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Bochner-Like Transform and Stepanov Almost Periodicity on Time Scales with Applications

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Abstract: By using Bochner transform, Stepanov almost periodic functions inherit some basic properties directly from almost periodic functions. Recently, this old work was extended to time scales. However, we show that Bochner transform is not valid on time scales. Then we present a revised version, called Bochner-like transform, for time scales, and prove that a function is Stepanov almost periodic if and only if its Bochner-like transform is almost periodic on time scales. Some basic properties including the composition theorem of Stepanov almost periodic functions are obtained by applying Bochner-like transform. Our results correct the recent results where Bochner transform is used on time scales. As an application, we give some results on dynamic equations with Stepanov almost periodic terms.

Keywords: Stepanov almost periodic; Bochner-like transform; time scale; dynamic equation

1. Introduction

The theory of time scales was established by S. Hilger [1] in 1988 in order to unify continuous and discrete problems. The theory provides a powerful tool for applications to economics, populations models, quantum physics among others and hence has been attracting the attention of lots of mathematicians. In 2011, Li and Wang [2,3] introduced almost periodicity on time scales. Since then many generalized types of almost periodicity have been introduced on time scales, such as almost automorphy [4], pseudo almost periodicity [5], weighted pseudo almost periodicity [6], weighted piecewise pseudo almost automorphy [7], etc.

To consider the almost periodicity of integrable functions, Stepanov [8] and Wiener [9] introduced Stepanov almost periodicity in 1926 by using Bochner transform. Namely, a function is Stepanov almost periodic if its Bochner transform is almost periodic. Then Stepanov almost periodic functions inherit some basic properties from almost periodic functions directly. In 2017, paper [10] tried to extend this work on time scales. Unfortunately, we show that Bochner transform is not valid on time scales (Example 1).

The main purpose of this work is to give a revised version of Bochner transform, called Bochner-like transform (Definition 15), for time scales. We prove that a function is Stepanov almost periodic if, and only if its Bochner-like transform is almost periodic on time scales (Theorem 1). Then some basic properties can be obtained by applying Bochner-like transform (Remark 8 and Theorem 2). Our results correct the results in [10] where Bochner transform was used on time scales (Remark 7).

We note that in 2015, Wang and Zhu [11] introduced Stepanov almost periodicity on time scales in Bohr sense avoiding Bochner transform. However, being lack of Bochner transform, Stepanov almost periodic functions can not inherit some important properties from almost periodic functions directly.

To pave the way to the main results, we give some notions of almost periodic functions and S^p-bounded functions, which themselves are important for further study. We first present an equivalent definition of almost periodic function on time scales (Definition 11), where condition "relatively dense in Π" is replaced by "relatively dense in $\mathbb{R}$". Then we regularize the norm of S^p-bounded functions, where the limits of the integration are fixed (Section 3.2).

As an application, we give some existence and uniqueness results on the almost periodic solutions to dynamic equations with Stepanov almost periodic terms (Theorems 3 and 4).

2. Preliminaries

The definitions and results in this section can be taken from [2,3,5,12–15]. From now on, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}$ indicate the sets of positive integers, integers, nonnegative real numbers and real numbers, respectively. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $\mathbb{E}^n$ be the Euclidian space $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidian norm $|\cdot|$, $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ be two Banach spaces, and Ωbe an open subset in $\mathbb{X}$.

Let $\mathbb{T} \subset \mathbb{R}$ be a time scale, namely, $\mathbb{T} \neq \emptyset$ is closed. The forward jump operators $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by $\sigma(s) = \inf\{t \in \mathbb{T} : t > s\}$, $\rho(s) = \sup\{t \in \mathbb{T} : t < s\}$ and $\mu(s) = \sigma(s) - s$, respectively. s is called left-scattered if $\rho(s) < s$. Otherwise, s is left-dense. Similarly, s is called right-scattered if $\sigma(s) > s$, Otherwise, s is right-dense.

2.1. Continuity and Differentiability

Definition 1. *(i)* *$f : \mathbb{T} \to \mathbb{X}$ is continuous on $\mathbb{T}$ if f is continuous at every right-dense point and at every left-dense point.*

(ii) *$f : \mathbb{T} \to \mathbb{X}$ is uniformly continuous on $\mathbb{T}$ if for $\varepsilon > 0$, there is a $\delta > 0$ such that $\|f(x_1) - f(x_2)\| < \varepsilon$ for $x_1, x_2 \in \mathbb{T}$ with $|x_1 - x_2| < \delta$.*

Denote by $C(\mathbb{T};\mathbb{X})$, $BC(\mathbb{T};\mathbb{X})$ and $UBC(\mathbb{T};\mathbb{X})$ the sets of all continuous functions, bounded continuous functions and bounded uniformly continuous functions $g : \mathbb{T} \to \mathbb{X}$, respectively. $BC(\mathbb{T};\mathbb{X})$ and $UBC(\mathbb{T};\mathbb{X})$ are Banach spaces with the sup norm $\|\cdot\|_\infty$.

If there is left-scattered maximum β in $\mathbb{T}$, then $\mathbb{T}^k = \mathbb{T} \setminus \{\beta\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If there is a right-scattered minimum β in $\mathbb{T}$, then $\mathbb{T}_k = \mathbb{T} \setminus \{\beta\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 2. *For $g : \mathbb{T} \to \mathbb{X}$ and $s \in \mathbb{T}^k$, $g^\Delta(t) \in \mathbb{X}$ is the delta derivative of g at s if for $\varepsilon > 0$, there is a neighborhood U of s such that for $t \in U$,*

$$\|g(\sigma(s)) - g(t) - g^\Delta(s)(\sigma(s) - t)\| < \varepsilon|\sigma(s) - t|.$$

Moreover, g is delta differentiable on $\mathbb{T}$ provided that $g^\Delta(s)$ exists for $s \in \mathbb{T}$.

2.2. Measure and Integral

For $t, s \in \mathbb{T}$ with $t \leq s$, Let (t,s), $[t,s]$, $(t,s]$, $[t,s)$ be the standard intervals in $\mathbb{R}$. We use the following symbols:

$$(t,s)_\mathbb{T} = (t,s) \cap \mathbb{T},\ [t,s]_\mathbb{T} = [t,s] \cap \mathbb{T},\ (t,s]_\mathbb{T} = (t,s] \cap \mathbb{T},\ [t,s)_\mathbb{T} = [t,s) \cap \mathbb{T}.$$

Note that in this paper, we use the above symbols only if $t, s \in \mathbb{T}$.

Let $\mathcal{F}_1 = \{[t,s)_\mathbb{T} : t, s \in \mathbb{T} \text{ with } t \leq s\}$. Define a countably additive measure m_1 on $\mathcal{F}_1$ by assigning to every $[t,s)_\mathbb{T} \in \mathcal{F}_1$ its length, i.e.,

$$m_1([t,s)_\mathbb{T}) = s - t.$$

Using m_1, we can generate the outer measure m_1^* on $\mathcal{P}(\mathbb{T})$ (the power set of $\mathbb{T}$): for $E \in \mathcal{P}(\mathbb{T})$,

$$m_1^*(E) = \begin{cases} \inf_{\mathcal{B}} \left\{ \sum_{i \in I_{\mathcal{B}}} (s_i - t_i) \right\} \in \mathbb{R}^+, & \beta \notin E, \\ +\infty, & \beta \in E, \end{cases}$$

where $\beta = \sup \mathbb{T}$ and

$$\mathcal{B} = \left\{ \{[t_i, s_i)_{\mathbb{T}} \in \mathcal{F}_1\}_{i \in I_{\mathcal{B}}} : I_{\mathcal{B}} \subset \mathbb{N}, E \subset \bigcup_{i \in I_{\mathcal{B}}} [t_i, s_i)_{\mathbb{T}} \right\}.$$

A set $A \subset \mathbb{T}$ is called Δ-measurable if for $E \subset \mathbb{T}$,

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)).$$

Let

$$\mathcal{M}(m_1^*) = \{A : A \text{ is a } \Delta\text{-measurable subset in } \mathbb{T}\}.$$

Restricting m_1^* to $\mathcal{M}(m_1^*)$, we get the Lebesgue Δ-measure, which is denoted by μ_Δ.

Definition 3. *$S : \mathbb{T} \to \mathbb{X}$ is said to be simple if S takes a finite number of values, $c_1, \cdots, c_N$. Let $E_j = \{s \in \mathbb{T} : S(s) = c_j\}$ [10,14]. Then*

$$S = \sum_{j=1}^{N} c_j \chi_{E_j}, \tag{1}$$

where χ_{E_j} is the characteristic function of E_j, namely

$$\chi_{E_j}(s) := \begin{cases} 1, & \text{if } s \in E_j, \\ 0, & \text{if } s \in \mathbb{T} \setminus E_j. \end{cases}$$

Definition 4. *Assume that E is a Δ-measurable subset of $\mathbb{T}$ and $S : \mathbb{T} \to \mathbb{X}$ is a Δ-measurable simple function given as (1) [14]. Then the Lebesgue Δ-integral of S on E is defined as*

$$\int_E S(s)\Delta s = \sum_{j=1}^{N} c_j \mu_\Delta(E_j \cap E).$$

Definition 5. *$g : \mathbb{T} \to \mathbb{X}$ is a Δ-measurable function if there exists a simple function sequence $\{g_k : k \in \mathbb{N}\}$ such that $g_k(s) \to g(s)$ a.e. in $\mathbb{T}$ [10].*

Definition 6. *$g : \mathbb{T} \to \mathbb{X}$ is a Δ-integrable function if there exists a simple function sequence $\{g_k : k \in \mathbb{N}\}$ such that $g_k(s) \to g(s)$ a.e. in $\mathbb{T}$ [10] and*

$$\lim_{k \to \infty} \int_{\mathbb{T}} \|g(s) - g_k(s)\|\Delta s = 0.$$

Then the integral of g is defined as

$$\int_{\mathbb{T}} g(s)\Delta s = \lim_{k \to \infty} \int_{\mathbb{T}} g_k(s)\Delta s.$$

Definition 7. *For $p \geq 1$, $g : \mathbb{T} \to \mathbb{X}$ is called locally L^p Δ-integrable if g is Δ-measurable and for any compact Δ-measurable set $E \subset \mathbb{T}$ [10], the Δ-integral*

$$\int_E \|g(s)\|^p \Delta s < \infty.$$

The set of all locally L^p Δ-integrable functions is denoted by $L^p_{loc}(\mathbb{T};\mathbb{X})$.

We remark that all the theorems of normal Lebesgue integration theory are also true for Δ-integrals on $\mathbb{T}$.

3. Notions of Almost Periodic Function and S^p-Bounded Function

To pave the way to the main results, we give some notions of almost periodic functions and S^p-bounded functions, which themselves are important for further study.

3.1. An Equivalent Definition of Almost Periodic Function

Definition 8. *A time scale $\mathbb{T}$ is said to be invariant under translations provided that [3]*

$$\Pi := \{\alpha \in \mathbb{R} : s \pm \alpha \in \mathbb{T} \text{ for } s \in \mathbb{T}\} \neq \{0\}.$$

We have the following result on the structure of Π.

Lemma 1. *Let $\mathbb{T}$ be a invariant under translations times scale, and let $K := \inf\{|\alpha| : \alpha \in \Pi \text{ for } \alpha \neq 0\}$. Then $K = 0$ iff $\mathbb{T} = \mathbb{R}$, and $K > 0$ iff $\mathbb{T} \neq \mathbb{R}$. Moreover, $\Pi = \mathbb{R}$ if $\mathbb{T} = \mathbb{R}$, and $\Pi = K\mathbb{Z}$ if $\mathbb{T} \neq \mathbb{R}$.*

Proof. If $\mathbb{T} = \mathbb{R}$, it is clear that $\Pi = \mathbb{R}$ and then $K = 0$. Then $K > 0$ implies $\mathbb{T} \neq \mathbb{R}$. If $\mathbb{T} \neq \mathbb{R}$, there is at least one right-scattered point. From [16] (Lemma 3.1), we get that $K \in \Pi$ and $K > 0$. Then $K = 0$ implies $\mathbb{T} = \mathbb{R}$. Now we need only to prove that $\Pi = K\mathbb{Z}$ if $\mathbb{T} \neq \mathbb{R}$. For $\alpha \in \Pi$, we get $\alpha = mK + r$ with $m \in \mathbb{Z}, 0 \leq r < K$. Since $\alpha, K \in \Pi$, we have that $r = \alpha - mK \in \Pi$. This implies that $r = 0$ by the definition of K. Thus, $\alpha = mK$, and then $\Pi = K\mathbb{Z}$. □

Remark 1. *It is clear that Π is also a time scale. Hereafter we always assume that $\mathbb{T}$ is invariant under translations.*

Definition 9. *$A \subset \mathbb{T}$ is called relatively dense in $\mathbb{T}$ if there exists $l > 0$ such that $[s, s+l]_{\mathbb{T}} \cap A \neq \emptyset, s \in \mathbb{T}$. We call l the inclusion length [7].*

Definition 10. *(i) $g \in C(\mathbb{T};\mathbb{X})$ is almost periodic on $\mathbb{T}$ if for $\varepsilon > 0$,*

$$T(g,\varepsilon) = \{\alpha \in \Pi : \|g(s+\alpha) - g(s)\| < \varepsilon \text{ for } s \in \mathbb{T}\}$$

is a relatively dense subset in Π. We call $T(g,\varepsilon)$ the ε-translation set of g and α the ε-translation period of g. The set of all almost periodic functions is denoted by $AP(\mathbb{T};\mathbb{X})$.

(ii) $AP(\mathbb{T} \times \Omega;\mathbb{Y})$ is the space consisting of all functions $g : \mathbb{T} \times \Omega \to \mathbb{Y}$ satisfying that $g(\cdot, y) \in AP(\mathbb{T};\mathbb{Y})$ uniformly for all $y \in A$, here A is any compact subset in Ω. That is, for $\varepsilon > 0$, $\bigcap_{x\in A} T(g(\cdot,x),\varepsilon)$ is a relatively dense subset in Π.

Remark 2. *Definition 10 (i) corrects the definition of almost periodicity given first in [3]. The correction is replacing the condition "$T(g,\varepsilon)$ is a relatively dense subset in $\mathbb{T}$" by "$T(g,\varepsilon)$ is a relatively dense subset in Π". This correction avoids some fatal errors such as the collision when $\Pi \cap \mathbb{T} = \emptyset$ (Notice that $T(g,\varepsilon) \subset \Pi$, and $T(g,\varepsilon)$ can never be dense in $\mathbb{T}$ if $\Pi \cap \mathbb{T} = \emptyset$). For more details of this correction we refer the readers to [7]. Definition 10 (ii) can be found in [10].*

We note that, to satisfy condition "$T(g,\varepsilon)$ is a relatively dense subset in Π", one needs to find the inclusion length l such that $t+l \in \Pi$ for each $t \in \Pi$. But it is convenient to find the inclusion length l in $\mathbb{R}$ and to verify the relative density of $T(g,\varepsilon)$ in $\mathbb{R}$. This is guaranteed by the following result.

Lemma 2. *For $\varepsilon > 0$, $T(g,\varepsilon)$ is relatively dense in $\mathbb{R}$ if and only if $T(g,\varepsilon)$ is relatively dense in Π.*

Proof. If $\Pi = \mathbb{R}$, the conclusion is obvious. So by Lemma 1, we only need to consider the case: $\Pi = K\mathbb{Z}$ with $K > 0$. If $T(g,\varepsilon)$ is relatively dense in $\mathbb{R}$, there exists $l \in \mathbb{R}, l > 0$ such that for each $t \in \mathbb{R}$ we have $[t,t+l] \cap T(g,\varepsilon) \neq \emptyset$. Let $l_0 \in \Pi$ such that $l_0 \geq l$. Noticing that $T(g,\varepsilon) \subset \Pi$, then for all $\alpha \in \Pi$,

$$[\alpha,\alpha+l_0]_\Pi \cap T(g,\varepsilon) \supset [\alpha,\alpha+l] \cap \Pi \cap T(g,\varepsilon) = [\alpha,\alpha+l] \cap T(g,\varepsilon) \neq \emptyset.$$

This means that $T(g,\varepsilon)$ is a relatively dense subset in Π.

Assume that $T(g,\varepsilon)$ is relatively dense in Π, that is, there exists $l > 0$ such that $l+\alpha \in \Pi$ and $[\alpha,\alpha+l]_\Pi \cap T(g,\varepsilon) \neq \emptyset$ for $\alpha \in \Pi$. Let $l_0 = l+K$. For each $a \in \mathbb{R}$, there is $n \in \mathbb{Z}$ such that $nK \leq a < (n+1)K$. Then

$$T(g,\varepsilon) \cap [a,a+l_0] \supset T(g,\varepsilon) \cap [(n+1)K,(n+1)K+l]_\Pi \neq \emptyset.$$

Hence, $T(g,\varepsilon)$ is a relatively dense subset in $\mathbb{R}$. □

Remark 3. *We note that the expression "$[t,t+l]_\mathbb{T}$" in Definition 9 implies that $t+l \in \mathbb{T}$ for each $t \in \mathbb{T}$. Moreover, from the proof of Lemma 2, the inclusion length l can be chosen in Π.*

Lemma 2 leads to the following definition of almost periodic function on time scales, which is convenient to be verified and is equivalent to Definition 10 (i).

Definition 11. *$g \in C(\mathbb{T};\mathbb{X})$ is an almost periodic function on $\mathbb{T}$ if for every $\varepsilon > 0$, the ε-translation set of g*

$$T(g,\varepsilon) = \{\alpha \in \Pi : \|g(s+\alpha) - g(s)\| < \varepsilon \text{ for all } s \in \mathbb{T}\}$$

is a relatively dense subset in $\mathbb{R}$.

The following proposition for almost periodicity on time scales is an extension of the corresponding results in [3,17] from Euclidian space $\mathbb{E}^n$ to $\mathbb{X}$ or $\mathbb{Y}$, and can be proved similarly. So we omit the details.

Proposition 1. *(i)* $AP(\mathbb{T};\mathbb{X}) \subset UBC(\mathbb{T};\mathbb{X})$.
(ii) *$AP(\mathbb{T};\mathbb{X})$ is a Banach space with supremum norm $\|\cdot\|_\infty$.*
(iii) *$g \in AP(\mathbb{T}\times\Omega,\mathbb{Y})$ if and only if for a sequence $\{t'_k\} \subset \Pi$, there is a subsequence $\{t_k\} \subset \{t'_k\}$ such that $\{g(s+t_k,x)\}$ converges uniformly on $\mathbb{T}\times S$ with S a compact subset in Ω.*
(iv) *Let $g \in C(\mathbb{T};\mathbb{X})$. Then $g \in AP(\mathbb{T};\mathbb{X})$ if and only if there is $g_1 \in AP(\mathbb{R};\mathbb{X})$ such that $g(s) = g_1(s)$ for $s \in \mathbb{T}$.*

By Proposition 1 (iv) and the well known fact that $g_1 \in AP(\mathbb{R};\mathbb{X})$ implies that $g_1(\mathbb{R})$ is a relatively compact set, we get the following result.

Proposition 2. *$g \in AP(\mathbb{T};\mathbb{X})$ implies that $g(\mathbb{T})$ is a relatively compact set.*

3.2. Regularization of the Norm of S^p-Bounded Functions

We always assume that $p \geq 1$ afterward without any further mentions. Let

$$\mathcal{K} := \begin{cases} K, & \text{if } \mathbb{T} \neq \mathbb{R}, \\ 1, & \text{if } \mathbb{T} = \mathbb{R}, \end{cases}$$

where K is as Lemma 1. Define $\|\cdot\|_{S^p} : L^p_{loc}(\mathbb{T};\mathbb{X}) \to \mathbb{R}^+$ as

$$\|g\|_{S^p} := \sup_{s\in\mathbb{T}} \left(\frac{1}{\mathcal{K}} \int_s^{s+\mathcal{K}} \|g(r)\|^p \Delta r \right)^{\frac{1}{p}} \quad \text{for } g \in L^p_{loc}(\mathbb{T};\mathbb{X}). \tag{2}$$

$g \in L^p_{loc}(\mathbb{T};\mathbb{X})$ is called S^p-bounded if $\|g\|_{S^p} < \infty$. The space of all S^p-bounded functions is denoted by $BS^p(\mathbb{T};\mathbb{X})$. It is easy to see that $\|\cdot\|_{S^p}$ is a norm, called Stepanov norm, of $BS^p(\mathbb{T};\mathbb{X})$.

Remark 4. *Given $\alpha \in \Pi$ with $\alpha > 0$, we can define another norm on $BS^p(\mathbb{T};\mathbb{X})$ as follows (see [10,11]):*

$$\|g\|_{S^p_\alpha} := \sup_{s\in\mathbb{T}} \left(\frac{1}{\alpha} \int_s^{s+\alpha} \|g(r)\|^p \Delta r \right)^{\frac{1}{p}} \quad \textit{for } g \in BS^p(\mathbb{T};\mathbb{X}). \tag{3}$$

It seems that the norms given by (3) may be different for different $\alpha \in \Pi$. Fortunately, the following result ensures that all the norms given by (3) are equivalent to the one given by (2).

Proposition 3. *For $\alpha \in \Pi$ with $\alpha > 0$, there exist $k_1, k_2 \in \mathbb{R}^+$ such that*

$$k_1\|g\|_{S^p} \leq \|g\|_{S^p_\alpha} \leq k_2\|g\|_{S^p} \quad \textit{for } g \in BS^p(\mathbb{T};\mathbb{X}).$$

Proof. If $\mathbb{T} = \mathbb{R}$, the conclusion is well-known (see [18]). If $\mathbb{T} \neq \mathbb{R}$, by Lemma 1, we have $\Pi = \mathcal{K}\mathbb{Z}$. Let $\alpha \in \Pi$ with $\alpha > 0$. Then $\alpha = m\mathcal{K}$ for some $m \in \mathbb{N}$.

$$\begin{aligned} \|g\|^p_{S^p_\alpha} &= \sup_{s\in\mathbb{T}} \frac{1}{m\mathcal{K}} \int_s^{s+m\mathcal{K}} \|g(r)\|^p \Delta r \\ &\geq \sup_{s\in\mathbb{T}} \frac{1}{m\mathcal{K}} \int_s^{s+\mathcal{K}} \|g(r)\|^p \Delta r = \frac{1}{m}\|g\|^p_{S^p}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|g\|^p_{S^p_\alpha} &= \sup_{s\in\mathbb{T}} \frac{1}{m\mathcal{K}} \int_s^{s+m\mathcal{K}} \|g(r)\|^p \Delta r \\ &\leq \frac{1}{m} \sum_{j=0}^{m-1} \sup_{s\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{s+j\mathcal{K}}^{s+(j+1)\mathcal{K}} \|g(r)\|^p \Delta r \\ &= \frac{1}{m} \cdot m\|g\|^p_{S^p} = \|g\|^p_{S^p}. \end{aligned}$$

Thus the conclusion holds with $k_1 = m^{-\frac{1}{p}}$ and $k_2 = 1$. □

The completeness of $BS^p(\mathbb{T};\mathbb{X})$ is given as the following proposition, which was also mentioned in [11], and can be proved by the same method as to prove the completeness of $BS^p(\mathbb{R};\mathbb{E}^n)$ in [19].

Proposition 4. *$BS^p(\mathbb{T};\mathbb{X})$ is a Banach space with norm $\|\cdot\|_{S^p}$.*

Remark 5. *(i) We can see easily that $BS^p(\mathbb{T};\mathbb{X})$ is translation invariant, namely, $g(\cdot+\alpha) \in BS^p(\mathbb{T};\mathbb{X})$ if $g \in BS^p(\mathbb{T};\mathbb{X})$ and $\alpha \in \Pi$. Moreover, we can get easily that $\|g(\cdot+\alpha)\|_{S^p} = \|g\|_{S^p}$ for $g \in BS^p(\mathbb{T};\mathbb{X})$.*
(ii) Let $1 \leq q \leq p < \infty$. We can verify easily that $BS^p(\mathbb{T};\mathbb{X}) \subset BS^q(\mathbb{T};\mathbb{X})$ and

$$\|g\|_{S^q} \leq \|g\|_{S^p} \quad \text{for } g \in BS^p(\mathbb{T};\mathbb{X}). \tag{4}$$

4. Bochner-Like Transform and S^p-Almost Periodic Functions

In this section, we give a revised version of Bochner transform, called Bochner-like transform, for time scales. We first show that Bochner transform is not valid on time scales. Then we give the definition of Bochner-like transform, and prove that the Stepanov almost periodicity of a function is equivalent to the almost periodicity of its Bochner-like transform on time scales. At last, we prove a theorem on the composition of Stepanov almost periodic functions by applying Bochner-like transform.

4.1. Problems in Bochner Transform on Time Scales

We first recall the Bochner transform and Stepanov almost periodicity on $\mathbb{R}$.

Definition 12 ([20])**.** *The Bochner transform $g^b : \mathbb{R} \times [0,1] \to \mathbb{X}$ of a function $g \in L^p_{loc}(\mathbb{R};\mathbb{X})$ is defined by $g^b(s,t) = g(s+t)$, $s \in \mathbb{R}, t \in [0,1]$.*

For $g \in L^p_{loc}(\mathbb{R};\mathbb{X})$, g^b is always regarded as a mapping from $\mathbb{T}$ to $L^p([0,1];\mathbb{X})$. i.e. $g^b(s,\cdot)$ is written as $g^b(s)$ and $g^b(s) \in L^p([0,1];\mathbb{X})$.

Definition 13 ([20])**.** *$g \in L^p_{loc}(\mathbb{R};\mathbb{X})$ is S^p-almost periodic (or Stepanov almost periodic) if $g^b \in AP(\mathbb{R}; L^p([0,1];\mathbb{X}))$. The space of all these functions g is denoted by $S^pAP(\mathbb{R};\mathbb{X})$ with Stepanov norm $\|g\|_{S^p} = \|g^b\|_\infty$.*

The definition of Stepanov almost periodic on time scales was given in [11], which was given in Bohr sense avoiding Bochner transform.

Definition 14. *(i) $g \in BS^p(\mathbb{T};\mathbb{X})$ is S^p-almost periodic on $\mathbb{T}$ if given $\varepsilon > 0$, the ε-translation set of g*

$$T(g,\varepsilon) = \{\alpha \in \Pi : \|g(\cdot+\alpha) - g\|_{S^p} < \varepsilon\}$$

is a relatively dense set in Π. The space of all these functions is denoted by $S^pAP(\mathbb{T},\mathbb{X})$ with norm $\|\cdot\|_{S^p}$.
(ii) $g : \mathbb{T} \times \Omega \to \mathbb{Y}$ is S^p-almost periodic in $s \in \mathbb{T}$ if $g(\cdot,y) \in S^pAP(\mathbb{T};\mathbb{Y})$ uniformly for $y \in S$ with S an arbitrary compact subset of Ω. Namely, for $\varepsilon > 0$, $\bigcap_{y\in S} T(g(\cdot,y),\varepsilon)$ is a relatively dense set in Π. The set of all these functions is denoted by $S^pAP(\mathbb{T}\times\Omega;\mathbb{Y})$.

Remark 6. *(i) Definition 14 (i) is a correction of the one introduced in [11], where condition "$T(g,\varepsilon)$ is a relatively dense set in $\mathbb{T}$ " is replaced by "$T(g,\varepsilon)$ is a relatively dense set in Π" (which is equivalent to "$T(g,\varepsilon)$ is a relatively dense set in $\mathbb{R}$" by Lemma 2). This correction avoids some fatal errors as shown in Remark 2. Definition 14 (ii) can be found in [10].*
(ii) If $\mathbb{T} = \mathbb{R}$, one can see easily that Definition 14 (i) is the same as Definition 13.
(iii) Let $g \in S^pAP(\mathbb{T},\mathbb{X})$ and $\alpha \in \Pi$. We can verify easily that $T(g(\cdot+\alpha),\varepsilon) = T(g,\varepsilon)$ for $\varepsilon > 0$. This yields that $g(\cdot+\alpha) \in S^pAP(\mathbb{T},\mathbb{X})$. Then $S^pAP(\mathbb{T},\mathbb{X})$ is translation invariant.
(iv) To emphasize the exponent p for the ε-translation set of $g \in S^pAP(\mathbb{T},\mathbb{X})$, we also write $T_p(g,\varepsilon)$ instead of $T(g,\varepsilon)$. Let $1 \leq q \leq p < \infty$. Then $S^pAP(\mathbb{T};\mathbb{X}) \subset S^qAP(\mathbb{T};\mathbb{X})$ and (4) holds (This was mentioned in [11]). In fact, for $g \in S^pAP(\mathbb{T};\mathbb{X})$ and $\varepsilon > 0$, it follows from (4) that $T_p(g,\varepsilon) \subset T_q(g,\varepsilon)$. This implies that $T_q(g,\varepsilon)$ is a relatively dense set in $\mathbb{R}$, and $g \in S^qAP(\mathbb{T};\mathbb{X})$.

We note that on $\mathbb{R}$, by using Bochner transform, Stepanov almost periodic functions inherit some basic properties from almost periodic functions directly. For example, $S^pAP(\mathbb{R};\mathbb{X})$ is a Banach space since $AP(\mathbb{R};L^p([0,1];\mathbb{X}))$ is a Banach space by Proposition 1 (i). Some more properties obtained by using Bochner transform can be found in [20]. But on $\mathbb{T}$, by Definition 14, the same process does not run being lack of Bochner transform.

Therefore it is natural and important to try to define Stepanov almost periodicity on time scales by Bochner transform. Unfortunately, Bochner transform is not valid on time scales. In fact, the aim of using Bochner transform is to get the following conclusion:

(A) $g \in S^pAP(\mathbb{T};\mathbb{X})$ is equivalent to $g^b \in AP(\mathbb{T};L^p([0,\mathcal{K}]_\Pi;\mathbb{X}))$.

In (A), to make the expression $g^b(s,t) = g(s+t)$ sense for all $s \in \mathbb{T}$, we have to restrict $t \in [0,\mathcal{K}]_\Pi$, and then $g^b(s) \in L^p([0,\mathcal{K}]_\Pi;\mathbb{X})$ for $s \in \mathbb{T}$. But the following example shows that conclusion (A) is false. Hence Bochner transform is not valid on time scales.

Example 1. *Let $\mathbb{X} = \mathbb{R}$ and $\mathbb{T} = \bigcup_{k\in\mathbb{Z}}[2k,2k+1]$. Then $\mathcal{K} = 2$, $\Pi = 2\mathbb{Z}$ and $[0,\mathcal{K}]_\Pi = \{0,2\}$. On time scale Π, $\mu(0) = \mu(2) = 2$. Let*

$$g(s) = \begin{cases} s, & s \in \Pi, \\ \sin \pi s, & s \in \mathbb{T} \setminus \Pi. \end{cases}$$

We can check easily that $g \in S^pAP(\mathbb{T};\mathbb{R})$. However,

$$\begin{aligned} \|g^b\|_\infty &= \sup_{s\in\mathbb{T}} \|g^b(s)\|_{L^p([0,\mathcal{K}]_\Pi;\mathbb{R})} = \sup_{s\in\mathbb{T}} \left(\int_{[0,\mathcal{K}]_\Pi} |g(s+t)|^p \Delta t \right)^{\frac{1}{p}} \\ &= \sup_{s\in\mathbb{T}} \left(\mu(0)|g(s)|^p + \mu(2)|g(s+2)|^p\right)^{\frac{1}{p}} \geq \sup_{s\in\mathbb{T}} |g(s)| = \infty. \end{aligned}$$

This implies that $g^b \notin AP(\mathbb{T};L^p([0,\mathcal{K}]_\Pi;\mathbb{X}))$ by Proposition 1 (ii). Then conclusion (A) is false.

Remark 7. *The integral $\int_s^{s+l} g(t)\Delta t$ with $l > 0, l \in \mathbb{R}$ was used in [10]. This should be corrected to $l \in \Pi$ according to the analysis in [7] (Problem 2). Then norm $\|\cdot\|_{S_l^p}, l \in \Pi$ is equivalent to $\|\cdot\|_{S^p}$ by Proposition 3, and the space $S_{lap}^p(\mathbb{T};\mathbb{X})$ defined by $\|\cdot\|_{S_l^p}$ in [10] is actually $S^pAP(\mathbb{T};\mathbb{X})$. As a result, Conclusion (A) is actually [10] (Lemma 2.10), where g^b is written by $\tilde{g}$ and $S^pAP(\mathbb{T};\mathbb{X})$ is replaced by $S_{lap}^p(\mathbb{T};\mathbb{X})$. Therefore our result (Theorem 1 below) is a correction of [10] (Lemma 2.10).*

4.2. S^p-Almost Periodicity Defined by Bochner-Like Transform

To overcome the problem mentioned in the last subsection, now we revise Bochner transform for general time scales.

If $\mathbb{T} \neq \mathbb{R}$, we fix a left scattered point $\omega \in \mathbb{T}$. Then for $s \in \mathbb{T}$, there is a unique $n_s \in \mathbb{Z}$ such that $s - n_s\mathcal{K} \in [\omega, \omega+\mathcal{K})_\mathbb{T}$. Let

$$\mathcal{N}_s = \begin{cases} s, & \mathbb{T} = \mathbb{R}, \\ n_s, & \mathbb{T} \neq \mathbb{R}. \end{cases}$$

Definition 15. *Let $g \in BS^p(\mathbb{T};\mathbb{X})$. The Bochner-like transform $g^c : \mathbb{T}\times\mathbb{T} \to \mathbb{X}$ of g is defined by $g^c(s,t) = g(\mathcal{N}_s\mathcal{K} + t)$ for $s,t \in \mathbb{T}$.*

We note that $g^c(s,t) = g(s+t)$ if $\mathbb{T} = \mathbb{R}$. It is easy to see that the Bochner-like transform is linear on $BS^p(\mathbb{T};\mathbb{X})$. Namely, let a,b be scalars and $h \in BS^p(\mathbb{T};\mathbb{X})$, then $(ag+bh)^c = ag^c + bh^c$. Moreover, g^c is always regarded as a mapping from $\mathbb{T}$ to $BS^p(\mathbb{T};\mathbb{X})$. That is $g^c(s,\cdot)$ is written as $g^c(s)$ and $g^c(s) \in BS^p(\mathbb{T};\mathbb{X})$.

Lemma 3. *Let $g \in BS^p(\mathbb{T}, \mathbb{X})$ with $\mathbb{T} \neq \mathbb{R}$. Then $g^c \in UBC(\mathbb{T}; BS^p(\mathbb{T}; \mathbb{X}))$. Moreover,*

$$\|g\|_{S^p} = \|g^c\|_\infty. \tag{5}$$

Proof. Let $g \in BS^p(\mathbb{T}; \mathbb{X})$. Then

$$\begin{aligned}
\|g\|_{S^p} &= \sup_{s\in\mathbb{T}} \left(\frac{1}{\mathcal{K}} \int_s^{s+\mathcal{K}} \|g(r)\|^p \Delta r \right)^{1/p} \\
&= \sup_{s\in\mathbb{T}} \sup_{t\in\mathbb{T}} \left(\frac{1}{\mathcal{K}} \int_{t+\mathcal{N}_s\mathcal{K}}^{t+(\mathcal{N}_s+1)\mathcal{K}} \|g(r)\|^p \Delta r \right)^{1/p} \\
&= \sup_{s\in\mathbb{T}} \sup_{t\in\mathbb{T}} \left(\frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} \|g(\mathcal{N}_s\mathcal{K} + r)\|^p \Delta r \right)^{1/p} \\
&= \sup_{s\in\mathbb{T}} \|g(\mathcal{N}_s\mathcal{K} + \cdot)\|_{S^p} = \sup_{s\in\mathbb{T}} \|g^c(s)\|_{S^p} = \|g^c\|_\infty.
\end{aligned}$$

That is (5) holds, and $\|g^c\|_\infty = \|g\|_{S^p} < \infty$. Let $s_1, s_2 \in \mathbb{T}$ with $|s_1 - s_2| < \omega - \rho(\omega)$, it is easy to see that $\mathcal{N}_{s_1} = \mathcal{N}_{s_2}$, and then $g^c(s_1) = g^c(s_2)$. This implies that $g^c \in UBC(\mathbb{T}; BS^p(\mathbb{T}; \mathbb{X}))$. □

The following example tells us that if the condition "ω is a left scattered point" is replaced by "ω is a right scattered point", the continuity of $s \mapsto g^c(s)$ may be lost.

Example 2. *Let $\mathbb{X} = \mathbb{R}$, $\mathbb{T} = \bigcup_{k\in\mathbb{Z}} [2k, 2k+1]$, and $g(s) = \sin(\pi s/2), s \in \mathbb{T}$. Then $g \in BS^p(\mathbb{T}; \mathbb{R})$, $\mathcal{K} = 2$, $\omega = 1$ is right scattered and is not left scattered. Applying Definition 15 to ω, then $\mathcal{N}_1 = 0$ and $\mathcal{N}_{1-\varepsilon} = -1$ for $\varepsilon \in (0,1)$, and*

$$\|g^c(1-\varepsilon) - g^c(1)\|_{S^p} = \sup_{s\in\mathbb{T}} \left(\frac{1}{2} \int_s^{s+2} |\sin \pi(r-2)/2 - \sin \pi r/2|^p \Delta r \right)^{\frac{1}{p}} \geq 1.$$

This implies that g^c is discontinuous at 1. Indeed, it is easy to verify that g^c is discontinuous at each $2k+1$ for $k \in \mathbb{Z}$.

Furthermore, the following example indicates that Lemma 3 doesn't hold when $\mathbb{T} = \mathbb{R}$.

Example 3. *Let*

$$g(t) = \begin{cases} (k - k^2 t + k^3)^{\frac{1}{p}}, & t \in [k, k+\frac{1}{k}], k \geq 2, k \in \mathbb{N}, \\ 0, & \text{others}. \end{cases}$$

Then $g \in BS^p(\mathbb{R}; \mathbb{R})$. But g^c is discontinuous at everywhere in $\mathbb{R}$. In fact, let $t \in \mathbb{R}$. For $h \in (0, 1/2)$, there is $k_h \in \mathbb{N}$ such that $h > 1/k_h$. Then $g(s+h) = 0$ for $s \in [k_h, k_h + 1/k_h]$, and

$$\begin{aligned}
\|g^c(t+h) - g^c(t)\|_{S^p}^p &= \sup_{r\in\mathbb{R}} \int_r^{r+1} |g(t+h+s) - g(t+s)|^p ds \\
&= \sup_{r\in\mathbb{R}} \int_r^{r+1} |g(h+s) - g(s)|^p ds \\
&\geq \int_{k_h}^{k_h+\frac{1}{k_h}} |g(h+s) - g(s)|^p ds \\
&= \int_{k_h}^{k_h+\frac{1}{k_h}} |g(s)|^p ds = \frac{1}{2},
\end{aligned}$$

which yields that g^c is discontinuous at t.

We see that $g \in BS^p(\mathbb{R};\mathbb{X})$ does not imply $g^c \in C(\mathbb{R}; BS^p(\mathbb{R};\mathbb{X}))$ in Example 3. But we have the following lemma.

Lemma 4. *Let $g \in BS^p(\mathbb{R};\mathbb{X})$. Then*

(i) $g^b \in UBC(\mathbb{R}; L^p([0,1];\mathbb{X}))$ *if and only if* $g^c \in UBC(\mathbb{R}; BS^p(\mathbb{R};\mathbb{X}))$.
(ii) $g^b \in BC(\mathbb{R}; L^p([0,1];\mathbb{X}))$ *if and only if* $g^c \in BC(\mathbb{R}; BS^p(\mathbb{R};\mathbb{X}))$.

Lemma 4 can be got from the following lemma immediately.

Lemma 5. *Let $g \in BS^p(\mathbb{R};\mathbb{X})$. Then*

$$\|g\|_{S^p} = \|g^b\|_\infty = \|g^c\|_\infty. \tag{6}$$

Proof. Let $g \in BS^p(\mathbb{R};\mathbb{X})$. Then

$$\begin{aligned} \sup_{s\in\mathbb{R}} \|g^b(s)\|_{L^p([0,1];\mathbb{X})} = \|g\|_{S^p} &= \sup_{s\in\mathbb{R}} \left(\int_s^{s+1} \|g(r)\|^p dr \right)^{1/p} \\ &= \sup_{s\in\mathbb{R}} \sup_{t\in\mathbb{R}} \left(\int_{t+s}^{t+s+1} \|g(r)\|^p dr \right)^{\frac{1}{p}} \\ &= \sup_{s\in\mathbb{R}} \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|g(s+r)\|^p dr \right)^{\frac{1}{p}} \\ &= \sup_{s\in\mathbb{R}} \|g(s+\cdot)\|_{S^p} = \sup_{s\in\mathbb{R}} \|g^c(s)\|_{S^p}. \end{aligned}$$

This leads to (6). □

Now we are ready to give the main result.

Theorem 1. $g \in S^pAP(\mathbb{T};\mathbb{X})$ *if and only if* $g^c \in AP(\mathbb{T}; BS^p(\mathbb{T};\mathbb{X}))$.

Proof. If $g \in S^pAP(\mathbb{R};\mathbb{X})$, $g^b \in UBC(\mathbb{R}; L^p([0,1];\mathbb{X}))$ by Proposition 1 (ii), and then by Lemma 4, $g^c \in UBC(\mathbb{R}; BS^p(\mathbb{R};\mathbb{X}))$. So by Lemma 3, $g^c \in UBC(\mathbb{T}; BS^p(\mathbb{T};\mathbb{X}))$ if $g \in S^pAP(\mathbb{T};\mathbb{X})$. Meanwhile, by (5) and (6), for $\varepsilon > 0$,

$$\begin{aligned} T(g,\varepsilon) &= \{\alpha \in \Pi : \|g(\cdot+\alpha) - g\|_{S^p} < \varepsilon\} \\ &= \{\alpha \in \Pi : \|g^c(\cdot+\alpha) - g^c\|_\infty < \varepsilon\} = T(g^c,\varepsilon). \end{aligned} \tag{7}$$

Thus $g \in S^pAP(\mathbb{T};\mathbb{X})$ if and only if $g^c \in AP(\mathbb{T}; BS^p(\mathbb{T};\mathbb{X}))$. □

Let $g \in S^pAP(\mathbb{T}\times\Omega;\mathbb{Y})$. Then $g^c(s,y) = g(\mathcal{N}_s\mathcal{K}+\cdot, y) \in BS^p(\mathbb{T};\mathbb{Y})$ for $(s,y) \in \mathbb{T}\times\Omega$. Then by Theorem 1, we have the following corollary.

Corollary 1. $g \in S^pAP(\mathbb{T}\times\Omega;\mathbb{Y})$ *if and only if* $g^c \in AP(\mathbb{T}\times\Omega; BS^p(\mathbb{T};\mathbb{Y}))$.

Theorem 1 ensures that Stepanov almost periodic functions can be defined by Bochner-like transform on time scales.

Definition 16. *(i)* $g \in BS^p(\mathbb{T};\mathbb{X})$ *is S^p-almost periodic if* $g^c \in AP(\mathbb{T}; BS^p(\mathbb{T};\mathbb{X}))$.
(ii) $g : \mathbb{T}\times\Omega \to \mathbb{Y}$ *is S^p-almost periodic in $s \in \mathbb{T}$ if* $g^c \in AP(\mathbb{T}\times\Omega; BS^p(\mathbb{T};\mathbb{Y}))$.

Remark 8. *Space $S^pAP(\mathbb{T};\mathbb{X})$ can inherit some important properties from $AP(\mathbb{T}; BS^p(\mathbb{T};\mathbb{X}))$ directly. For example, it is easy to obtain the following statements by using Bochner-like transform.*

(i) $S^pAP(\mathbb{T};\mathbb{X})$ *is a Banach space.*

(ii) $g \in S^pAP(\mathbb{T};\mathbb{X})$ *if and only if for each sequence* $\{\alpha'_k\} \subset \Pi$*, there is a subsequence* $\{\alpha_k\} \subset \{\alpha'_k\}$ *such that* $\{g(\cdot + \alpha_k)\}$ *converges in* $S^pAP(\mathbb{T};\mathbb{X})$.

4.3. Composition Theorem

Let us begin with a lemma on the uniform almost periodicity.

Lemma 6. *Let* $g \in S^pAP(\mathbb{T}\times\mathbb{X};\mathbb{X})$ *and* $u \in AP(\mathbb{T};\mathbb{X})$*. Then for* $\varepsilon > 0$,

$$\mathcal{G} = \left(\bigcap_{x\in\overline{u(\mathbb{T})}} T(g(\cdot,x),\varepsilon) \right) \cap T(u,\varepsilon)$$

is relatively dense in $\mathbb{R}$.

Proof. $g \in S^pAP(\mathbb{T}\times\mathbb{X};\mathbb{X})$ means that $g^c \in AP(\mathbb{T}\times\mathbb{X};BS^p(\mathbb{T};\mathbb{X}))$. Let $\mathbb{Y} = \mathbb{X}\times BS^p(\mathbb{T};\mathbb{X})$ with norm

$$\|\cdot\|_{\mathbb{Y}} = \|\cdot\|_{\mathbb{X}} + \|\cdot\|_{S^p} \tag{8}$$

and $G : \mathbb{T}\times\mathbb{X} \to \mathbb{Y}$ be defined by $G(s,x) = (u(s), g^c(s,x))$, $s \in \mathbb{T}, x \in \mathbb{X}$. Then $\mathbb{Y}$ is a Banach space. It follows from Proposition 1 (iii) that for each sequence $\{\alpha'_k\} \subset \Pi$, there are a subsequence $\{\alpha_k\} \subset \{\alpha'_k\}$ and $\hat{u} \in AP(\mathbb{T};\mathbb{X}), \hat{g}^c \in AP(\mathbb{T}\times\mathbb{X};BS^p(\mathbb{T};\mathbb{X}))$ such that $\lim_{k\to\infty} u(\cdot+\alpha_k) = \hat{u}$ and

$$\lim_{k\to\infty} g(s+\alpha_k,x) = \hat{g}^c \quad \text{uniformly on } \mathbb{T}\times S \text{ for compact } S \subset \mathbb{X}.$$

Let $\hat{G}(s,x) = (\hat{u}(s), \hat{g}(s,x)), (s,x) \in \mathbb{T}\times\mathbb{X}$. Then

$$\lim_{k\to\infty} G(s+\alpha_k,x) = \hat{G} \quad \text{uniformly on } \mathbb{T}\times S \text{ for compact } S \subset \mathbb{X}.$$

Again by Proposition 1 (iii), $G \in AP(\mathbb{T}\times\mathbb{X};\mathbb{Y})$. Now we invoke Proposition 2 to get that $\overline{u(\mathbb{T})}$ is compact. Then for $\varepsilon > 0$, $\bigcap_{x\in\overline{u(\mathbb{T})}} T(G(\cdot,x),\varepsilon)$ is relatively dense in $\mathbb{R}$. Note that (7) and (8) imply that

$$T(G(\cdot,x),\varepsilon) \subset T(g^c(\cdot,x),\varepsilon) \cap T(u,\varepsilon) = T(g(\cdot,x),\varepsilon) \cap T(u,\varepsilon), \quad x \in \mathbb{X}.$$

Thus $\mathcal{G}$ is a relatively dense set in $\mathbb{R}$. □

Remark 9. *Let* $a > 0$ *and* $g \in AP(\mathbb{T};\mathbb{X})$*. For* $\varepsilon > 0$*, we can check easily that* $T(ag,\varepsilon) = T(g,a^{-1}\varepsilon)$*. So* $\mathcal{G}$ *in Lemma 6 can be replaced by* $\left(\bigcap_{x\in\overline{u(\mathbb{T})}} T(g(\cdot,x),\varepsilon_2)\right) \cap T(u,\varepsilon_1)$ *for* $\varepsilon_1,\varepsilon_2 > 0$.

Theorem 2. *Assume that* $u \in AP(\mathbb{T};\mathbb{X})$*,* $g \in S^pAP(\mathbb{T}\times\mathbb{X};\mathbb{X})$ *and for some* $L \in BS^p(\mathbb{T};\mathbb{R}^+)$,

$$\|g(s,x) - g(s,y)\| \le L(s)\|x-y\| \quad \text{for } x,y \in \mathbb{X}. \tag{9}$$

Then $g(\cdot,u(\cdot)) \in S^pAP(\mathbb{T};\mathbb{X})$.

Proof. By (9),

$$\|g(\cdot,u(\cdot)) - g(\cdot,0)\|_{S^p} \le \|L\|_{S^p}\|u\|_\infty.$$

Then

$$\|g(\cdot,u(\cdot))\|_{S^p} \le \|g(\cdot,0)\|_{S^p} + \|L\|_{S^p}\|u\|_\infty < \infty.$$

That is $g(\cdot,u(\cdot)) \in BS^p(\mathbb{T};\mathbb{X})$.

Assume that $\|L\|_{S^p} > 0$. Note that $u(\mathbb{T})$ is a relatively compact set. Then for $\varepsilon > 0$, there is a finite set $\{t_i\}_{i=1}^m \subset \mathbb{T}$ such that

$$u(\mathbb{T}) \subset \bigcup_{i=1}^{m} B\left(u(t_i), \frac{\varepsilon}{8\|L\|_{S^p}}\right), \tag{10}$$

where $B(x,a)$ with $x \in \mathbb{X}, r > 0$ denotes an open ball with radius a and center x. Let

$$\mathcal{B} = T\left(u, \frac{\varepsilon}{2\|L\|_{S^p}}\right) \cap \left(\bigcap_{x\in \overline{u(\mathbb{T})}} T\left(g(\cdot,x), \frac{\varepsilon}{4m}\right)\right). \tag{11}$$

Then $\mathcal{B}$ is relatively dense in $\mathbb{R}$ by Remark 9. Let $s \in \mathbb{T}$ and $\alpha \in \mathcal{B}$, there is $k \in \{1,2,\cdots,m\}$ such that $u(s) \in B\left(u(t_k), \frac{\varepsilon}{8\|L\|_{S^p}}\right)$. Thus by (9),

$$\begin{aligned}
&\|g(r+\alpha,u(r)) - g(r,u(r))\| \\
&\le \|g(r+\alpha,u(r)) - g(r+\alpha,u(t_k))\| + \|g(r+\alpha,u(t_k)) - g(r,u(t_k))\| \\
&\quad + \|g(r,u(t_k)) - g(r,u(r))\| \\
&\le L(r+\alpha)\frac{\varepsilon}{8\|L\|_{S^p}} + \|g(r+\alpha,u(t_k)) - g(r,u(t_k))\| + L(r)\frac{\varepsilon}{8\|L\|_{S^p}} \\
&\le (L(r+\alpha)+L(r))\frac{\varepsilon}{8\|L\|_{S^p}} + \sum_{i=1}^{m}\|g(r+\alpha,u(t_i)) - g(r,u(t_i))\|.
\end{aligned}$$

By Remark 5 (i), we have $L, L(\cdot+\alpha) \in BS^p(\mathbb{T};\mathbb{X})$ and $\|L(\cdot+\alpha)\|_{S^p} = \|L\|_{S^p}$. Hence

$$\begin{aligned}
&\|g(\cdot+\alpha,u(\cdot)) - g(\cdot,u(\cdot))\|_{S^p} \\
&= \sup_{s\in\mathbb{T}}\left(\frac{1}{\mathcal{K}}\int_s^{s+\mathcal{K}} \|g(r+\alpha,u(r)) - g(r,u(r))\|^p \Delta r\right)^{\frac{1}{p}} \\
&\le 2\|L\|_{S^p}\frac{\varepsilon}{8\|L\|_{S^p}} + \sum_{i=1}^{m}\|g(\cdot+\alpha,u(t_i)) - g(\cdot,u(t_i))\|_{S^p} \\
&< \frac{\varepsilon}{4} + m\cdot\frac{\varepsilon}{4m} = \frac{\varepsilon}{2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|g(\cdot+\alpha,u(\cdot+\alpha)) - g(\cdot,u(\cdot))\|_{S^p} \\
&\le \|g(\cdot+\alpha,u(\cdot+\alpha)) - g(\cdot+\alpha,u(\cdot))\|_{S^p} + \|g(\cdot+\alpha,u(\cdot)) - g(\cdot,u(\cdot))\|_{S^p} \\
&< \|L\|_{S^p}\|u(\cdot+\alpha) - u\|_\infty + \frac{\varepsilon}{2} \\
&< \|L\|_{S^p}\|\frac{\varepsilon}{2\|L\|_{S^p}} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This implies that $\mathcal{B} \subset T(g(\cdot,u(\cdot)),\varepsilon)$. Then $T(g(\cdot,u(\cdot)),\varepsilon)$ is relatively dense in $\mathbb{R}$, and $g(\cdot,u(\cdot)) \in S^pAP(\mathbb{T};\mathbb{X})$. If $\|L\|_{S^p} = 0$, replacing $\frac{\varepsilon}{8\|L\|_{S^p}}$ in (10) and $\frac{\varepsilon}{2\|L\|_{S^p}}$ in (11) by ε, a slight modification of the above process leads to the same conclusion. □

5. Dynamic Equations

Applying the results obtained above, we consider the following nonlinear dynamic equation:

$$x^\Delta(s) = A(s)x(s) + g(s,x(s)) \quad \text{for } s \in \mathbb{T}, \tag{12}$$

where $g \in S^pAP(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n) \cap C(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n)$ and A is an $n \times n$ continuous matrix function.

We first recall the concept of exponential functions on $\mathbb{T}$. $h : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(s)h(s) \neq 0, s \in \mathbb{T}^k$. We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}; \mathbb{R})$ the set of all regressive and rd-continuous (i.e., continuous at each right-dense point) functions $h : \mathbb{T} \to \mathbb{R}$. Let $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}; \mathbb{R}) = \{h \in \mathcal{R} : 1 + \mu(s)h(s) > 0 \text{ for } s \in \mathbb{T}\}$. The set of regressive functions on time scales is an Abelian group with addition $\oplus$ given by $a \oplus b \triangleq a + b + \mu(t)ab$. Meanwhile, we denote by $\ominus a \triangleq -\dfrac{a}{1 + \mu(t)a}$ the additive inverse of a in the group.

Definition 17. *Let $h \in \mathcal{R}$. The exponential function is defined as*

$$e_h(s, t) = \exp\left(\int_t^s \eta_{\mu(\alpha)}(h(\alpha))\Delta\alpha\right), \quad t, s \in \mathbb{T}$$

with the cylinder transformation

$$\eta_p(v) = \begin{cases} \dfrac{1}{p} Log(1 + pv), & p \neq 0, \\ v, & p = 0. \end{cases}$$

Here Log indicate the principal logarithm.

Let matrix $X(s)$ be the fundamental solution of the homogeneous linear equation of (12):

$$x^{\Delta}(s) = A(s)x(s), \quad s \in \mathbb{T}. \tag{13}$$

Definition 18. *(13) is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist a projection P and constants $\alpha > 0, M > 0$ such that [3]*

$$|X(s)PX^{-1}(t)| \leq Me_{\ominus\alpha}(s, t), \quad s, t \in \mathbb{T}, s \geq t,$$
$$|X(s)(I - P)X^{-1}(t)| \leq Me_{\ominus\alpha}(t, s), \quad s, t \in \mathbb{T}, s \leq t.$$

Let

$$\Gamma(s, t) = \begin{cases} X(s)PX^{-1}(t), & s, t \in \mathbb{T}, s \geq t, \\ -X(s)(I - P)X^{-1}(t), & s, t \in \mathbb{T}, s \leq t. \end{cases}$$

Note that $e_{\ominus\alpha}(s, t) \leq 1$ for $s \geq t$. Then $|\Gamma(s, t)| \leq M$.

Definition 19. *A continuous bounded function $g : \mathbb{T} \times \mathbb{T} \to \mathbb{E}^n$ is bi-almost periodic if each sequence $\{\alpha'_k\} \subset \Pi$ has a subsequence $\{\alpha_k\}$ such that $\{g(s + \alpha_k, t + \alpha_k)\}$ converges uniformly for all $(s, t) \in \mathbb{T} \times \mathbb{T}$.*

We will use the following assumptions later:

(H1) (13) admits an exponential dichotomy with projection P and constants $\alpha > 0, M > 0$.
(H2) $\Gamma(t, s)$ is bi-almost periodic.

To consider (12), we consider first the following linear equation:

$$x^{\Delta}(s) = A(s)x(s) + g(s) \quad \text{for } s \in \mathbb{T}, \tag{14}$$

where A is as (12) and $g \in S^p AP(\mathbb{T}; \mathbb{E}^n) \cap C(\mathbb{T}; \mathbb{E}^n)$. We obtain the following theorem.

Theorem 3. *Suppose that (H1) and (H2) are satisfied. Then (14) admits a unique almost periodic solution given as*

$$u(s) = \int_{\mathbb{T}} \Gamma(s, \sigma(r))g(r)\Delta r, \quad s \in \mathbb{T}. \tag{15}$$

Proof. By condition (H1) and [3] (Lemma 4.17), it is easy to verify that u given by (15) is the unique continuous bounded solution of (14). Now we only need to prove that $u \in AP(\mathbb{T};\mathbb{E}^n)$. In fact, for $s \in \mathbb{T}$, let

$$\phi(s) = \int_{-\infty}^{t} \Gamma(s,\sigma(r))g(r)\Delta r = \sum_{j=1}^{\infty} \phi_j(r)$$

with

$$\phi_j(s) = \int_{s-j\mathcal{K}}^{s-(j-1)\mathcal{K}} \Gamma(s,\sigma(r))g(r)\Delta r, \quad j \in \mathbb{N}.$$

By (H2) and Remark 8 (ii), for each sequence $\{\alpha_k'\} \subset \Pi$, we can find subsequence $\{\alpha_k\} \subset \{\alpha_k'\}$ and functions $\bar{g} \in S^pAP(\mathbb{T};\mathbb{E}^n)$ and $\bar{\Gamma}$ such that

$$\lim_{k\to\infty} \sup_{s,r\in\mathbb{T}} |\Gamma(s+\alpha_k,\sigma(r)+\alpha_k) - \bar{\Gamma}(s,\sigma(r))| = 0, \quad \lim_{k\to\infty} \|g(\cdot+\alpha_k) - \bar{g}\|_{S^p} = 0. \tag{16}$$

Let

$$\bar{\phi}_j(s) := \int_{s-j\mathcal{K}}^{s-(j-1)\mathcal{K}} \bar{\Gamma}(s,\sigma(r))\bar{g}(r)\Delta r, \quad j \in \mathbb{N}, s \in \mathbb{T}.$$

By (H1), (16) and Hölder inequality,

$$\begin{aligned}
&\|\phi_j(\cdot+\alpha_k) - \bar{\phi}_j\| \\
&= \sup_{s\in\mathbb{T}} \int_{s-j\mathcal{K}}^{s-(j-1)\mathcal{K}} (|\Gamma(s+\alpha_k,\sigma(r+\alpha_k))g(r+\alpha_k) - \bar{\Gamma}(s,\sigma(r))\bar{g}(r)|\Delta r \\
&\le \sup_{s\in\mathbb{T}} \int_{s-j\mathcal{K}}^{s-(j-1)\mathcal{K}} (|\Gamma(s+\alpha_k,\sigma(r)+\alpha_k)||g(r+\alpha_k) - \bar{g}(r)| \\
&\qquad + |\Gamma(s+\alpha_k,\sigma(r)+\alpha_k) - \bar{\Gamma}(s,\sigma(r))||\bar{g}(r)|)\Delta r \\
&\le M\mathcal{K}\|g(\cdot+\alpha_k) - \bar{g}\|_{S^p} + \sup_{s,r\in\mathbb{T}} |\Gamma(s+\alpha_k,\sigma(r)+\alpha_k) - \bar{\Gamma}(s,\sigma(r))|\mathcal{K}\|\bar{g}\|_{S^p} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Thus $\phi_j \in AP(\mathbb{T};\mathbb{E}^n)$ for each $j \in \mathbb{N}$. Then $\phi \in AP(\mathbb{T};\mathbb{E}^n)$. Let

$$\psi = \int_{s}^{\infty} \Gamma(s,\sigma(r))g(r)\Delta r.$$

Similarly, we can prove that $\psi \in AP(\mathbb{T};\mathbb{E}^n)$. Hence $u = \phi + \psi \in AP(\mathbb{T};\mathbb{E}^n)$. □

For nonlinear dynamics Equation (12), we have the following theorem.

Theorem 4. *Suppose that $g \in S^pAP(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n) \cap C(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n)$ satisfying (9), (H1) and (H2). Then (12) admits a unique almost periodic solution $u(s)$ satisfying*

$$u(s) = \int_{\mathbb{T}} \Gamma(s,\sigma(r))g(r,u(r))\Delta r, \quad s \in \mathbb{T}, \tag{17}$$

provided that

$$\|L\|_{S^p} < \begin{cases} \dfrac{1-e^{-\alpha\mathcal{K}}}{2M\mathcal{K}}, & \text{if } \mathbb{T} = \mathbb{R}, \\ \dfrac{1}{M\mathcal{K}}\left(3 + \alpha\bar{\mu} + \dfrac{2}{\alpha\bar{\mu}}\right)^{-1}, & \text{if } \mathbb{T} \ne \mathbb{R}, \end{cases} \tag{18}$$

where $\bar{\mu} := \sup_{s\in\mathbb{T}} \mu(s)$.

Proof. Let $\varphi \in AP(\mathbb{T};\mathbb{E}^n)$. By Theorem 2, we have $g(\cdot,\varphi(\cdot)) \in S^pAP(\mathbb{T};\mathbb{E}^n)$. Define

$$T(\varphi)(s) := \int_{\mathbb{T}} \Gamma(s,\sigma(r))g(r,\varphi(r))\Delta r \quad \text{for } s \in \mathbb{T}.$$

By Theorem 3, $T(\varphi) \in AP(\mathbb{T};\mathbb{E}^n)$. That is $T : AP(\mathbb{T};\mathbb{E}^n) \to AP(\mathbb{T};\mathbb{E}^n)$. By (H1), for $s \in \mathbb{T}$, $\varphi_1, \varphi_2 \in AP(\mathbb{T};\mathbb{E}^n)$,

$$\begin{aligned}
&|T(\varphi_1)(s) - T(\varphi_2)(s)| \\
&\leq \int_{\mathbb{T}} |\Gamma(s,\sigma(r))||g(r,\varphi_1(r)) - g(r,\varphi_2(r))|\Delta r \\
&\leq \left(\int_{-\infty}^{s} e_{\ominus\alpha}(s,\sigma(r))|L(r)|\Delta r + \int_{s}^{\infty} e_{\ominus\alpha}(\sigma(r),s)|L(r)|\Delta r\right) M\|\varphi_1 - \varphi_2\|_\infty \\
&= M(L_1(s) + L_2(s))\|\varphi_1 - \varphi_2\|_\infty,
\end{aligned} \tag{19}$$

where

$$L_1(s) = \int_{-\infty}^{s} e_{\ominus\alpha}(s,\sigma(r))|L(r)|\Delta r, \quad L_2(s) = \int_{s}^{\infty} e_{\ominus\alpha}(\sigma(r),s)|L(r)|\Delta r.$$

If $\mathbb{T} = \mathbb{R}$,

$$\begin{aligned}
L_1(s) &= \int_{-\infty}^{s} e^{-\alpha(s-r)}|L(r)|\Delta r = \sum_{j=1}^{\infty} \int_{s-j\mathcal{K}}^{s-(j-1)\mathcal{K}} e^{-\alpha(s-r)}|L(r)|dr \\
&\leq \mathcal{K}\|L\|_{S^p} \sum_{j=1}^{\infty} e^{-\alpha\mathcal{K}(j-1)} = \frac{\mathcal{K}\|L\|_{S^p}}{1 - e^{-\alpha\mathcal{K}}}.
\end{aligned}$$

If $\mathbb{T} \neq \mathbb{R}$, $\mathbb{T}$ contains at least one right scattered point. Let $\bar{\mu} := \sup_{s\in\mathbb{T}} \mu(s)$. Then $\bar{\mu} \leq \mathcal{K}$, and it is easy to see that there is a right scattered point s_0 satisfying that $\mu(s_0) = \bar{\mu}$. Then for $s \in \mathbb{T}$ and $j \geq 3$, $[\sigma(s) - j\mathcal{K} + \mathcal{K}, s)_{\mathbb{T}}$ contains at least $j-2$ right scattered points with form $s_0 + \mathcal{L}_s\mathcal{K}$, $\mathcal{L}_s \in \mathbb{Z}$, and $\mu(s_0 + \mathcal{L}_s\mathcal{K}) = \mu(s_0) = \bar{\mu}$. Note that for $j \geq 3$,

$$e_{\ominus\alpha}(s,\sigma(s) - (j-1)\mathcal{K}) \leq (e_{\ominus\alpha}(\sigma(s_0) + \mathcal{L}_s\mathcal{K}, s_0 + \mathcal{L}_s\mathcal{K}))^{j-2} = (1 + \alpha\bar{\mu})^{2-j}.$$

Thus

$$\begin{aligned}
L_1(s) &= \sum_{j=1}^{\infty} \int_{s-j\mathcal{K}}^{s-(j-1)\mathcal{K}} e_{\ominus\alpha}(s,\sigma(r))|L(r)|\Delta r \\
&\leq \mathcal{K}\|L\|_{S^p} \sum_{j=1}^{\infty} e_{\ominus\alpha}(s,\sigma(s) - (j-1)\mathcal{K}) \\
&\leq \mathcal{K}\|L\|_{S^p} \left(e_{\ominus\alpha}(s,\sigma(s)) + e_{\ominus\alpha}(s,\sigma(s) - \mathcal{K}) + \sum_{j=3}^{\infty} (1 + \alpha\bar{\mu})^{2-j}\right) \\
&\leq \mathcal{K}\|L\|_{S^p} \left(2 + \alpha\bar{\mu} + \frac{1}{\alpha\bar{\mu}}\right).
\end{aligned}$$

Similarly, we can prove that

$$L_2(s) \leq \begin{cases} \dfrac{\mathcal{K}\|L\|_{S^p}}{1 - e^{-\alpha\mathcal{K}}}, & \text{if } \mathbb{T} = \mathbb{R}, \\ \mathcal{K}\|L\|_{S^p}\left(1 + \dfrac{1}{\alpha\bar{\mu}}\right), & \text{if } \mathbb{T} \neq \mathbb{R}. \end{cases}$$

Let

$$\lambda = \begin{cases} \dfrac{2M\mathcal{K}\|L\|_{S^p}}{1-e^{-\alpha\mathcal{K}}}, & \text{if } \mathbb{T}=\mathbb{R}, \\ M\mathcal{K}\|L\|_{S^p}\left(3+\alpha\bar{\mu}+\dfrac{2}{\alpha\bar{\mu}}\right), & \text{if } \mathbb{T}\neq\mathbb{R}. \end{cases}$$

Then $M(L_1(s)+L_2(s)) \leq \lambda < 1$ by (18). This together with (19) implies that

$$\|T(\varphi_1)-T(\varphi_2)\|_\infty \leq \lambda\|\varphi_1-\varphi_2\|_\infty.$$

That is T is a contraction operator. Hence T admits a unique fixed point $u \in AP(\mathbb{T};\mathbb{E}^n)$. Then that (12) admits a unique almost periodic solution u satisfying (17). □

6. Conclusions

By using Bochner transform, Stepanov almost periodic functions inherit some basic properties from almost periodic functions. But we show that this old work can not be extended directly to time scales. We revised the classic Bochner transform, and give the Bochner-like transform for time scales. Then the old work can be extended to time scales by using this method. Namely, we prove that a function is Stepanov almost periodic if and only if its Bochner-like transform is almost periodic on time scales. Some basic properties including the composition theorem of Stepanov almost periodic functions are obtained by applying Bochner-like transform. We correct some recent results where Bochner transform was used directly on time scales. Moreover, we apply the results to get some existence results of almost periodic solutions for some dynamic equations with Stepanov almost periodic terms. We expect some more results based on the new method Bochner-like transform in further study.

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