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# Generalized Shifted Chebyshev Koornwinder's Type Polynomials: Basis Transformations

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Received: 13 November 2018; Accepted: 30 November 2018; Published: 2 December 2018

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**Abstract:** Approximating continuous functions by polynomials is vital to scientific computing and numerous numerical techniques. On the other hand, polynomials can be characterized in several ways using different bases, where every form of basis has its advantages and power. By a proper choice of basis, several problems will be removed; for instance, stability and efficiency can be improved, and numerous complications can be resolved. In this paper, we provide an explicit formula of the generalized shifted Chebyshev Koornwinder's type polynomial of the first kind, $\mathscr{T}_r^{*(K_0,K_1)}(x)$, using the Bernstein basis of fixed degree. Moreover, a Bézier's degree elevation was used to rewrite $\mathscr{T}_r^{*(K_0,K_1)}(x)$ in terms of a higher degree Bernstein basis without altering the shapes. In addition, explicit formulas of conversion matrices between generalized shifted Chebyshev Koornwinder's type polynomials and Bernstein polynomial bases were given.

**Keywords:** transformation; basis; Bernstein; Bézier curves

## 1. Introduction

Bernstein polynomials $\mathbb{B}_j^m(x) = \frac{m!}{j!(m-j)!}x^j(1-x)^{m-j}$, $j = 0, 1, \ldots, m$, form a standard basis for Bézier surfaces and curves [1]. Farouki in [2] examined Bernstein basis properties and described key characteristics and algorithms related to the Bernstein basis. Remarkable properties and features of Bernstein polynomials [2] make them essential in the development of Bézier surfaces and curves in many areas of computer-aided and geometric designs (CAGDs). They have been studied thoroughly (see [2] for more details), and there exist great enduring works (see [3] and references therein).

Though higher order Bézier curves need extra time to process, their bases are optimally stable, and flexible in designing shapes. In addition, numerous applications (see [1,4]) contain two or more Bézier curves of different degrees that require an equal or higher degree for all involved Bézier curves. Knowing that the degree elevation of Bézier curves defined by [4] does not alter the shapes, the degree elevation can be used to express all comprised Bézier curves and Bernstein polynomials of deg $\leq n$ in respect of the $n$th-degree Bernstein polynomials using

$$\mathbb{B}_j^v(x) = \sum_{i=j}^{n-v+j} \frac{\binom{v}{j}\binom{n-v}{i-j}}{\binom{n}{i}} \mathbb{B}_i^n(x), \quad j = 0, 1, \ldots, v. \tag{1}$$

For extra information, see [1,5,6].

However, Bernstein polynomials are not orthogonal, so they cannot be used efficiently and effectively in approximation problems [7]. Therefore, calculations performed [7–9] to obtain the least squares polynomial of deg $= m$ using Bernstein polynomials do not reduce the calculations to obtain the least squares approximation polynomial of deg $= m + 1$.

For example [9], for $g(t) \in C[0,1]$, the least squares approximation requires finding a least squares polynomial, $\mathbf{B}g_m^*(t) = \sum_{k=0}^m c_k \phi_k(t)$, such that $\{\phi_k(t)\}_{k=0}^m$ is a basis that minimizes the error

$$E(c_0, c_1, \ldots, c_m) = \int_0^1 [g(t) - \sum_{k=0}^m c_k \phi_k(t)]^2 dt. \tag{2}$$

Now, $\frac{\partial E}{\partial c_k} = 0$, for $k = 0, \ldots, m$, is the necessary condition for Equation (2) to have a minimum over all $c_i$. Thus, for $i = 0, 1, \ldots, m$, the values $c_i$ that minimize $\|g(t) - \sum_{k=0}^m c_k \phi_k(t)\|_2$ satisfy

$$\int_0^1 g(t)\phi_i(t)dt = \sum_{k=0}^m c_k \int_0^1 \phi_k(t)\phi_i(t)dt, \tag{3}$$

which leads to a system of $(m+1)$ normal equations defined using $(m+1)$ unknown coefficients of $\mathbf{B}g_m^*(t)$, namely, $c_i$, $i = 0, \ldots, m$. By choosing natural powers $\phi_i(t) = t^i$, $i = 0, \ldots, m$ as a basis, Equation (3) equals $\sum_{k=0}^m \frac{c_k}{i+k+1}$, and the resulting coefficients form a Hilbert matrix, which has a notoriously ill condition for even modest values of $m$, and possess round-off error difficulties.

Therefore, approximations accompanied with an orthogonal polynomials basis, where many computations have been simplified, have been introduced instead [7,8], and they have turned out to be effective.

For instance, such calculations can be made computationally effective by using orthogonal polynomials [9], such as generalized shifted Chebyshev polynomials of the first kind. Thus, choosing our basis, $\phi_i(t)$, to act as orthogonal polynomials will simplify the approximation problem, where the resulting matrix will be diagonal. Since then, approximation using orthogonal polynomials has been introduced and has received more attention. Moreover, knowing $\mathbf{B}g_m^*(t)$ is enough to compute $c_{m+1}$ to obtain $\mathbf{B}g_{m+1}^*(t)$. Using orthogonal polynomials has shown to be computationally effective (see [7,9] for more details).

Characterization using the Bernstein basis of generalized shifted Chebyshev Koornwinder's type polynomials of the first and second kind was discussed in [10,11], respectively. Rababah [8] considered the transformation of the Bernstein polynomial basis with classical Chebyshev polynomials. A preliminary abstract of this manuscript was presented in [12], where a generalization of the work in [8] is given by providing an explicit formula of the generalized shifted Chebyshev Koornwinder's type polynomials of the first kind. We will refer to these as generalized shifted Chebyshev-I polynomials throughout this article. Then, we write the generalized shifted Chebyshev-I polynomials of degree $r \leq n$ using the Bernstein basis of degree $n$, and an explicit form of the transformation and the inverse transformation of the generalized shifted Chebyshev-I polynomials to Bernstein polynomial bases are presented.

*The Generalized Shifted Chebyshev-I Koornwinder's Type Polynomials*

Chebyshev's polynomials of the first kind $T_k(x)$ of degree $k \geq 0$ in $x$, are defined as $T_k(x) = \cos(k \arccos x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^k + \left(x - i\sqrt{1-x^2}\right)^k\right]$, $x \in [-1,1]$. They are solutions of the well-known Chebyshev's differential equation,

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2 y = 0, \qquad k = 0, 1, 2, \ldots,$$

and form a set of orthogonal polynomials [13,14], except for a constant factor, on $-1 \leq x \leq 1$ with respect to $W(x) = \frac{1}{\sqrt{1-x^2}}$. They are a special case of the classical Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, and the interrelation is given by [13]

$$T_k(x) = \frac{2^{2k}(k!)^2}{(2k)!} P_k^{(-\frac{1}{2},-\frac{1}{2})}(x). \tag{4}$$

For more details, see [13,14] and references therein.

Although these polynomials are traditionally defined for $-1 \leq x \leq 1$, for analysis and numerical computational purposes, it is more convenient to use the interval $0 \leq x \leq 1$. Thus, shifted Chebyshev polynomials of the first kind, $T_k^*(x)$, which is defined [15] as $T_k^*(x) = \cos[k \arccos(2x-1)] = T_k(2x-1)$ for $x \in [0,1]$. The shifted Chebyshev polynomials of the first kind, $T_k^*(x)$, are orthogonal on the support interval $0 \leq x \leq 1$ with $(x-x^2)^{-1/2}$ as a weight function.

Generalized orthogonal polynomials were first considered by [16] and developed by [17]. For $K_0, K_1 \geq 0$, a characterization of the orthogonal generalized shifted Chebyshev-I polynomials of degree $j$ are defined on the interval $0 \leq x \leq 1$ in [10] by

$$\mathscr{T}_j^{*(K_0,K_1)}(x) = \frac{(2j-1)!!}{(2j)!!} T_j^*(x) + \sum_{i=0}^{j} \frac{(2i)!\lambda_i}{2^{2i}(i!)^2} T_i^*(x), \quad j = 0, \ldots, \tag{5}$$

with respect to the measure $\frac{1}{\pi}(1-x^2)^{-1/2} + K_0\delta_0 + K_1\delta_1$, where $\delta_x$ is a singular Dirac measure, $T_j^*(x)$ is the $j$th degree shifted Chebyshev-I polynomial, and

$$\lambda_k = \left[ \frac{2K_0\Gamma(k+\frac{1}{2})}{\Gamma(k-\frac{1}{2})} + \frac{2K_1\Gamma(k+\frac{1}{2})}{\Gamma(k-\frac{1}{2})} + 4K_0K_1 \right]. \tag{6}$$

The double factorial, $j!!$, of an integer $j$ is defined in [18] as $(2j-1)!! = (2j-1)(2j-3)(2j-5)\ldots(3)(1)$ when $j$ is odd and as $j!! = (j)(j-2)(j-4)\ldots(4)(2)$ when $j$ is even, which can be written as

$$j!! = \begin{cases} 2^{\frac{j}{2}}(\frac{j}{2})! & \text{if } j \text{ is even} \\ \frac{j!}{2^{\frac{j-1}{2}}(\frac{j-1}{2})!} & \text{if } j \text{ is odd} \end{cases} \tag{7}$$

where $0!! = (-1)!! = 1$. From the double factorial definition [18] and the fact that $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, we can write $\Gamma(j+\frac{1}{2}) = \frac{(2j-1)!!}{2^j}\sqrt{\pi}$ and $\Gamma(\frac{1}{2}-j) = \frac{(-2)^j}{(2j-1)!!}\sqrt{\pi}$. In addition, from Equation (7), we can derive

$$(2j)!! = [2(j)][2(j-1)]\ldots[2.1] = 2^j j! \tag{8}$$

and

$$(2j)! = [(2j-1)(2j-3)\ldots(1)].([2(j)][2(j-1)][2(j-2)]\ldots[2(1)]) = (2j-1)!!2^j j!. \tag{9}$$

Theorem 1 [10] illustrates how generalized shifted Chebyshev-I polynomials $\mathscr{T}_r^{*(K_0,K_1)}(x)$ of $\deg = r$ can be expressed as a span of Bernstein polynomial basis.

**Theorem 1** ([10]). *For $K_0, K_1 \geq 0$, the rth degree generalized shifted Chebyshev-I polynomials $\mathscr{T}_r^{*(K_0,K_1)}(x)$ have the next representation using Bernstein polynomial basis,*

$$\mathscr{T}_r^{*(K_0,K_1)}(x) = \frac{(2r-1)!!}{(2r)!!} \sum_{i=0}^{r} (-1)^{r-i} \eta_{i,r} \mathbb{B}_i^r(x) + \sum_{j=0}^{r} \frac{(2j)!\lambda_j}{2^{2j}(j!)^2} \sum_{l=0}^{j} (-1)^{j-l} \eta_{l,j} \mathbb{B}_l^j(x) \tag{10}$$

*where $\lambda_j$ is defined in (6), and $\eta_{i,r} = \frac{\binom{2r}{r}\binom{2r}{2i}}{2^{2r}\binom{r}{i}}$, $i = 0, 1, \ldots, r$. Furthermore, we can write $\frac{(2r-1)!!}{(2r)!!} = \frac{(2r)!}{2^{2r}(r!)^2}$ using Equations (8) and (9).*

Interestingly, it is worth mentioning that the above representation is related to the semicircular law of the random matrix distribution [19].

In this section, explicit forms of the transformation matrix for the generalized shifted Chebyshev-I polynomials basis into the Bernstein polynomials basis, and the inverse transformation matrix that converts the Bernstein polynomials basis into the generalized shifted Chebyshev-I polynomials basis were provided.

## 2. Results: Bases Transformations

At the beginning, we provide the matrix transformation of the generalized shifted Chebyshev-I to the Bernstein basis in Section 2.1, and in Section 2.2 we provide the transformation matrix of the Bernstein basis to the generalized shifted Chebyshev-I basis.

### *2.1. Generalized Shifted Chebyshev-I to Bernstein*

In the following, we generalize the technique introduced by A. Rababah in [8], where some results concerning the classical Chebyshev case were provided. Theorem 2 will be needed to associate the superb performance of the least squares of the generalized shifted Chebyshev-I polynomials with the geometric perceptions of the Bernstein polynomials basis.

**Theorem 2.** *The entries $\boldsymbol{A}^n_{i,r}$, $r, i = 0, 1, \ldots, n$ of the transformation matrix of the generalized shifted Chebyshev-I polynomials basis into the nth degree Bernstein polynomial basis is given by*

$$\begin{aligned}\boldsymbol{A}^n_{i,r} =& \frac{(2r)!}{2^{2r}\binom{r}{i}(r!)^2}\sum_{l=\max(0,i+r-n)}^{\min(i,r)}(-1)^{r-l}\binom{n-r}{i-l}\binom{r-\frac{1}{2}}{l}\binom{r-\frac{1}{2}}{r-l}\\ &+\sum_{j=0}^{r}\lambda_j\frac{(2j)!}{2^{2j}\binom{j}{i}(j!)^2}\sum_{k=\max(0,i+j-n)}^{\min(i,j)}(-1)^{j-k}\binom{n-j}{i-k}\binom{j-\frac{1}{2}}{k}\binom{j-\frac{1}{2}}{j-k}.\end{aligned} \tag{11}$$

**Proof.** Express an $n$th degree polynomial, $g_n(x)$, as a span of the Bernstein polynomial basis as $g_n(x) = \sum_{r=0}^{n} c_r \mathbb{B}^n_r(x)$, and as a linear combination of the generalized shifted Chebyshev-I polynomials as $g_n(x) = \sum_{i=0}^{n} d_i \mathscr{T}_i^{*(K_0,K_1)}(x)$.

We want to obtain a matrix, $\boldsymbol{A}$, which converts the coefficients $\{d_i\}_{i=0}^n$ of the generalized shifted Chebyshev-I polynomials into Bernstein coefficients $\{c_r\}_{r=0}^n$, in $g_n(x) = \sum_{r=0}^{n} c_r \mathbb{B}^n_r(x) = \sum_{i=0}^{n} d_i \mathscr{T}_i^{*(K_0,K_1)}(x)$, i.e., $c_i = \sum_{r=0}^{n} \boldsymbol{A}^n_{i,r} d_r$. Further, express $\mathscr{T}_r^{*(K_0,K_1)}(x)$ with respect to the $n$th degree Bernstein polynomial basis as

$$\mathscr{T}_r^{*(K_0,K_1)}(x) = \sum_{i=0}^{n} \boldsymbol{D}^n_{r,i}\mathbb{B}^n_i(x), \qquad \text{for} \quad r = 0, 1, \ldots, n, \tag{12}$$

where $\boldsymbol{D}^n_{r,i}$ denote the entries of conversion matrix $\boldsymbol{D}$, of dimension $(n+1)\times(n+1)$. Thus, we can write the elements of vector $d$ as $d_i = \sum_{r=0}^{n} c_r \boldsymbol{D}^n_{r,i}$, where it is clear that $\boldsymbol{D}^T = \boldsymbol{A}$ by comparing $d_i = \sum_{r=0}^{n} c_r \boldsymbol{D}^n_{r,i}$ and $c_i = \sum_{r=0}^{n} \boldsymbol{A}^n_{i,r} d_r$.

Use values of $\eta_{i,r}$ and $\eta_{l,j}$ defined in Theorem 1 to rewrite Equation (10) as

$$\mathscr{T}_r^{*(K_0,K_1)}(x) = \frac{(2r)!}{2^{2r}(r!)^2}\sum_{i=0}^{r}(-1)^{r-i}\frac{\binom{2r}{r}\binom{2r}{2i}}{2^{2r}\binom{r}{i}}\mathbb{B}^r_i(x) + \sum_{j=0}^{r}\frac{(2j)!\lambda_j}{2^{2j}(j!)^2}\sum_{l=0}^{j}(-1)^{j-l}\frac{\binom{2j}{j}\binom{2j}{2l}}{2^{2j}\binom{j}{l}}\mathbb{B}^j_l(x)$$

and then apply the combinatorial identity $\binom{r-\frac{1}{2}}{r-l}\binom{r-\frac{1}{2}}{l} = \frac{\binom{2r}{r}\binom{2r}{2l}}{2^{2r}}$ and the Bernstein symmetry relation $\mathbb{B}^n_i(x) = \mathbb{B}^n_{n-i}(1-x)$ to obtain

$$\mathscr{T}_r^{*(K_0,K_1)}(x) = \frac{(2r)!}{2^{2r}(r!)^2}\sum_{i=0}^{r}(-1)^{r-i}\frac{\binom{r-\frac{1}{2}}{i}\binom{r-\frac{1}{2}}{r-i}}{\binom{r}{i}}\mathbb{B}_i^r(x) + \sum_{j=0}^{r}\frac{(2j)!\lambda_j}{2^{2j}(j!)^2}\sum_{l=0}^{j}(-1)^{j-l}\frac{\binom{j-\frac{1}{2}}{l}\binom{j-\frac{1}{2}}{j-l}}{\binom{j}{l}}\mathbb{B}_l^j(x)$$
$$= \frac{(2r)!}{2^{2r}(r!)^2}\sum_{i=0}^{r}\frac{\binom{r-\frac{1}{2}}{r-i}\binom{r-\frac{1}{2}}{i}}{\binom{r}{r-i}}\mathbb{B}_{r-i}^r(x) + \sum_{j=0}^{r}\frac{(2j)!\lambda_j}{2^{2j}(j!)^2}\sum_{l=0}^{j}\frac{\binom{j-\frac{1}{2}}{j-l}\binom{j-\frac{1}{2}}{l}}{\binom{j}{j-l}}\mathbb{B}_{j-l}^j(x).$$

Now, use degree elevation defined in Equation (1) introduced in [4] to exchange Bernstein polynomials $\mathbb{B}_{r-i}^r(x)$ and $\mathbb{B}_{j-l}^j(x)$ of degrees $r, j$, respectively, to the $n$th degree Bernstein polynomial, reorder the summations, and compare it with Equation (12) to attain the entries of the matrix $\boldsymbol{D}$ as

$$\boldsymbol{D}_{r,i}^n = \frac{(2r)!}{2^{2r}\binom{r}{i}(r!)^2}\sum_{l=\max(0,i+r-n)}^{\min(i,r)}(-1)^{r-l}\binom{n-r}{i-l}\binom{r-\frac{1}{2}}{l}\binom{r-\frac{1}{2}}{r-l}$$
$$+\sum_{j=0}^{r}\lambda_j\frac{(2j)!}{2^{2j}\binom{j}{i}(j!)^2}\sum_{k=\max(0,i+j-n)}^{\min(i,j)}(-1)^{j-k}\binom{n-j}{i-k}\binom{j-\frac{1}{2}}{k}\binom{j-\frac{1}{2}}{j-k}.$$

Transposing will get the desired entries $\boldsymbol{A}_{i,r}^n$ of the matrix $\boldsymbol{A}$. □

Many applications in the numerical analysis [1,4] might have Bézier curves of different degrees or require a Bézier curve of a higher degree. Corollary 1 uses Bézier's degree elevation defined by [4] to express the $r$th degree, $r \le n$, $\mathscr{T}_r^{*(K_0,K_1)}(x)$ with respect to the Bernstein basis of a higher degree, say $n$, which will help in improving the numerical stability and the efficiency of calculations.

**Corollary 1.** *The generalized shifted Chebyshev-I polynomials $\mathscr{T}_r^{*(K_0,K_1)}(x)$ of degree $r$ where $0 \le r \le n$ can be written with respect to $n$th degree Bernstein basis as*

$$\mathscr{T}_r^{*(K_0,K_1)}(x) = \sum_{i=0}^{n}\boldsymbol{D}_{r,i}^n\mathbb{B}_i^n(x), \quad r = 0, 1, \ldots, n,$$

*where $\boldsymbol{D}_{r,i}^n = \mu_{r,i}^n + \sum_{l=0}^{r}\lambda_l\mu_{l,i}^n$ and $\mu_{r,i}^n$ is defined as*

$$\mu_{r,i}^n = \frac{(2r)!}{2^{2r}(r!)^2}\sum_{l=\max(0,i+r-n)}^{\min(i,r)}(-1)^{r-l}\frac{\binom{n-r}{i-l}\binom{2r}{r}\binom{2r}{2l}}{2^{2r}\binom{n}{i}}.$$

**Proof.** From Equation (12) and the proof of Theorem 2, the generalized shifted Chebyshev-I polynomials $\mathscr{T}_r^{*(K_0,K_1)}(x)$ of degree $r$, where $r \le n$ can be stated with respect to the fixed $n$th degree Bernstein polynomial basis as $\mathscr{T}_r^{*(K_0,K_1)}(x) = \sum_{i=0}^{n}\boldsymbol{D}_{r,i}^n\mathbb{B}_i^n(x)$, for $r = 0, \ldots, n$; such that the entries of the matrix $\boldsymbol{A}$ can be obtained by transposing the matrix $\boldsymbol{D}$ defined in (11). Note that

$$\binom{r-\frac{1}{2}}{r-l}\binom{r-\frac{1}{2}}{l} = \frac{(2r-1)!!}{2^r(r-l)!}\frac{2^l}{(2l-1)!!}\frac{(2r-1)!!}{2^r l!}\frac{2^{r-l}}{(2(r-l)-1)!!}$$
$$= \frac{1}{2^r(r-l)!l!}\frac{(2r-1)!!}{(2l-1)!!}\frac{(2r-1)!!}{(2(r-l)-1)!!}.$$

and using $(2r)! = (2r-1)!!2^r r!$, we attain

$$\frac{\binom{n-r}{i-l}\binom{r-\frac{1}{2}}{r-l}\binom{r-\frac{1}{2}}{l}}{\binom{n}{i}} = \frac{\binom{n-r}{i-l}\binom{2r}{r}\binom{2r}{2l}}{2^{2r}\binom{n}{i}}. \tag{13}$$

Therefore, entries $\boldsymbol{D}^n_{r,i}$ can be rewritten as

$$\boldsymbol{D}^n_{r,i} = \mu^n_{r,i} + \sum_{l=0}^{r} \lambda_l \mu^n_{l,i}$$

where

$$\mu^n_{r,i} = \frac{(2r)!}{2^{2r}(r!)^2} \sum_{l=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-l} \frac{\binom{n-r}{i-l}\binom{2r}{r}\binom{2r}{2l}}{2^{2r}\binom{n}{i}}.$$

□

*2.2. Bernstein to Generalized Shifted Chebyshev-I*

In the introduction, significant analytical and geometrical properties for Bernstein polynomials are discussed. It is noteworthy that $\int_0^1 \mathbb{B}^n_v(t)dt = \frac{1}{n+1}$, $v = 0,1,\ldots,n$, and the product of two Bernstein polynomials $\mathbb{B}^n_i(x) * \mathbb{B}^m_j(x)$ is a Bernstein polynomial that equals $\frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}}\mathbb{B}^{n+m}_{i+j}(x)$. Now, Theorem 3 provides the orthogonality relation between the Bernstein basis and the generalized shifted Chebyshev-I polynomials, which will be used in the proof of the next conclusion (Theorem 4).

**Theorem 3** ([10])**.** *Let $\mathbb{B}^n_r(x)$ be the nth degree Bernstein polynomial and $\mathscr{T}_i^{*(K_0,K_1)}(x)$ be the ith degree generalized shifted Chebyshev-I polynomials. For $r,i = 0,1,\ldots,n$, we obtain*

$$\begin{aligned}
&\int_0^1 (x-x^2)^{-\frac{1}{2}} \mathbb{B}^n_r(x) \mathscr{T}_i^{*(K_0,K_1)}(x)dx \\
&= \binom{n}{r} \frac{(2i)!}{2^{2i}(i!)^2} \sum_{l=0}^{i} (-1)^{i-l} \binom{i-\frac{1}{2}}{l} \binom{i-\frac{1}{2}}{i-l} \frac{\Gamma(r+l+\frac{1}{2})\Gamma(n+i-r-l+\frac{1}{2})}{\Gamma(n+i+1)} \\
&+ \sum_{d=0}^{i} \lambda_d \binom{n}{r} \frac{(2d)!}{2^{2d}(d!)^2} \sum_{j=0}^{d} (-1)^{d-j} \binom{d-\frac{1}{2}}{j} \binom{d-\frac{1}{2}}{d-j} \frac{\Gamma(r+j+\frac{1}{2})\Gamma(n+d-r-j+\frac{1}{2})}{\Gamma(n+d+1)}.
\end{aligned}$$

**Proof.** For the proof, see [10]. □

Now, Theorem 4, we find the entries of $\boldsymbol{A}^{-1}$, the inverse of the transformation matrix $\boldsymbol{A}$ found in Theorem 2.

**Theorem 4.** *The entries $\boldsymbol{A}^{n^{-1}}_{i,r}$, $i,r = 0,1,\ldots,n$ of the inverse of the transformation matrix, $\boldsymbol{A}^{-1}$, which converts the Bernstein polynomial basis into the nth degree generalized shifted Chebyshev-I polynomials, are written as*

$$\boldsymbol{A}^{n^{-1}}_{i,r} = \frac{\Phi_i}{(1+\lambda_i)^2}\binom{n}{r}\left[\frac{2^{2i}(i!)^2}{(2i)!}\Psi^{n,r}_{l,i} + \left(\frac{2^{2i}(i!)^2}{(2i)!}\right)^2 \sum_{d=0}^{i} \frac{(2d)!\lambda_d}{2^{2d}(d!)^2}\Psi^{n,r}_{j,d}\right] \tag{14}$$

*where $\lambda_i$ defined in (6),* $\Phi_i = \begin{cases} 2/\pi & \text{if } i = 0 \\ 1/\pi & \text{if } i \neq 0 \end{cases}$,

$$\Psi^{n,r}_{l,i} = \sum_{l=0}^{i} \frac{(-1)^{i-l}}{2^{2i}} \binom{2i}{i} \binom{2i}{2l} \frac{\Gamma(r+l+\frac{1}{2})\Gamma(n+i-r-l+\frac{1}{2})}{\Gamma(n+i+1)}.$$

**Proof.** To be able to transform the Bernstein polynomial basis to the $n$th degree generalized shifted Chebyshev-I polynomials basis, we invert the transformation $c = \boldsymbol{A}.d$. Let $\boldsymbol{A}^{n^{-1}}_{i,r}, \boldsymbol{D}^{n^{-1}}_{i,r}$, $r,i = 0,1,\ldots,n$ be the entries of the matrices $\boldsymbol{A}^{-1}$ and $\boldsymbol{D}^{-1}$, respectively. Then, the change in basis transformation of the Bernstein polynomial into the $n$th degree generalized shifted Chebyshev-I polynomials is written as

$$\mathbb{B}_r^n(x) = \sum_{i=0}^{n} \boldsymbol{D}_{r,i}^{n^{-1}} \mathscr{T}_i^{*(K_0,K_1)}(x). \tag{15}$$

The entries $\boldsymbol{D}_{r,i}^{n^{-1}}, i, r = 0, 1, \ldots, n$ can be set by multiplying (15) by $(x - x^2)^{-\frac{1}{2}} \mathscr{T}_i^{*(K_0,K_1)}(x)$ and integrating over $[0, 1]$ to obtain

$$\int_0^1 (x - x^2)^{-\frac{1}{2}} \mathbb{B}_r^n(x) \mathscr{T}_i^{*(K_0,K_1)}(x) dx = \sum_{i=0}^{n} \boldsymbol{D}_{r,i}^{n^{-1}} \int_0^1 (x - x^2)^{-\frac{1}{2}} \mathscr{T}_i^{*(K_0,K_1)}(x) \mathscr{T}_i^{*(K_0,K_1)}(x) dx \tag{16}$$

where $\mathscr{T}_j^{*(K_0,K_1)}(x)$ defined in (5) by $\mathscr{T}_j^{*(K_0,K_1)}(x) = \frac{(2j)!}{2^{2j}(j!)^2} T_j^*(x) + \sum_{i=0}^{j} \frac{(2i)!\lambda_i}{2^{2i}(i!)^2} T_i^*(x)$.

Substituting $\mathscr{T}_i^{*(K_0,K_1)}(x)$ into Equation (16), and using the orthogonality relation [13,20] of the univeriate shifted Chebyshev's polynomials of the first kind, $T_i^*(x) = \cos[i \arccos(2x - 1)]$ given as

$$\int_0^1 (x - x^2)^{-\frac{1}{2}} T_i^*(x) T_j^*(x) dx = \begin{cases} 0 & \text{if } j \neq i \\ \pi & \text{if } j = i = 0 \\ \frac{\pi}{2} & \text{if } j = i = 1, 2, \ldots \end{cases}.$$

We then obtain

$$\int_0^1 (x - x^2)^{-\frac{1}{2}} \mathbb{B}_r^n(x) \mathscr{T}_i^{*(K_0,K_1)}(x) dx = \begin{cases} \pi \boldsymbol{D}_{r,i}^{n^{-1}} \left(\frac{(2i)!}{2^{2i}(i!)^2}\right)^2 (1 + \lambda_i)^2 & \text{if } i = 0 \\ \frac{\pi}{2} \boldsymbol{D}_{r,i}^{n^{-1}} \left(\frac{(2i)!}{2^{2i}(i!)^2}\right)^2 (1 + \lambda_i)^2 & \text{if } i \neq 0 \end{cases}.$$

Thus, by using Theorem 3, we obtain

$$\begin{aligned} \boldsymbol{D}_{r,i}^{n^{-1}} &= \frac{\Phi_i \binom{n}{r}}{(1 + \lambda_i)^2} [\frac{2^{2i}(i!)^2}{(2i)!} \sum_{l=0}^{i} \frac{(-1)^{i-l}}{2^{2i}} \binom{2i}{i} \binom{2i}{2l} \frac{\Gamma(r + l + \frac{1}{2})\Gamma(n + i - r - l + \frac{1}{2})}{\Gamma(n + i + 1)} \\ &+ \left(\frac{2^{2i}(i!)^2}{(2i)!}\right)^2 \sum_{d=0}^{i} \frac{(2d)!\lambda_d}{2^{2d}(d!)^2} \sum_{j=0}^{d} \frac{(-1)^{d-j}}{2^{2d}} \binom{2d}{d} \binom{2d}{2j} \frac{\Gamma(r + j + \frac{1}{2})\Gamma(n + d - r - j + \frac{1}{2})}{\Gamma(n + d + 1)}]. \end{aligned}$$

The terms can be rearranged to obtain the entries of the matrix $\boldsymbol{D}^{-1}$ in the form

$$\boldsymbol{D}_{r,i}^{n^{-1}} = \frac{\Phi_i}{(1 + \lambda_i)^2} \binom{n}{r} \left[ \frac{2^{2i}(i!)^2}{(2i)!} \Psi_{l,i}^{n,r} + \left(\frac{2^{2i}(i!)^2}{(2i)!}\right)^2 \sum_{d=0}^{i} \frac{(2d)!\lambda_d}{2^{2d}(d!)^2} \Psi_{j,d}^{n,r} \right], \tag{17}$$

where $\Phi_i = \begin{cases} 1/\pi & \text{if } i = 0 \\ 2/\pi & \text{if } i \neq 0 \end{cases}$, $\lambda_i$ defined in (6), and

$$\Psi_{l,i}^{n,r} = \sum_{l=0}^{i} \frac{(-1)^{i-l}}{2^{2i}} \binom{2i}{i} \binom{2i}{2l} \frac{\Gamma(r + l + \frac{1}{2})\Gamma(n + i - r - l + \frac{1}{2})}{\Gamma(n + i + 1)}.$$

The desired entries $\boldsymbol{A}_{i,r}^{n^{-1}}, i, r = 0, 1, \ldots, n$ of the matrix $\boldsymbol{A}^{-1}$ are then found by transposing the matrix $\boldsymbol{D}^{-1}$. □

Hence, we applied Theorem 2, the matrix transformation, $\boldsymbol{A}$, of the generalized shifted Chebyshev-I polynomials basis to a fixed $n$th degree Bernstein polynomial basis. Moreover, Corollary 1 will enable us to improve stability and efficiency by rewriting $\mathscr{T}_r^{*(K_0,K_1)}(x)$ in terms of a Bézier curve of higher degrees. We conclude the section with Theorem 4, with the entries $\boldsymbol{A}_{i,r}^{n^{-1}}, i, r = 0, 1, \ldots, n$ of the inverse of $\boldsymbol{A}$.

## 3. Discussion

Research in the area of orthogonal polynomials has gained great attention. They are vital to the efficiency and stability of numerical techniques. In this article, an interrelation between ordinary Chebyshev polynomials $T_r(x)$, ordinary shifted Chebyshev polynomials $T_r^*(x)$, and Jacobi polynomials $P_r^{(\alpha,\beta)}(x)$ are given. In addition, an explicit form of generalized shifted Chebyshev-I polynomials $\mathscr{T}_r^{*(K_0,K_1)}(x)$ using ordinary Chebyshev polynomials is provided. In addition, the definition of the orthogonal polynomials using cosine function leads to new discoveries in trigonometry identities. Moreover, a characterization of the generalized shifted Chebyshev-I polynomials of degree $r$ using the Bernstein basis of degree $r \leq n$ is discussed, where degree elevation can be used to rewrite $\mathscr{T}_r^{*(K_0,K_1)}(x)$ in terms of a higher degree Bernstein basis, since applications (see [1,4]) might have two or more Bézier curves of different degrees that require equal degree or higher degree Bézier curves. In addition, an explicit form of the entries of the transformation matrix, $\boldsymbol{A}$, $\boldsymbol{A}_{i,r}^{n}$, $i, r = 0, 1, \ldots, n$, can transform the generalized shifted Chebyshev-I polynomials basis into the Bernstein polynomials basis (Theorem 2). However, Bernstein polynomials are not orthogonal and cannot be used efficiently in approximation problems [7]. Therefore, approximations using orthogonal polynomials as bases have an advantage. An explicit form of the entries of the transformation matrix, $\boldsymbol{A}^{-1}$, $\boldsymbol{A}_{i,r}^{n^{-1}}$, $i, r = 0, 1, \ldots, n$, can transform the Bernstein polynomial basis into the basis of the generalized shifted Chebyshev-I polynomials of $\deg = n$.

*Applications*

Exploring new systems of orthogonal polynomials helps in the discovery of applications in many areas, such as integro-differential and Fredholm integral equations, spectral element methods for ODEs and PDEs, splines, computation probability and data integration, fractional differential equations, stochastic differential equations, and stochastic dynamics. Future developments and numerous ideas to expand the scope of this article exist. Some ideas are mentioned at the end of Section 1: constructing bivariate generalized shifted Jacobi polynomials on a simplex, formulating basis transformations for generalized Jacobi Koornwinder's type polynomials, constructing various degree elevation/reductions of Bézier surfaces and curves, and computing numerical differentiations/integrations, integral transforms, cubature formulas, and Fourier integrals/transforms.

**Author Contributions:** M.A.A. and M.N.A. contributed equally to this work.

**Funding:** This research received no external funding.

**Acknowledgments:** The authors would like to thank the anonymous referees for their comments that helped us to improve this article.

**Conflicts of Interest:** The authors declare no conflict of interest.

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