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Positive Solutions for a Three-Point Boundary Value Problem of Fractional *Q*-Difference Equations

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Abstract: In this work, a three-point boundary value problem of fractional q-difference equations is discussed. By using fixed point theorems on mixed monotone operators, some sufficient conditions that guarantee the existence and uniqueness of positive solutions are given. In addition, an iterative scheme can be made to approximate the unique solution. Finally, some interesting examples are provided to illustrate the main results.

Keywords: fractional q-difference equation; existence and uniqueness; positive solutions; fixed point theorem on mixed monotone operators

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1. Introduction

We will deal with a fractional q-difference equation subject to three-point boundary conditions

$$\begin{cases} D_q^{\alpha}x(t)+f(t,x(t),x(t))+g(t,x(t))=0, \quad 0<t<1,\ 2<\alpha<3, \\ x(0)=D_qx(0)=0,\ D_qx(1)=\beta D_qx(\eta), \end{cases} \tag{1}$$

where $0<\beta\eta^{\alpha-2}<1$, $0<q<1$, D_q^{α} is the Riemann–Liouville fractional q-derivative of order α.

Due to fast development in fractional calculus, many researchers studied q-difference calculus or quantum calculus. For this topic, the earlier results can be seen in Al-Salam [1] and Agarwal [2], and some recent results related to q-difference calculus in [3–15] and some references therein. Nowadays, fractional q-difference calculus has been given in wide applications of different science areas, which include basic hyper-geometric functions, mechanics, the theory of relativity, combinatorics and discrete mathematics. So many mathematical models have been abstracted out(see [16–18]) and problem (1) is one of the models. Therefore, fractional q-difference calculus has been of great interest and many good results can be found in [5–8] and references therein. Recently, the fruits about fractional q-difference equation boundary value problems emerge continuously. For different problems of fractional q-difference equations, the existence and the uniqueness of solutions have been always considered in literature. To solve these boundary value problems, some techniques have been applied, such as the monotone iterative technique, the lower-upper solution method, the Schauder fixed point theorem and the Krasnoselskii fixed point theorem. For details, one can see [13–15,19–25].

In [15], Liang and Zhang considered the existence and uniqueness of positive nondecreasing solutions for a fractional q-difference equation involving three-point boundary conditions

$$\begin{cases} D_q^{\alpha}x(t)+f(t,x(t))=0, \quad 2<\alpha<3,\ 0<t<1, \\ x(0)=D_qx(0)=0,\ D_qx(1)=\beta D_qx(\eta), \end{cases} \tag{2}$$

where $0 < \beta\eta^{\alpha-2} < 1$. They gave some sufficient conditions for Label (2), and their tool is a fixed point theorem in partially ordered sets.

In [19], Sriphanomwan et al. investigated the problem of fractional q-difference equations

$$\begin{cases} D_q^{\alpha}(D_q^{\beta}(1+p(t)))x(t) = f(t, x(t), D_{\theta}^{\mu}x(t), \Psi_{\omega}^{v}x(t)), \\ x(0) = x(\eta), \quad I_r^{\gamma}x(T) = \int_0^T \frac{(T-rs)^{(\gamma-1)}}{\Gamma_r(\gamma)}x(s)d_rs = g(x), \end{cases} \tag{3}$$

where $t \in I_\chi^T := \{\chi^k T : k \in \mathbf{N} \cup \{0, T\}\}, 0 < \alpha, \beta, \mu \leq 1, 1 < \alpha + \beta \leq 2, v, \gamma > 0, \eta \in I_\chi^T - \{0, T\}$, and p, q, r, θ, ω are simple fractions. The existence and uniqueness of solutions for Label (3) was obtained. The used methods are the Banach contraction mapping principle and Krasnosel'skii fixed point theorem.

By using Schauder fixed point theorem and the Banach fixed point theorem, Yang [25] discussed a fractional q-difference equation with three-point boundary conditions:

$$\begin{cases} D_q^{\alpha}x(t) + f(t, x(t)) = 0, \quad 0 \leq t \leq 1, \ 1 < \alpha \leq 2, \\ x(0) = 0, \ x(1) = \beta x(\xi), \end{cases} \tag{4}$$

where $0 < \beta\xi^{\alpha-1} < 1, \ 0 < \xi < 1$. The author gave the existence and uniqueness of positive solutions for Label (4).

In a very recent paper [24], the authors considered a special fractional q-difference equation with a three-point problem

$$\begin{cases} D_q^{\alpha}u(t) + f(t, u(t)) = b, \quad 0 < t < 1, \ 2 < \alpha < 3, \\ u(0) = D_q u(0) = 0, \ D_q u(1) = \beta D_q u(\eta), \end{cases} \tag{5}$$

where $0 < \beta\eta^{\alpha-2} < 1, \ 0 < q < 1, \ b \geq 0$ is a constant. The existence and uniqueness of solutions for Label (5) by using fixed point theorems for ψ-(h, r)-concave operators.

Motivated by [15,26], we consider the existence and uniqueness of positive solutions for Label (1). Different from the methods mentioned above, our tools are two fixed point theorems for mixed monotone operators. To the authors' knowledge, Label (1) is a new form of fractional q-difference equations. We can give the existence and uniqueness of solutions for Label (1). Furthermore, we can make an iteration to approximate the unique solution.

2. Preliminaries

Here, we list some concepts and lemmas of fractional q-calculus. One can see [1–8], for example. For $0 < q < 1$ and f defined on $[a, b]$, let

$$(I_q f)(t) = \int_0^t f(s)d_qs = (1-q)\sum_{n=0}^{\infty} f(tq^n)tq^n, \quad t \in [0, b].$$

Then,

$$\int_a^b f(t)d_qt = \int_a^c f(t)d_qt + \int_c^b f(t)d_qt, \quad \forall c \in [a, b].$$

Definition 1. *(See [3]). $\alpha \geq 0$ and f is defined on $[0, 1]$. The Riemann–Liouville fractional q-integral is $(I_q^0 f)(t) = f(t)$ and*

$$(I_q^{\alpha} f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s)d_qs, \ \alpha > 0.$$

Clearly, $(I_q^{\alpha} f)(t) = (I_q f)(t)$ when $\alpha = 1$.

Lemma 1. *(See [22]). If f, g are continuous on $[0, s]$ and $f(t) \leq g(t)$ for $t \in [0, s]$, then*

(i) $\int_0^s f(t)d_qt \leq \int_0^s g(t)d_qt$. *In addition, if $\alpha > 1$, then $I_q^\alpha f(s) \leq I_q^\alpha g(s)$, $t \in [0, s]$,*
(ii) $\left|\int_0^s f(t)d_qt\right| \leq \int_0^s |f(t)|d_qt$, $t \in [0, s]$.

Definition 2. *(See [3]). The Riemann–Liouville fractional q-derivative of order $\alpha \geq 0$ is*

$$(D_q^\alpha f)(t) = (D_q^n I_q^{n-\alpha} f)(t), \ \ \alpha > 0, \ t \in [0, 1],$$

where n denotes the smallest integer greater than or equal to α.

When $\alpha = 1$, $(D_q^\alpha f)(t) = D_q f(t)$. Furthermore,

$$(I_q^\alpha D_q^p f)(t) = (D_q^p I_q^\alpha f)(t) - \sum_{n=0}^{p-1} \frac{t^{\alpha-p+n}}{\Gamma_q(\alpha - p + n + 1)} (D_q^n f)(0), \ \ p \in \mathbf{N}.$$

Lemma 2. *If $f(t)$ is continuous with $f(t) \geq 0$ for $t \in [0, 1]$, and there is $t_0 \in (0, 1)$ such that $f(t_0) \neq 0$. Then,*

$$\int_0^1 f(t)d_qt > 0, \ t \in [0, 1],$$

where

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} q^n f(q^n), \ q \in (0, 1).$$

Proof. Because $f(t) \geq 0$ and $f(t_0) \neq 0$, there is $n_0 \in \mathbf{N}$ such that $t_0 = q^{n_0}$, then

$$f(q^{n_0})q^{n_0} > 0, 0 < q < 1,$$

and thus

$$(1-q)\sum_{n=0}^{\infty} q^n f(q^n) \geq (1-q) f(q^{n_0})q^{n_0} = (1-q) f(t_0)t_0 > 0.$$

Hence, we have $\int_0^1 f(t)d_qt > 0$. □

Here, we list other facts that are important in the sequel. See [26–30] for instance.

$(X, \|\cdot\|)$ is a real Banach space, its partial order induced by a cone K of X, i.e., $x \leq y$ if and only if $y - x \in K$. If there is $N > 0$ such that $\|x\| \leq N\|y\|$ for $\theta \leq x \leq y$, $x, y \in X$, then K is called normal, where θ denotes the zero element of X. The notation x–y denotes that there exist $\mu, \nu > 0$ such that $\mu x \leq y \leq \nu x$, $\forall\, x, y \in X$. For fixed $h > \theta$, define a set $K_h = \{x \in E \mid x \sim h\}$. Then, $K_h \subset K$.

Definition 3. *(See [27]). Suppose $T : K \to K$ is a given operator. If*

$$T(tx) \geq tTx, \ \forall t \in (0, 1), \ x \in K, \tag{6}$$

then T is said to be sub-homogeneous.

Definition 4. *(See [27]). Let $0 \leq \gamma < 1$. An operator $T : K \to K$ satisfies*

$$T(tx) \geq t^\gamma Tx, \ \forall t \in (0, 1), \ x \in K. \tag{7}$$

Then, T is said to be γ-concave.

Lemma 3. *(See [27]). Let $h > \theta$, $0 < \gamma < 1$, $T_1 : K \times K \to K$ be a mixed monotone operator and*

$$T_1(tx, t^{-1}y) \geq t^{\gamma} T_1(x, y), \; \forall t \in (0, 1), \; x, y \in K. \tag{8}$$

$T_2 : K \to K$ is an increasing sub-homogeneous operator. Moreover,

(i) there exists $h_0 \in K_h$ such that $T_1(h_0, h_0), T_2h_0 \in K_h$;
(ii) there exists $\sigma > 0$ such that $T_1(x, y) \geq \sigma T_2x, \; x, y \in K$.

Then:

(a) $T_1 : K_h \times K_h \to K_h$ and $T_2 : K_h \to K_h$;
(b) there are $u_0, v_0 \in K_h$ and $\tau \in (0, 1)$ satisfying

$$\tau v_0 \leq u_0 < v_0, \; u_0 \leq T_1(u_0, v_0) + T_2u_0 \leq T_1(v_0, u_0) + T_2v_0 \leq v_0;$$

(c) $T_1(x, x) + T_2x = x$ exists a unique solution x^* in K_h;
(d) for $x_0, y_0 \in K_h$, set

$$x_n = T_1(x_{n-1}, y_{n-1}) + T_2x_{n-1}, \; y_n = T_1(y_{n-1}, x_{n-1}) + T_2y_{n-1}, \; n = 1, 2, \ldots,$$

then $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.

Lemma 4. *(See [27]). Let $h > \theta$, $0 < \gamma < 1$, $T_1 : K \times K \to K$ be a mixed monotone operator and*

$$T_1(tx, t^{-1}y) \geq t T_1(x, y), \; \forall t \in (0, 1), \; x, y \in K. \tag{9}$$

$T_2 : K \to K$ is an increasing γ-concave operator. Moreover,

(i) there exists $h_0 \in K_h$ such that $T_1(h_0, h_0), T_2h_0 \in K_h$;
(ii) there exists $\sigma > 0$ such that $T_1(x, y) \leq \sigma T_2x, \; x, y \in K$.

Then:

(a) $T_1 : K_h \times K_h \to K_h$ and $T_2 : K_h \to K_h$;
(b) there are $u_0, v_0 \in K_h$ and $\tau \in (0, 1)$ satisfying

$$\tau v_0 \leq u_0 < v_0, \; u_0 \leq T_1(u_0, v_0) + T_2u_0 \leq T_1(v_0, u_0) + T_2v_0 \leq v_0;$$

(c) $T_1(x, x) + T_2x = x$ exists a unique solution x^* in K_h;
(d) for $x_0, y_0 \in K_h$, set

$$x_n = T_1(x_{n-1}, y_{n-1}) + T_2x_{n-1}, \; y_n = T_1(y_{n-1}, x_{n-1}) + T_2y_{n-1}, \; n = 1, 2, \ldots,$$

then $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.

Remark 1. *From Lemmas 3 and 4, we have two special cases:*

(i) Let $T_2 = \theta$ in Lemma 3, we get the corresponding conclusion (see Corollary 2.2 in [27]);
(ii) Let $T_1 = \theta$ in Lemma 4, we have the corresponding conclusion (see Theorem 2.7 in [31]).

3. Main Results

By using Lemmas 3 and 4, we will establish our main results for Label (1). Consider a Banach space $X = C[0, 1]$, the norm is $\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$. Set $K = \{x \in C[0, 1] | x(t) \geq 0, \; t \in [0, 1]\}$, a normal cone.

Lemma 5. *(See [15]). Let $g \in C[0,1]$, $\beta\eta^{\alpha-2} \neq 1$ and $0 < \eta < 1$, then the unique solution of following three-point problem*

$$\begin{cases} D_q^\alpha x(t) + g(t) = 0,\ 0 < t < 1,\ 2 < \alpha < 3, \\ x(0) = D_q x(0) = 0,\ \ D_q x(1) = \beta D_q x(\eta) \end{cases} \tag{10}$$

is

$$x(t) = \int_0^1 G(t,qs)g(s)d_qs + \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)g(s)d_qs, \tag{11}$$

where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} (1-s)^{(\alpha-2)}t^{\alpha-1} - (t-s)^{(\alpha-1)},\ 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-2)}t^{\alpha-1},\ 0 \le t \le s \le 1, \end{cases} \tag{12}$$

$$\begin{aligned} H(t,s) &= \ {}_tD_qG(s,t) \\ &= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\begin{cases} (1-s)^{(\alpha-2)}t^{\alpha-2} - (t-s)^{(\alpha-2)},\ 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-2)}t^{\alpha-2},\ 0 \le t \le s \le 1. \end{cases} \end{aligned}$$

Lemma 6. *(See [15]). For $G(t,qs)$ in (11), we obtain*

(1) $G(t,qs)$ is continuous and $G(t,qs) \ge 0,\ t,s \in [0,1]\times[0,1]$;
(2) $G(t,qs)$ is strictly increasing in $t \in [0,1]$.

Remark 2. *For $G(t,qs)$ in (11), we can easily get*

$$G(t,qs) \le \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-1},\ t,s \in [0,1]\times[0,1].$$

By (2) in Lemma 6, we have ${}_tD_qG(qs,t) \ge 0$, that is, $H(t,qs) \ge 0$. Obviously,

$$H(t,qs) \le \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-2} \le \frac{[\alpha-1]_q}{\Gamma_q(\alpha)},\ t,s \in [0,1]\times[0,1].$$

Next, four assumptions are listed:

(H_1) $f : [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ and $g : [0,1]\times[0,+\infty) \to [0,+\infty)$ are continuous;
(H_2) $f(t,u,v)$ is increasing relative to u for fixed $t \in [0,1]$ and $v \in [0,+\infty)$, decreasing relative to v for fixed $t \in [0,1]$ and $u \in [0,+\infty)$; $g(t,u)$ is increasing relative to u for fixed $t \in [0,1]$;
(H_3) for $\lambda \in (0,1), t \in [0,1], u \ge 0$, $g(t,\lambda u) \ge \lambda g(t,u)$ is satisfied, and there is $\gamma \in (0,1)$ such that $f(t,\lambda u,\lambda^{-1}v) \ge \lambda^\gamma f(t,u,v)$ for $u,v \ge 0$. In addition, $g(t,0) \not\equiv 0$;
(H_4) there exists $\sigma > 0$ such that $f(t,u,v) \ge \sigma g(t,u),\ \forall t \in [0,1], u,v \in [0,+\infty)$.

Theorem 1. *Let $(H_1)-(H_4)$ be satisfied, then*

(a) there are $u_0, v_0 \in K_h$ and $\tau \in (0,1)$ satisfying $\tau v_0 \le u_0 < v_0$ and

$$\begin{aligned} u_0(t) \ \ \le \ \ & \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]d_qs \\ & + \int_0^1 G(t,qs)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]d_qs,\ t \in [0,1], \end{aligned}$$

$$\begin{aligned} v_0(t) \ \ \ge \ \ & \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]d_qs \\ & + \int_0^1 G(t,qs)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]d_qs,\ t \in [0,1], \end{aligned}$$

where $h(t) = t^{\alpha-1}$ *and* $G(t, qs),\ H(t, qs)$ *are defined as in Lemma 5;*

(b) *BVP (1) has a unique positive solution* $u^* \in K_h$*;*
(c) *for* $x_0, y_0 \in K_h$*, set*

$$\begin{aligned} x_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta, qs)[f(s, x_n(s), y_n(s)) + g(s, x_n(s))]d_q s \\ &\quad + \int_0^1 G(t, qs)[f(s, x_n(s), y_n(s)) + g(s, x_n(s))]d_q s,\ n = 1, 2, \ldots, \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta, qs)[f(s, y_n(s), x_n(s)) + g(s, y_n(s))]d_q s \\ &\quad + \int_0^1 G(t, qs)[f(s, y_n(s), x_n(s)) + g(s, y_n(s))]d_q s,\ n = 1, 2, \ldots, \end{aligned}$$

then $\|x_n - u^*\| \to 0,\ \|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Proof. By Lemma 5, the solution u of BVP (1) can be written by

$$\begin{aligned} u(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta, qs)[f(s, u(s), u(s)) + g(s, u(s))]d_q s \\ &\quad + \int_0^1 G(t, qs)[f(s, u(s), u(s)) + g(s, u(s))]d_q s. \end{aligned}$$

Now, we give two operators $T_1 : K \times K \to X$ and $T_2 : K \to X$ by

$$\begin{aligned} T_1(u, v)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta, qs) f(s, u(s), v(s))d_q s \\ &\quad + \int_0^1 G(t, qs) f(s, u(s), v(s))d_q s, \end{aligned}$$

$$\begin{aligned} (T_2 u)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta, qs) g(s, u(s))d_q s \\ &\quad + \int_0^1 G(t, qs) g(s, u(s))d_q s. \end{aligned}$$

Obviously, u is a solution of Label (1) if and only if $u = T_1(u, u) + T_2 u$. By (H_1), one has $T_1 : K \times K \to K$ and $T_2 : K \to K$. We will prove that $T_1,\ T_2$ satisfy all the assumptions of Lemma 3. The proof consists of three steps.

Step 1. The aim of this step is to prove that T_1 is a mixed monotone operator.

For $u_i, v_i \in K,\ i = 1,2$ with $u_1 \geq u_2,\ v_1 \leq v_2$, then $u_1(t) \geq u_2(t),\ v_1(t) \leq v_2(t)$ for $t \in [0,1]$. From (H_2) and Lemma 6,

$$\begin{aligned} T_1(u_1,v_1)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,u_1(s),v_1(s))d_qs \\ &\quad + \int_0^1 G(t,qs)f(s,u_1(s),v_1(s))d_qs \\ &\geq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,u_2(s),v_2(s))d_qs \\ &\quad + \int_0^1 G(t,qs)f(s,u_2(s),v_2(s))d_qs \\ &= T_1(u_2,v_2)(t). \end{aligned}$$

Thus, $T_1(u_1,v_1) \geq T_1(u_2,v_2)$, that is, T_1 is mixed monotone.

Step 2. Our aim of this step is to show that T_1 satisfies the condition (8) and the operator T_2 is sub-homogeneous.

From (H_2) and Lemma 6, T_2 is increasing. Furthermore, for $\lambda \in (0,1)$ and $u,v \in P$, by (H_3),

$$\begin{aligned} T_1(\lambda u,\lambda^{-1}v)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,\lambda u(s),\lambda^{-1}v(s))d_qs \\ &\quad + \int_0^1 G(t,qs)f(s,\lambda u(s),\lambda^{-1}v(s))d_qs \\ &\geq \frac{\lambda^\gamma\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,u(s),v(s))d_qs \\ &\quad +\lambda^\gamma \int_0^1 G(t,qs)f(s,u_2(s),v_2(s))d_qs \\ &= \lambda^\gamma T_1(u,v)(t), \end{aligned}$$

and thus $T_1(\lambda u,\lambda^{-1}v) \geq \lambda^\gamma T_1(u,v)$ for $\lambda \in (0,1),\ u,v \in K$. Hence, the operator T_1 satisfies (8). In addition, for any $\lambda \in (0,1),\ u \in K$, by (H_3),

$$\begin{aligned} T_2(\lambda u)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)g(s,\lambda u(s))d_qs + \int_0^1 G(t,qs)g(s,\lambda u(s))d_qs \\ &\geq \frac{\lambda\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)g(s,u(s))d_qs + \lambda\int_0^1 G(t,qs)g(s,u(s))d_qs \\ &= \lambda T_2u(t), \end{aligned}$$

that is, $T_2(\lambda u) \geq \lambda T_2 u,\ u \in P$. Thus, the operator T_2 is sub-homogeneous.

Step 3. The purpose of this step is to prove that $T_1(h,h),\ T_2h \in K_h$. Furthermore, we also prove that there exists $\sigma > 0$ such that $T_1(x,y) \geq \sigma T_2 x,\ \forall x,y \in K$.

Firstly, in view of $(H_1),(H_2)$ and Lemma 6, for $t \in [0,1]$,

$$\begin{aligned} T_1(h,h)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,h(s),h(s))d_qs + \int_0^1 G(t,qs)f(s,h(s),h(s))d_qs \\ &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,s^{\alpha-1},s^{\alpha-1})d_qs + \int_0^1 G(t,qs)f(s,s^{\alpha-1},s^{\alpha-1})d_qs \\ &\leq \frac{\beta h(t)}{(1-\beta\eta^{\alpha-2})\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}f(s,1,0)d_qs + \frac{h(t)}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}f(s,1,0)d_qs. \end{aligned}$$

By the same arguments, for $t \in [0,1]$,

$$\begin{aligned} T_1(h,h)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,s^{\alpha-1},s^{\alpha-1})d_qs + \int_0^1 G(t,qs)f(s,s^{\alpha-1},s^{\alpha-1})d_qs \\ &\geq \frac{h(t)}{\Gamma_q(\alpha)}\int_0^1 [(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}]f(s,0,1)d_qs. \end{aligned}$$

From $(H_2), (H_4)$,

$$\int_0^1 f(s,1,0)d_qs \geq \int_0^1 f(s,0,1)d_qs \geq \sigma \int_0^1 g(s,0)d_qs > 0.$$

Set

$$l_1 = \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\beta}{(1-\beta\eta^{\alpha-2})\Gamma_q(\alpha)}\right)\int_0^1 (1-qs)^{(\alpha-2)} f(s,1,0)d_qs,$$

$$l_2 = \frac{1}{\Gamma_q(\alpha)}\int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s,0,1)d_qs.$$

Then, $l_2h(t) \leq T_1(h,h)(t) \leq l_1h(t), t \in [0,1]$. It follows that $T_1(h,h) \in K_h$. Similarly,

$$T_2h(t) \geq \frac{h(t)}{\Gamma_q(\alpha)}\int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] g(s,0)d_qs,$$

and

$$T_2h(t) \leq \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\beta}{(1-\beta\eta^{\alpha-2})\Gamma_q(\alpha)}\right) h(t)\int_0^1 (1-qs)^{(\alpha-2)} g(s,1)d_qs.$$

Since $g(t,0) \not\equiv 0$, we also get $T_2h \in K_h$. Thus, the condition (i) of Lemma 3 holds. Next, we will indicate that (ii) of Lemma 3 is still satisfied. For $t \in [0,1]$, $u, v \in K$, from (H_4),

$$\begin{aligned} T_1(u,v)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,u(s),v(s))d_qs + \int_0^1 G(t,qs)f(s,u(s),v(s))d_qs \\ &\geq \frac{\sigma\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)g(s,u(s))d_qs + \sigma\int_0^1 G(t,qs)g(s,u(s))d_qs \\ &= \sigma T_2u(t). \end{aligned}$$

Then, $T_1(u,v) \geq \sigma T_2u$ for $u, v \in K$. Therefore, by Lemma 3, we have: $u_0, v_0 \in K_h$ and $\tau \in (0,1)$ satisfying $\tau v_0 \leq u_0 < v_0$, $u_0 \leq T_1(u_0,v_0) + T_2u_0 \leq T_1(v_0,u_0) + T_2v_0 \leq v_0$; the equation $T_1(u,u) + T_2u = u$ has a unique solution u^* in K_h; for $x_0, y_0 \in K_h$, set

$$x_n = T_1(x_{n-1},y_{n-1}) + T_2x_{n-1},\ y_n = T_1(y_{n-1},x_{n-1}) + T_2y_{n-1},\ n = 1,2,\ldots,$$

one obtains $x_n \to u^*$, $y_n \to u^*$ as $n \to \infty$. Namely,

$$\begin{aligned} u_0(t) &\leq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]d_qs,\ t \in [0,1], \end{aligned}$$

$$\begin{aligned} v_0(t) &\geq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]d_qs,\ t \in [0,1]; \end{aligned}$$

Label (1) has a unique positive solution $u^* \in K_h$; for $x_0, y_0 \in K_h$, the sequences

$$\begin{aligned} x_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,x_n(s),y_n(s))+g(s,x_n(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,x_n(s),y_n(s))+g(s,x_n(s))]d_qs,\ n=1,2,\ldots, \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,y_n(s),x_n(s))+g(s,y_n(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,y_n(s),x_n(s))+g(s,y_n(s))]d_qs,\ n=1,2,\ldots \end{aligned}$$

satisfy $\|x_n - u^*\| \to 0,\ \|y_n - u^*\| \to 0$ as $n \to \infty$. □

Theorem 2. *Let $(H_1), (H_2)$ and the following conditions be satisfied:*

(H_5) *for $t \in [0,1], \lambda \in (0,1), u \geq 0$, there is $\gamma \in (0,1)$ such that $g(t,\lambda u) \geq \lambda^\gamma g(t,u)$ and $f(t,\lambda u,\lambda^{-1}v) \geq \lambda f(t,u,v)$ for $t \in [0,1], \lambda \in (0,1), u, v \geq 0$;*

(H_6) *$f(t,0,1) \not\equiv 0$ for $t \in [0,1]$, and there is $\sigma > 0$ satisfying $f(t,u,v) \leq \sigma g(t,u),\ \forall t \in [0,1], u, v \geq 0$.*

Then:

(a) there is $u_0, v_0 \in P_h$ and $\tau \in (0,1)$ such that $\tau v_0 \leq u_0 < v_0$ and

$$\begin{aligned} u_0(t) &\leq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,u_0(s),v_0(s))+g(s,u_0(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,u_0(s),v_0(s))+g(s,u_0(s))]d_qs,\ t \in [0,1], \end{aligned}$$

$$\begin{aligned} v_0(t) &\geq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,v_0(s),u_0(s))+g(s,v_0(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,v_0(s),u_0(s))+g(s,v_0(s))]d_qs,\ t \in [0,1], \end{aligned}$$

where $h(t) = t^{\alpha-1}$ and $G(t,qs),\ H(t,qs)$ are defined as in Lemma 5;

(b) BVP (1) has a unique positive solution $u^* \in K_h$;

(c) for any $x_0, y_0 \in K_h$, set

$$\begin{aligned} x_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,x_n(s),y_n(s))+g(s,x_n(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,x_n(s),y_n(s))+g(s,x_n(s))]d_qs,\ n=1,2,\ldots, \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,y_n(s),x_n(s))+g(s,y_n(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,y_n(s),x_n(s))+g(s,y_n(s))]d_qs,\ n=1,2,\ldots, \end{aligned}$$

and we get $\|x_n - u^*\| \to 0,\ \|y_n - u^*\| \to 0$ as $n \to \infty$.

Proof. We also consider two operators T_1, T_2. Given in the proof of Theorem 1, it has been shown that $T_1 : K \times K \to K$ is mixed monotone and $T_2 : K \to K$ is increasing. By (H_5),

$$T_1(\lambda u, \lambda^{-1}v) \geq \lambda T_1(u,v),\ T_2(\lambda u) \geq \lambda^{\gamma} T_2 u,\ \lambda \in (0,1), u, v \in K.$$

From $(H_2), (H_6)$,

$$g(s,0) \geq \frac{1}{\sigma} f(s,0,1),\ f(s,1,0) \geq f(s,0,1),\ s \in [0,1].$$

Since $f(t,0,1) \not\equiv 0$, we obtain

$$\int_0^1 f(s,1,0)d_q s \geq \int_0^1 f(s,0,1)d_q s > 0,\ \int_0^1 g(s,1)d_q s \geq \int_0^1 g(s,0)d_q s \geq \frac{1}{\sigma}\int_0^1 f(s,0,1)d_q s > 0,$$

so

$$\begin{aligned} &\left(\tfrac{1}{\Gamma_q(\alpha)} + \tfrac{\beta}{(1-\beta\eta^{\alpha-2})\Gamma_q(\alpha)}\right) \textstyle\int_0^1 (1-qs)^{(\alpha-2)} f(s,1,0)d_q s \\ \geq\ & \tfrac{1}{\Gamma_q(\alpha)} \textstyle\int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s,0,1)d_q s > 0, \end{aligned}$$

and

$$\begin{aligned} &\left(\tfrac{1}{\Gamma_q(\alpha)} + \tfrac{\beta}{(1-\beta\eta^{\alpha-2})\Gamma_q(\alpha)}\right) \textstyle\int_0^1 (1-qs)^{(\alpha-2)} g(s,1)d_q s \\ \geq\ & \tfrac{1}{\Gamma_q(\alpha)} \textstyle\int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] g(s,0)d_q s > 0. \end{aligned}$$

It can easily prove that $T_1(h,h),\ T_2 h \in K_h$. Furthermore, by (H_6),

$$\begin{aligned} T_1(u,v)(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs) f(s,u(s),v(s))d_q s + \int_0^1 G(t,qs) f(s,u(s),v(s))d_q s \\ &\leq \frac{\sigma\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs) g(s,u(s))d_q s + \sigma\int_0^1 G(t,qs) g(s,u(s))d_q s \\ &= \sigma T_2 u(t). \end{aligned}$$

Hence, $T_1(u,v) \leq T_2 u$, for $u, v \in K$. By Lemma 4, we can claim: there are $u_0, v_0 \in P_h$ and $\tau \in (0,1)$ satisfying $\tau v_0 \leq u_0 < v_0,\ u_0 \leq T_1(u_0,v_0) + T_2 u_0 \leq T_1(v_0,u_0) + T_2 v_0 \leq v_0$; the equation $T_1(u,u) + T_2 u = u$ has a unique solution u^* in K_h; for $x_0, y_0 \in K_h$, set

$$x_n = T_1(x_{n-1}, y_{n-1}) + T_2 x_{n-1},\ y_n = T_1(y_{n-1}, x_{n-1}) + T_2 y_{n-1},\ n = 1, 2, \ldots,$$

one has $x_n \to u^*,\ y_n \to u^*$ as $n \to \infty$. Namely,

$$\begin{aligned} u_0(t) &\leq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]d_q s \\ &\quad + \int_0^1 G(t,qs)[f(s,u_0(s),v_0(s)) + g(s,u_0(s))]d_q s,\ t \in [0,1], \end{aligned}$$

$$\begin{aligned} v_0(t) &\geq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]d_q s \\ &\quad + \int_0^1 G(t,qs)[f(s,v_0(s),u_0(s)) + g(s,v_0(s))]d_q s,\ t \in [0,1]; \end{aligned}$$

Label (1) has a unique positive solution $u^* \in K_h$; for $x_0, y_0 \in P_h$, the sequences

$$\begin{aligned} x_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,x_n(s),y_n(s))+g(s,x_n(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,x_n(s),y_n(s))+g(s,x_n(s))]d_qs,\ n=1,2,\ldots, \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)[f(s,y_n(s),x_n(s))+g(s,y_n(s))]d_qs \\ &\quad + \int_0^1 G(t,qs)[f(s,y_n(s),x_n(s))+g(s,y_n(s))]d_qs,\ n=1,2,\ldots \end{aligned}$$

satisfy $\|x_n - u^*\| \to 0,\ \|y_n - u^*\| \to 0$ as $n \to \infty$. $\square$

In the sequel, we consider special cases of Label (1) with $g \equiv 0$ or $f \equiv 0$. Similar to the proofs of Theorems 1 and 2 and according to Remark 1, we can draw the following conclusions:

Corollary 1. *Assume f satisfies $(H_1)-(H_4)$ and $f(t,0,1) \not\equiv 0$, for $t \in [0,1]$. Then: (a) there are $u_0, v_0 \in K_h$ and $\tau \in (0,1)$ such that $\tau v_0 \le u_0 < v_0$ and*

$$\begin{aligned} u_0(t) &\le \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,u_0(s),v_0(s))d_qs \\ &\quad + \int_0^1 G(t,qs)f(s,u_0(s),v_0(s))d_qs,\ t\in[0,1], \end{aligned}$$

$$\begin{aligned} v_0(t) &\ge \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,v_0(s),u_0(s))d_qs \\ &\quad + \int_0^1 G(t,qs)f(s,v_0(s),u_0(s))d_qs,\ t\in[0,1], \end{aligned}$$

where $h(t) = t^{\alpha-1}$ and $G(t,qs),\ H(t,qs)$ are given as in Lemma 5; (b) the following BVP

$$\begin{cases} D_q^\alpha x(t) + f(t,x(t),x(t)) = 0, \quad 0<t<1,\ 2<\alpha<3, \\ x(0) = D_qx(0) = 0,\ D_qx(1) = \beta D_qx(\eta), \end{cases} \tag{13}$$

has a unique positive solution $u^* \in K_h$; (c) for $x_0, y_0 \in K_h$, set

$$\begin{aligned} x_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,x_n(s),y_n(s))d_qs \\ &\quad + \int_0^1 G(t,qs)f(s,x_n(s),y_n(s))d_qs,\ n=1,2,\ldots, \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})}\int_0^1 H(\eta,qs)f(s,y_n(s),x_n(s))d_qs \\ &\quad + \int_0^1 G(t,qs)f(s,y_n(s),x_n(s))d_qs,\ n=1,2,\ldots, \end{aligned}$$

and we get $\|x_n - u^*\| \to 0,\ \|y_n - u^*\| \to 0$ as $n \to \infty$.

Corollary 2. *Assume g satisfies $(H_1), (H_2)$ and $(H_5), (H_6)$, $g(t, 0) \not\equiv 0$, for $t \in [0, 1]$. Then: (a) there are $u_0, v_0 \in K_h$ and $\tau \in (0, 1)$ such that $\tau v_0 \leq u_0 < v_0$ and*

$$\begin{aligned} u_0(t) &\leq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs) g(s, u_0(s)) d_q s \\ &+ \int_0^1 G(t, qs) g(s, u_0(s)),\ t \in [0, 1], \end{aligned}$$

$$\begin{aligned} v_0(t) &\geq \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs) g(s, v_0(s)) d_q s \\ &+ \int_0^1 G(t, qs) g(s, v_0(s)) d_q s,\ t \in [0, 1], \end{aligned}$$

where $h(t) = t^{\alpha-1}$ and $G(t, qs),\ H(t, qs)$ are given as in Lemma 5; (b) the following problem

$$\begin{cases} D_q^\alpha x(t) + g(t, x(t)) = 0, & 0 < t < 1,\ 2 < \alpha < 3, \\ x(0) = D_q x(0) = 0,\ D_q x(1) = \beta D_q x(\eta), \end{cases} \tag{14}$$

has a unique positive solution $u^* \in K_h$; (c) for $x_0, y_0 \in K_h$, set

$$\begin{aligned} x_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs) g(s, x_n(s)) d_q s \\ &+ \int_0^1 G(t, qs) g(s, x_n(s)) d_q s,\ n = 1, 2, \ldots, \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) &= \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\eta^{\alpha-2})} \int_0^1 H(\eta, qs) g(s, y_n(s)) d_q s \\ &+ \int_0^1 G(t, qs) g(s, y_n(s)) d_q s,\ n = 1, 2, \ldots, \end{aligned}$$

and we obtain $\|x_n - u^*\| \to 0,\ \|y_n - u^*\| \to 0$ as $n \to \infty$.

Remark 3. *In literature, we have not found such results as Theorems 1 and 2, and Corollaries 1 and 2 on fractional q-difference equation boundary value problems. The used methods in literature were not fixed point theorems for mixed monotone operators. Thus, our method is different from previous ones. We should point out that we can not only give the existence and uniqueness of solutions but also make an iteration to approximate the unique solution.*

4. Examples

Example 1. *We consider a problem:*

$$\begin{cases} D_q^{\frac{5}{2}} u(t) + u^{\frac{1}{5}}(t) + [u(t) + 4]^{-\frac{1}{3}} + \frac{u(t)}{2+u(t)} t^3 + 3a = 0, & t \in (0, 1), \\ u(0) = D_q u(0) = 0,\ D_q u(1) = \frac{1}{2} D_q u(\frac{1}{2}), \end{cases} \tag{15}$$

where $q = \frac{1}{2},\ \alpha = \frac{5}{2},\ \beta = \eta = \frac{1}{2},\ a > 0$. Take $0 < b < a$ and let

$$f(t, u, v) = u^{\frac{1}{5}} + [v + 4]^{-\frac{1}{3}} + b,\ g(t, u) = \frac{u}{2+u} t^3 + 3a - b,\ \gamma = \frac{1}{3}.$$

Then, $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ and $g : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are continuous, $g(t, 0) = 3a - b > 0$. Furthermore, $f(t, u, v)$ is increasing relative to u for fixed $t \in [0, 1]$ and $v \in [0, +\infty)$, *decreasing relative to v for fixed $t \in [0,1]$ and $u \in [0,+\infty)$, $g(t,u)$ is increasing relative to u for fixed $t \in [0,1]$. On the other hand, for $\lambda \in (0,1), t \in [0,1], u, v \geq 0$,*

$$g(t,\lambda u) = \frac{\lambda u(t)}{2+\lambda u(t)}t^3 + 3a - b \geq \frac{\lambda u(t)}{2+u(t)}t^3 + \lambda(3a-b) = \lambda g(t,u),$$

and

$$f(t,\lambda u,\lambda^{-1}v) = \lambda^{\frac{1}{5}}u^{\frac{1}{5}} + \lambda^{\frac{1}{3}}[v+4\lambda]^{-\frac{1}{3}} + b \geq \lambda^{\frac{1}{3}}\left\{u^{\frac{1}{5}} + [v+4]^{-\frac{1}{3}} + b\right\} = \lambda^{\gamma}f(t,u,v).$$

Then, (H_1)–(H_3) holds. Moreover, taking $\sigma \in (0, \frac{b}{3a-b}]$, one has

$$f(t,u,v) = u^{\frac{1}{5}} + [v+4]^{-\frac{1}{3}} + b \geq b = \frac{b}{3a-b}\cdot(3a-b) \geq \sigma\left[\frac{u}{2+u}t^3 + 3a - b\right] = \sigma g(t,u),$$

then (H_4) holds. By means of Theorem 1, problem (15) has a unique positive solution $u^* \in K_h$, where $h(t) = t^{\frac{3}{2}}, t \in [0,1]$.

Example 2. *In Example 4.1, we replace the nonlinear term $u^{\frac{1}{5}}(t) + [u(t)+4]^{-\frac{1}{3}} + \frac{u(t)}{2+u(t)}t^3 + 3a$ by*

$$\sin^2 t + u^{\frac{1}{3}}(t) + \frac{1}{2+u(t)} + \frac{u(t)}{1+u(t)} + 3.$$

By Theorem 2, we can also show that problem (4.1) has a unique positive solution $u^* \in K_h$, where $h(t) = t^{\frac{3}{2}}, t \in [0,1]$. In fact, let

$$f(t,u,v) = \sin^2 t + \frac{1}{2+v} + \frac{u}{1+u},\ g(t,u) = u^{\frac{1}{3}} + 3,\ \gamma = \frac{1}{3}.$$

It is easy to check that $(H_1), (H_2)$ hold. We only show $(H_5), (H_6)$ are satisfied. For $\lambda \in (0,1), t \in [0,1], u, v \geq 0$,

$$g(t,\lambda u) = \lambda^{\frac{1}{3}}u^{\frac{1}{3}} + 3 \geq \lambda^{\frac{1}{3}}[u^{\frac{1}{3}} + 3] = \lambda^{\gamma}g(t,u),$$

and

$$f(t,\lambda u,\lambda^{-1}v) = \sin^2 t + \frac{1}{2+\lambda^{-1}v} + \frac{\lambda u}{1+\lambda u} \geq \sin^2 t + \frac{\lambda}{2+v} + \frac{\lambda u}{1+u} \geq \lambda f(t,u,v).$$

Furthermore, $f(t,0,1) = \sin^2 t + \frac{1}{3} \not\equiv 0$ and

$$f(t,u,v) \leq 3 \leq u^{\frac{1}{3}} + 3 = g(t,u).$$

Take $\sigma \in [1,\infty)$ and then $(H_5), (H_6)$ hold.

Remark 4. *From Theorems 1 and 2 and Examples 1 and 2, we see that many boundary value problems can be studied by our methods under mixed monotone conditions. We can find that there are many functions that satisfy our conditions. In some works, the nonlinear terms required were super-linearity, sub-linearity or boundness, which guarantee existence of solutions, but the uniqueness has not been obtained.*

5. Conclusions

In this article, we investigate a fractional q-difference equation with three-point boundary conditions (1). We obtain the existence and uniqueness of positive solutions in a special K_h, where $h(t) = t^{\alpha-1}$. The used methods here are some theorems for operator equation $T_1(x,x) + T_2x = x$, where T_1 is a mixed monotone operator and T_2 is an increasing operator. Our methods are new to fractional q-difference equation boundary value problems. Thus, we can claim that we give an alternative answer to fractional problems and our results are very limited in the literature. Finally, two interesting examples are presented to illustrate the main results. We should note that, to get the uniqueness, we must need the conditions of mixed monotonicity and monotonicity for nonlinear terms.

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