = ((0.25, 0.3), (0.6, 0.4))$. *Then, for* $\varrho = 4$, *we have*

$$\left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\gamma)}}{\hat{r}_{\sigma(\gamma)}}\right)^4\right\}^{\frac{1}{4}} = 2.330, \left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\gamma)}}{\hat{\theta}_{\sigma(\gamma)}}\right)^4\right\}^{\frac{1}{4}} = 1.709,$$

$$\left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\gamma)}}{1-\hat{k}_{\sigma(\gamma)}}\right)^4\right\}^{\frac{1}{4}} = 1.173, \left\{\sum_{\gamma=1}^{4} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\gamma)}}{1-\hat{\phi}_{\sigma(\gamma)}}\right)^4\right\}^{\frac{1}{4}} = 0.603.$$

This implies that

$$\begin{aligned} CIFDOWG(a_1, a_2, a_3, a_4) &= \otimes_{\gamma=1}^{4} \left(a_{\sigma(\gamma)}\right)^{\xi_\gamma} \\ &= ((0.300, 0.369), (0.540, 0.376)) \end{aligned}$$

This leads us to the conclusion that the outcome of the preceding discussion is also a CIFN.

Theorem 14. *(Idempotency property) If* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right) (\gamma = 1, 2, 3, \ldots, m)$ *are CIFNs such that* $a_\gamma = a_0 \forall \gamma$, *where* $a_0 = \left((\hat{r}_0, \hat{\theta}_0), \left(\hat{k}_0, \hat{\phi}_0\right)\right)$ *is a CIFN. Then* $CIFDOWG(a_1, a_2, \ldots, a_m) = a_0$.

Proof. Since $a_\gamma = a_0 \forall\gamma$. Then, by Definition 4, $\hat{r}_\gamma = \hat{r}_0, \hat{\theta}_\gamma = \hat{\theta}_0, \hat{k}_\gamma = \hat{k}_0$ and $\hat{\phi}_\gamma = \hat{\phi}_0\ \forall\gamma$. By substituting the above relations in Theorem 13, we obtain

$$CIFDOWG(a_1, a_2, \ldots, a_m)$$

$$= \left(\begin{array}{c} \left(\frac{1}{1+\left(\frac{1-\hat{r}_0}{\hat{r}_0}\right)\left\{\sum_{\gamma=1}^m \xi_\gamma\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left(\frac{1-\hat{\theta}_0}{\hat{\theta}_0}\right)\left\{\sum_{\gamma=1}^m \xi_\gamma\right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left(\frac{\hat{k}_0}{1-\hat{k}_0}\right)\left\{\sum_{\gamma=1}^m \xi_\gamma\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left(\frac{\hat{\phi}_0}{1-\hat{\phi}_0}\right)\left\{\sum_{\gamma=1}^m \xi_\gamma\right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

$$= \left(\left(\frac{1}{1+\left(\frac{1-\hat{r}_0}{\hat{r}_0}\right)}, \frac{1}{1+\left(\frac{1-\hat{\theta}_0}{\hat{\theta}_0}\right)} \right), \left(1 - \frac{1}{1+\left(\frac{\hat{k}_0}{1-\hat{k}_0}\right)}, 1 - \frac{1}{1+\left(\frac{\hat{\phi}_0}{1-\hat{\phi}_0}\right)} \right) \right)$$

$$= \left((\hat{r}_0, \hat{\theta}_0), (\hat{k}_0, \hat{\phi}_0) \right)$$

This shows that

$$CIFDOWG(a_1, a_2, \ldots, a_m) = a_0.$$

□

Theorem 15. *(Boundedness property) Let* $a_\gamma = \left((\hat{r}_\gamma, \hat{\theta}_\gamma), (\hat{k}_\gamma, \hat{\phi}_\gamma) \right) (\gamma = 1, 2, 3 \ldots m)$ *be CIFNs. Let* $a^- = \min_\gamma\{a_\gamma\} = \left((\hat{r}^-, \hat{\theta}^-), (\hat{k}^-, \hat{\phi}^-) \right)$ *and* $a^+ = \max_\gamma\{a_\gamma\} = \left((\hat{r}^+, \hat{\theta}^+), (\hat{k}^+, \hat{\phi}^+) \right)$ *where* $\hat{r}^- = \min_\gamma\{\hat{r}_\gamma\}$, $\hat{\theta}^- = \min_\gamma\{\hat{\theta}_\gamma\}$, $\hat{k}^- = \max_\gamma\left\{\hat{k}_\gamma\right\}$, $\hat{\phi}^- = \max_\gamma\{\hat{\phi}_\gamma\}$, $\hat{r}^+ = \max_\gamma\{\hat{r}_\gamma\}$, $\hat{\theta}^+ = \max_\gamma\{\hat{\theta}_\gamma\}, \hat{k}^+ = \min_\gamma\left\{\hat{k}_\gamma\right\}, \hat{\phi}^+ = \min_\gamma\{\hat{\phi}_\gamma\}$. *Then,* $a^- \leq CIFDOWG(a_1, a_2, \ldots, a_m) \leq a^+$.

Proof. In view of the given conditions, we have

$$\frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{1-\hat{r}^-}{\hat{r}^-}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \leq \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\gamma)}}{\hat{r}_{\sigma(\gamma)}}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \leq \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{1-\hat{r}^+}{\hat{r}^+}\right)^\varrho\right\}^{\frac{1}{\varrho}}},$$

$$\frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{1-\hat{\theta}^-}{\hat{\theta}^-}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \leq, \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\gamma)}}{\hat{\theta}_{\sigma(\gamma)}}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \leq \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{1-\hat{\theta}^+}{\hat{\theta}^+}\right)^\varrho\right\}^{\frac{1}{\varrho}}},$$

$$1 - \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{\hat{k}^-}{1-\hat{k}^-}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \geq 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{\hat{k}_{\sigma(\gamma)}}{1-\hat{k}_{\sigma(\gamma)}}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \geq 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{\hat{k}^+}{1-\hat{k}^+}\right)^\varrho\right\}^{\frac{1}{\varrho}}},$$

$$1 - \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{\hat{\phi}^-}{1-\hat{\phi}^-}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \geq 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\gamma)}}{1-\hat{\phi}_{\sigma(\gamma)}}\right)^\varrho\right\}^{\frac{1}{\varrho}}} \geq 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^m \xi_\gamma \left(\frac{\hat{\phi}^+}{1-\hat{\phi}^+}\right)^\varrho\right\}^{\frac{1}{\varrho}}}.$$

In light of Definition 4 and the above relations, we get

$$a^{-} \leq CIFDOWG(a_1, a_2, \ldots, a_m) \leq a^{+}.$$

□

Theorem 16. *(Monotonicity property) Let* $a_\gamma = \left(\left(\hat{r}_\gamma, \hat{\theta}_\gamma\right), \left(\hat{k}_\gamma, \hat{\phi}_\gamma\right)\right)$ *and* $a'_\gamma = \left(\left(\hat{r}'_\gamma, \hat{\theta}'_\gamma\right), \left(\hat{k}'_\gamma, \hat{\phi}'_\gamma\right)\right)$ *for* $\gamma = 1, 2, 3, \ldots, m$ *be two collections of CIFNs. If* $\hat{r}_{\sigma(\gamma)} \leq \hat{r}'_{\sigma(\gamma)}$, $\hat{k}_{\sigma(\gamma)} \geq \hat{k}'_{\sigma(\gamma)}$, $\hat{\theta}_{\sigma(\gamma)} \leq \hat{\theta}'_{\sigma(\gamma)}$ *and* $\hat{\phi}_{\sigma(\gamma)} \geq \hat{\phi}'_{\sigma(\gamma)}$ $\forall\gamma$. *Then* $CIFDOWG(a_1, a_2, \ldots, a_m) \leq CIFDOWG(a'_1, a'_2, \ldots, a'_m)$.

Proof. The proof is a straightforward implementation of Definition 4. □

5. Application of Proposed CIF Dombi Aggregation Operators in MADM Problem

In this section, we present a technique to solve a MADM problem with CIF information by applying CIF Dombi aggregation operators. Let $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_p\}$ be the set of distinct alternatives; $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_q\}$ be the set of attributes; and $\xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_q)$ be the associated weight vector of the attributes, where $\xi_\gamma > 0$ for all $\gamma = 1, 2, 3, \ldots q$ such that $\sum_{\gamma=1}^{q} \xi_\gamma = 1$. Suppose $\mathcal{D} = (\varphi_{\varsigma\gamma})_{p\times q} = \left(\left(\hat{r}_{\varsigma\gamma}, \hat{\theta}_{\varsigma\gamma}\right), \left(\hat{k}_{\varsigma\gamma}, \hat{\phi}_{\varsigma\gamma}\right)\right)_{p\times q}$ is the CIF decision matrix, where $\hat{r}_{\varsigma\gamma}, \hat{\theta}_{\varsigma\gamma}$ and $\hat{k}_{\varsigma\gamma}, \hat{\phi}_{\varsigma\gamma}$ are the membership and non-membership grades assigned by an expert under which an alternative $\mathcal{A}_\varsigma$ satisfies the criteria $\mathcal{C}_\gamma$. Moreover, $\hat{r}_{\varsigma\gamma}, \hat{k}_{\varsigma\gamma}, \hat{\theta}_{\varsigma\gamma}, \hat{\phi}_{\varsigma\gamma} \in [0,1]$ such that $0 \leq \hat{r}_{\varsigma\gamma} + \hat{k}_{\varsigma\gamma}, \hat{\theta}_{\varsigma\gamma} + \hat{\phi}_{\varsigma\gamma} \leq 1$. The algorithm to solve the MADM problem is designed in the following way:

Step 1.

The decision-maker's preferences, summed up in the CIF decision matrix, are represented as

$$\mathcal{D} = \begin{pmatrix} \left(\left(\hat{r}_{11}, \hat{\theta}_{11}\right), \left(\hat{k}_{11}, \hat{\phi}_{11}\right)\right) & \left(\left(\hat{r}_{12}, \hat{\theta}_{12}\right), \left(\hat{k}_{12}, \hat{\phi}_{12}\right)\right) & \cdots & \left(\left(\hat{r}_{1q}, \hat{\theta}_{1q}\right), \left(\hat{k}_{1q}, \hat{\phi}_{1q}\right)\right) \\ \left(\left(\hat{r}_{21}, \hat{\theta}_{21}\right), \left(\hat{k}_{21}, \hat{\phi}_{21}\right)\right) & \left(\left(\hat{r}_{22}, \hat{\theta}_{22}\right), \left(\hat{k}_{22}, \hat{\phi}_{22}\right)\right) & \cdots & \left(\left(\hat{r}_{2q}, \hat{\theta}_{2q}\right), \left(\hat{k}_{2q}, \hat{\phi}_{2q}\right)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\left(\hat{r}_{p1}, \hat{\theta}_{p1}\right), \left(\hat{k}_{p1}, \hat{\phi}_{p1}\right)\right) & \left(\left(\hat{r}_{p2}, \hat{\theta}_{p2}\right), \left(\hat{k}_{p2}, \hat{\phi}_{p2}\right)\right) & \cdots & \left(\left(\hat{r}_{pq}, \hat{\theta}_{pq}\right), \left(\hat{k}_{pq}, \hat{\phi}_{pq}\right)\right) \end{pmatrix}$$

Step 2.

Calculate the aggregated values $\varphi_\varsigma$for all $\varsigma = 1, 2, 3, \ldots, p$ of the alternatives $\mathcal{A}_\varsigma$ by applying $CIFDWA$ operator as follows:

$$\varphi_\varsigma = \left(\left(\hat{r}_\varsigma, \hat{\theta}_\varsigma\right), \left(\hat{k}_\varsigma, \hat{\phi}_\varsigma\right)\right) = CIFDWA(\varphi_{\varsigma 1}, \varphi_{\varsigma 2}, \ldots, \varphi_{\varsigma q}) = \bigoplus(\xi_\gamma \varphi_{\varsigma\gamma})$$

By applying Theorem 1 to the aforementioned relationship, we get that

$$\varphi_\varsigma = \left(\begin{array}{c} \left(1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{r}_{\varsigma\gamma}}{1-\hat{r}_{\varsigma\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{\theta}_{\varsigma\gamma}}{1-\hat{\theta}_{\varsigma\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right), \\ \left(\frac{1}{1+\left\{\sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{k}_{\varsigma\gamma}}{\hat{k}_{\varsigma\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{\sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{\phi}_{\varsigma\gamma}}{\hat{\phi}_{\varsigma\gamma}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right) \end{array} \right)$$

Similarly, the aggregated values φ_ς in the framework of the $CIFDOWA$ operators are calculated in the following way:

$$\varphi_\varsigma = \left((\hat{r}_\varsigma, \hat{\theta}_\varsigma), \left(\hat{k}_\varsigma, \hat{\phi}_\varsigma\right) \right) = CIFDOWA(\varphi_{\varsigma 1}, \varphi_{\varsigma 2}, \ldots, \varphi_{\varsigma q}) = \bigoplus \left(\xi_\gamma \varphi_{\sigma(\varsigma\gamma)} \right)$$

By applying Theorem 5 to the aforementioned relationship, we get that

$$\varphi_\varsigma = \left(\begin{array}{c} \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{r}_{\sigma(\varsigma\gamma)}}{1-\hat{r}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{\theta}_{\sigma(\varsigma\gamma)}}{1-\hat{\theta}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{k}_{\sigma(\varsigma\gamma)}}{\hat{k}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{\phi}_{\sigma(\varsigma\gamma)}}{\hat{\phi}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

Likewise, the aggregated values φ_ς in the framework of the $CIFDWG$ operators are calculated in the following way:

$$\varphi_\varsigma = \left((\hat{r}_\varsigma, \hat{\theta}_\varsigma), \left(\hat{k}_\varsigma, \hat{\phi}_\varsigma\right) \right) = CIFDWG(\varphi_{\varsigma 1}, \varphi_{\varsigma 2}, \ldots, \varphi_{\varsigma q}) = \otimes (\varphi_{\varsigma\gamma})^{\xi_\gamma}$$

By applying Theorem 9 to the aforementioned relationship, we get that

$$\varphi_\varsigma = \left(\begin{array}{c} \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\varsigma\gamma)}}{\hat{r}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\varsigma\gamma)}}{\hat{\theta}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\varsigma\gamma)}}{1-\hat{k}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\varsigma\gamma)}}{1-\hat{\phi}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

Moreover, the aggregated values φ_ς in the framework of the $CIFDOWG$ operators are calculated in the following way:

$$\varphi_\varsigma = \left((\hat{r}_\varsigma, \hat{\theta}_\varsigma), \left(\hat{k}_\varsigma, \hat{\phi}_\varsigma\right) \right) = CIFDOWG(\varphi_{\varsigma 1}, \varphi_{\varsigma 2}, \ldots, \varphi_{\varsigma q}) = \bigoplus \left(\varphi_{\sigma(\varsigma\gamma)} \right)^{\xi_\gamma}$$

By applying Theorem 13 to the aforementioned relationship, we get that

$$\varphi_\varsigma = \left(\begin{array}{c} \left(\frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{r}_{\sigma(\varsigma\gamma)}}{\hat{r}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{1-\hat{\theta}_{\sigma(\varsigma\gamma)}}{\hat{\theta}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right), \\ \left(1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{k}_{\sigma(\varsigma\gamma)}}{1-\hat{k}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{ \sum_{\gamma=1}^{q} \xi_\gamma \left(\frac{\hat{\phi}_{\sigma(\varsigma\gamma)}}{1-\hat{\phi}_{\sigma(\varsigma\gamma)}} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right) \end{array} \right)$$

Step 3.
Compute the score values for each φ_ς for all $\varsigma = 1, 2, 3, \ldots, p$ by using Definition 11.
Step 4.
Select the best alternative among all.

Implementation

Water is the most essential thing for the living body to survive. However, now that industries are increasing, the availability of pure water is a major problem. Contaminated water causes different diseases, some of which lead to death. So, it is the most important task for the government to supply pure water to domestic areas. Here, we discuss some methods to purify water at a commercial scale and some factors affecting these methods. We present a step-by-step procedure to select the best alternative within the framework of CIF Dombi aggregation operators.

Let $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$ be the set of alternatives to purifying water at a commercial scale;

1. $\mathcal{A}_1$: Boiling;
2. $\mathcal{A}_2$: Reverse osmosis;
3. $\mathcal{A}_3$: Distillation;
4. $\mathcal{A}_4$: Filtration;
5. $\mathcal{A}_5$: Deionization.

Let $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ be the four factors affecting these methods:

1. $\mathcal{C}_1$: Environmental Factor;
2. $\mathcal{C}_2$: Economic Factor;
3. $\mathcal{C}_3$: Technical Factor;
4. $\mathcal{C}_4$: Socio-political Factor.

To make a CIFN, these factors are further categorized into two characteristics given below:

- Environmental factors consist of land requirements and waste disposal management.
- Economic factors consist of financial resources and investment costs.
- Technical factors consist of risk factors and feasibility.
- Socio-political factors further consist of social and political acceptance.

Table 2 summarizes the decision-maker's expert opinion on each alternative for each attribute in the form of a CIFN.

Table 2. CIF decision matrix.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\mathcal{A}_1$	((0.55,0.45),(0.42,0.40))	((0.33,0.45),(0.42,0.50))	((0.45,0.60),(0.45,0.15))	((0.45,0.15),(0.20,0.50))
$\mathcal{A}_2$	((0.46,0.50),(0.48,0.45))	((0.85,0.56),(0.15,0.15))	((0.75,0.65),(0.20,0.15))	((0.50,0.60),(0.45,0.20))
$\mathcal{A}_3$	((0.59,0.45),(0.38,0.40))	((0.20,0.45),(0.75,0.55))	((0.50,0.60),(0.50,0.25))	((0.40,0.45),(0.50,0.40))
$\mathcal{A}_4$	((0.72,0.55),(0.22,0.40))	((0.33,0.30),(0.42,0.40))	((0.40,0.40),(0.50,0.60))	((0.50,0.65),(0.40,0.10))
$\mathcal{A}_5$	((0.48,0.40),(0.47,0.60))	((0.46,0.45),(0.45,0.50))	((0.49,0.60),(0.45,0.15))	((0.75,0.45),(0.10,0.40))

The weight vector assigned by the decision-maker is $\xi = (0.2, 0.3, 0.4, 0.1)^T$, where $\sum_{\gamma=1}^{4} \xi_\gamma = 1$.

Now we solve the decision matrix to choose the best alternative by applying the $CIFDWA$, $CIFDOWA$, $CIFDWG$, and $CIFDOWG$ operators in the following discussion.

The above MADM problem in the framework of the $CIFDWA$ operator is solved as follows:

Step 1. The application of the $CIFDWA$ operator on the values listed in Table 2 for a specific value of $\varrho = 4$ yields Table 3

Table 3. Aggregated values of alternatives under the CIFDWA operator.

Alternatives	φ_ς
$\mathcal{A}_1$	((0.477,0.550),(0.302,0.181))
$\mathcal{A}_2$	((0.811,0.613),(0.182,0.161))
$\mathcal{A}_3$	((0.515,0.552),(0.459,0.293))
$\mathcal{A}_4$	((0.634,0.534),(0.293,0.165))
$\mathcal{A}_5$	((0.632,0.550),(0.165,0.182))

Step 2. By applying Definition 11, determine the score values for all CIFNs acquired in step 1.
$\mathbb{K}(\mathcal{A}_1) = 0.6832$, $\mathbb{K}(\mathcal{A}_2) = 0.7949$, $\mathbb{K}(\mathcal{A}_3) = 0.5938$, $\mathbb{K}(\mathcal{A}_4) = 0.7203$ and $\mathbb{K}(\mathcal{A}_5) = 0.7709$.

Step 3. Since $\mathbb{K}(\mathcal{A}_2) > \mathbb{K}(\mathcal{A}_5) > \mathbb{K}(\mathcal{A}_4) > \mathbb{K}(\mathcal{A}_1) > \mathbb{K}(\mathcal{A}_3)$, therefore, the ranking order of alternatives is $\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$.

Step 4. $\mathcal{A}_2$ is the best alternative.

Similarly, the above MADM problem in the framework of the $CIFDWG$ operator is solved as follows:

Step 1. The application of the $CIFDWG$ operator on the values listed in Table 2 for a specific value of $\varrho = 4$ gives us Table 4

Table 4. Aggregated values of alternatives under the CIFDWG operator.

Alternatives	φ_ς
$\mathcal{A}_1$	((0.387,0.238), (0.429,0.449))
$\mathcal{A}_2$	((0.545,0.562), (0.398,0.354))
$\mathcal{A}_3$	((0.252,0.479), (0.691,0.481))
$\mathcal{A}_4$	((0.379,0.354), (0.457,0.547))
$\mathcal{A}_5$	((0.483,0.457), (0.448,0.518))

Step 2. By applying Definition 11, determine the score values for all CIFNs acquired in step 1.
$\mathbb{K}(\mathcal{A}_1) = 0.4086$, $\mathbb{K}(\mathcal{A}_2) = 0.5959$, $\mathbb{K}(\mathcal{A}_3) = 0.3822$, $\mathbb{K}(\mathcal{A}_4) = 0.4241$ and $\mathbb{K}(\mathcal{A}_5) = 0.4943$.

Step 3. Since $\mathbb{K}(\mathcal{A}_2) > \mathbb{K}(\mathcal{A}_5) > \mathbb{K}(\mathcal{A}_4) > \mathbb{K}(\mathcal{A}_1) > \mathbb{K}(\mathcal{A}_3)$, therefore, the ranking order of alternatives is $\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$.

Step 4. $\mathcal{A}_2$ is the best alternative.

Moreover, the above MADM problem in the framework of the $CIFDOWA$ operator is solved as follows:

Step 1. The permuted CIF decision matrix is given is Table 5.

Table 5. Permuted CIF decision matrix.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\mathcal{A}_1$	((0.45,0.60), (0.45,0.15))	((0.55,0.45), (0.42,0.40))	((0.45,0.15), (0.20,0.50))	((0.33,0.45), (0.42,0.50))
$\mathcal{A}_2$	((0.85,0.56), (0.15,0.15))	((0.75,0.65), (0.20,0.15))	((0.50,0.60), (0.45,0.20))	((0.46,0.50), (0.48,0.45))
$\mathcal{A}_3$	((0.50,0.60), (0.50,0.25))	((0.59,0.45), (0.38,0.40))	((0.40,0.45), (0.5,0.40))	((0.20,0.45), (0.75,0.55))
$\mathcal{A}_4$	((0.50,0.65), (0.40,0.10))	((0.72,0.55), (0.22,0.40))	((0.33,0.30), (0.42,0.40))	((0.40,0.40), (0.50,0.60))
$\mathcal{A}_5$	((0.75,0.45), (0.10,0.40))	((0.49,0.60), (0.45,0.15))	((0.46,0.45), (0.45,0.50))	((0.48,0.40), (0.47,0.60))

Step 2. The application of the $CIFDOWA$ operator on the values listed in Table 5 for a specific value of $\varrho = 4$ yields Table 6.

Table 6. Aggregated values of alternatives under the CIFDOWA operator.

Alternatives	φ_ς
$\mathcal{A}_1$	((0.496,0.511),(0.238,0.208))
$\mathcal{A}_2$	((0.796,0.612),(0.196,0.167))
$\mathcal{A}_3$	((0.528,0.520),(0.438,0.322))
$\mathcal{A}_4$	((0.657,0.570),(0.273,0.142))
$\mathcal{A}_5$	((0.669,0.537),(0.142,0.192))

Step 3. By applying Definition 11, determine the score values for all CIFNs acquired in step 2.
$\text{Ꝁ}(\mathcal{A}_1) = 0.6933$, $\text{Ꝁ}(\mathcal{A}_2) = 0.7857$, $\text{Ꝁ}(\mathcal{A}_3) = 0.5833$, $\text{Ꝁ}(\mathcal{A}_4) = 0.7502$ and $\text{Ꝁ}(\mathcal{A}_5) = 0.7783$.

Step 4. Since $\text{Ꝁ}(\mathcal{A}_2) > \text{Ꝁ}(\mathcal{A}_5) > \text{Ꝁ}(\mathcal{A}_4) > \text{Ꝁ}(\mathcal{A}_1) > \text{Ꝁ}(\mathcal{A}_3)$, therefore, the ranking order of alternatives is $\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$.

Step 5. $\mathcal{A}_2$ is the best alternative.

Furthermore, the above MADM problem in the framework of the $CIFDOWG$ operator is solved as follows:

Step 1. The permuted CIF decision matrix is given is Table 5.

Step 2. The application of the $CIFDOWG$ operator on the values listed in Table 5 for a specific value of $\varrho = 4$ yields Table 7.

Table 7. Aggregated values of alternatives under the CIFDOWG operator.

Alternatives	φ_ς
$\mathcal{A}_1$	((0.428,0.182), (0.401,0.464))
$\mathcal{A}_2$	((0.533,0.579), (0.415,0.318))
$\mathcal{A}_3$	((0.303,0.462), (0.632,0.437))
$\mathcal{A}_4$	((0.377,0.347), (0.415,0.473))
$\mathcal{A}_5$	((0.486,0.460), (0.439,0.497))

Step 3. By applying Definition 11, determine the score values for all CIFNs acquired in step 2.
$\text{Ꝁ}(\mathcal{A}_1) = 0.3944$, $\text{Ꝁ}(\mathcal{A}_2) = 0.6047$, $\text{Ꝁ}(\mathcal{A}_3) = 0.4206$, $\text{Ꝁ}(\mathcal{A}_4) = 0.4499$ and $\text{Ꝁ}(\mathcal{A}_5) = 0.5034$.

Step 4. Since $\text{Ꝁ}(\mathcal{A}_2) > \text{Ꝁ}(\mathcal{A}_5) > \text{Ꝁ}(\mathcal{A}_4) > \text{Ꝁ}(\mathcal{A}_3) > \text{Ꝁ}(\mathcal{A}_1)$, therefore, the ranking order of alternatives is $\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1$.

Step 5. $\mathcal{A}_2$ is the best alternative.

All the information obtained from the above procedure in the light of newly defined CIF Dombi aggregation operators is summarized in Table 8.

Table 8. Score values and ranking of alternatives under newly defined techniques.

Operators	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	Ranking
CIFDWA	0.6832	0.7949	0.5938	0.7203	0.7709	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
CIFDWG	0.4086	0.5959	0.3822	0.4241	0.4943	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
CIFDOWA	0.6933	0.7857	0.5833	0.7502	0.7783	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
CIFDOWG	0.3944	0.6047	0.4206	0.4499	0.5034	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1$

6. Comparative Analysis

In the following section, we solve the above MADM problem through various existing operators in IF and CIF environments, namely, IFDWA, IFDWG, IFDOWA, IFDOWG, CIFWA, CIFWG, CIFOWA, and CIFOWG operators. The computed results by applying these operators are summarized in Table 9 and are ranked in Table 10.

Table 9. Aggregated values obtained from different existing operators.

	IFDWA [27]	IFDWG [27]	CIFWA [33]	CIFWG [33]
$\mathcal{A}_1$	(0.477,0.302)	(0.387,0.429)	((0.439,0.494),(0.401,0.295))	((0.427,0.452),(0.414,0.359))
$\mathcal{A}_1$	(0.811,0.182)	(0.545,0.398)	((0.732,0.592),(0.237,0.192))	((0.678,0.585),(0.280,0.226))
$\mathcal{A}_1$	(0.515,0.459)	(0.252,0.691)	((0.437,0.516),(0.535,0.365))	((0.384,0.505),(0.576,0.398))
$\mathcal{A}_1$	(0.634,0.293)	(0.379,0.457)	((0.477,0.438),(0.394,0.410))	((0.434,0.411),(0.418,0.469))
$\mathcal{A}_1$	(0.632,0.165)	(0.483,0.448)	((0.515,0.507),(0.391,0.313))	((0.450,0.493),(0.427,0.398))
	IFDOWA [27]	IFDOWG [27]	CIFOWA [33]	CIFOWG [33]
$\mathcal{A}_1$	(0.496,0.238)	(0.428,0.401)	((0.472,0.386),(0.316,0.368))	((0.463,0.307),(0.347,0.413))
$\mathcal{A}_1$	(0.796,0.196)	(0.533,0.415)	((0.678,0.599),(0.285,0.188))	((0.623,0.595),(0.332,0.206))
$\mathcal{A}_1$	(0.528,0.438)	(0.303,0.632)	((0.469,0.484),(0.480,0.376))	((0.439,0.477),(0.502,0.390))
$\mathcal{A}_1$	(0.657,0.273)	(0.377,0.415)	((0.519,0.474),(0.349,0.316))	((0.462,0.432),(0.371,0.375))
$\mathcal{A}_1$	(0.669,0.142)	(0.486,0.439)	((0.547,0.496),(0.335,0.339))	((0.519,0.485),(0.395,0.405))

Table 10. Score values and ranking of alternatives under existing and newly proposed strategies.

Operators	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	**Ranking**
IFDWA [27]	0.5248	0.5022	0.5007	0.5134	0.5595	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3$
IFDWG [27]	0.4953	0.5044	0.4867	0.4923	0.5013	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
IFDOWA [27]	0.5467	0.5024	0.5016	0.5145	0.5614	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3$
IFDOWG [27]	0.5028	0.5032	0.4886	0.4950	0.5019	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
CIFWA [33]	0.5753	0.7503	0.5204	0.5319	0.5939	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
CIFWG [33]	0.5328	0.7123	0.4818	0.4885	0.5331	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
CIFOWA [33]	0.5549	0.7306	0.5297	0.5986	0.6067	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
CIFOWG [33]	0.4967	0.6979	0.5097	0.5449	0.5562	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
CIFDWA	0.6832	0.7949	0.5938	0.7203	0.7709	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
CIFDWG	0.4086	0.5959	0.3822	0.4241	0.4943	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
CIFDOWA	0.6933	0.7857	0.5833	0.7502	0.7783	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
CIFDOWG	0.3944	0.6047	0.4206	0.4499	0.5034	$\mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1$

From the above discussion, it is quite evident that the proposed strategies in this article are more generalized than the other existing techniques because the best preference changes due to the loss of information in the framework of existing IF operators, whereas, the CIF Dombi aggregation operators tackle this situation effectively. Moreover, the newly defined operators are more flexible as they involve a parametric value. It is also important to note that the operators proposed by Sheik and Mandal [27] are special cases of these operators by making the second dimension constant.

7. Conclusions

In this research, the concepts of CIFDWA, CIFDOWA, CIFDWG, and CIFDOWG operators in a CIF environment have been introduced. A new score function has been determined to rank and choose the best alternative. Moreover, a real-life MADM problem has been formulated in the light of newly defined aggregation operators. Finally, a comparative analysis has been established to show the validity and feasibility of the proposed techniques with existing methods. The first primary objective of future work will be the development of a comprehensive decision-analysis aid based on Dombi operators to maximize its practical relevance and usability. Although the method presented in this article has numerous benefits, it still has certain limitations, particularly in MADM scenarios where the sum of membership and non-membership exceeds one or involves neutral membership. Moreover, one of our primary research objectives is to devise novel approaches for improving the structure of fuzzy settings.

Moreover, the suggested strategies of this article will effectively be applied to counter the energy crises in developing countries and risk management in construction projects. In addition, Dombi operators can be applied to the orthotriple fuzzy rough sets [16,17], cubic intuitionistic fuzzy sets [18,19], complex Pythagorean fuzzy sets, and complex bipolar fuzzy sets [47] in future studies to solve many important MCDM problems economically. These initiatives will make it possible to solve a number of crucial MADM issues effectively and affordably.

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