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Fixed Points Results in Algebras of Split Quaternion and Octonion

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Abstract: Fixed points of functions have applications in game theory, mathematics, physics, economics and computer science. The purpose of this article is to compute fixed points of a general quadratic polynomial in finite algebras of split quaternion and octonion over prime fields $\mathbb{Z}_p$. Some characterizations of fixed points in terms of the coefficients of these polynomials are also given. Particularly, cardinalities of these fixed points have been determined depending upon the characteristics of the underlying field.

Keywords: fixed point; split-quaternion; quadratic polynomial; split-octonion

Subject Classification (2010): 30C35; 05C31

1. Introduction

Geometry of space-time can be understood by the choice of convenient algebra which reveals hidden properties of the physical system. These properties are best describable by the reflections of symmetries of physical signals that we receive and of the algebra using in the measurement process [1–3]. Thus, we need normed division algebras with a unit element for the better understanding of these systems. For these reasons, higher dimension algebras have been an immense source of inspiration for mathematicians and physicists as their representations pave the way towards easy understanding of universal phenomenons. These algebras present nice understandings towards general rotations and describe some easy ways to consider geometric problems in mechanics and dynamical systems [4,5].

Quaternion algebra have been playing a central role in many fields of sciences such as differential geometry, human imaging, control theory, quantum physics, theory of relativity, simulation of particle motion, 3D geo-phones, multi-spectral images, signal processing including seismic velocity analysis, seismic waveform de-convolution, statistical signal processing and probability distributions (see [6–8] and references therein). It is known that rotations of 3D-Minkowski spaces can be represented by the algebra of split quaternions [5]. Applications of these algebras can be traced in the study of Graphenes, Black holes, quantum gravity and Gauge theory. A classical application of split quaternion is given in [1] where Pavsic discussed spin gauge theory. Quantum gravity of $2+1$ dimension has been described by Carlip in [2] using split quaternions. A great deal of research is in progress where authors are focused on considering matrices of quaternions and split-quaternions [9–12]. The authors in [13] gave a fast structure-preserving method to compute singular value decomposition of quaternion matrices. Split quaternions play a vital role in geometry and physical models in four-dimensional spaces as the elements of split quaternion are used to express Lorentzian rotations [14]. Particularly, the geometric and physical applications of split quaternions require solving split quaternionic equations [15,16]. Similarly, octonion and split octonion algebras play important role in mathematical physics. In [8], authors discussed ten dimensional space-time with help of these eight dimensional algebras. In [16], authors gave comprehensive applications of split octonions in geometry. Anastasiou developed M-theory algebra with the help of octonions [3].

This article mainly covers finite algebras of split quaternion and split octonion over prime fields $\mathbb{Z}_p$. Split quaternion algebra over $\mathbb{R}$ was in fact introduced by James Cockle in 1849 on already established quaternions by Hamilton in 1843. Both of these algebras are actually associative, but non-commutative, non-division ring generated by four basic elements. Like quaternion, it also forms a four dimensional real vector space equipped with a multiplicative operation. However, unlike the quaternion algebra, the split quaternion algebra contains zero divisors, nilpotent and nontrivial idempotents. For a detailed description of quaternion and its generalization (octonions), please follow [15–18]. As mathematical structures, both are algebras over the real numbers which are isomorphic to the algebra of 2×2 real matrices. The name split quaternion is used due to the division into positive and negative terms in the modulus function. The set $(1, \hat{i}, \hat{j}, \hat{k})$ forms a basis. The product of these elements are $\hat{i}^2 = -1$, $\hat{j}^2 = 1 = \hat{k}^2$, $\hat{i}\hat{j} = \hat{k} = -\hat{j}\hat{i}$, $\hat{j}\hat{k} = -\hat{i} = -\hat{k}\hat{j}$, $\hat{k}\hat{i} = \hat{j} = -\hat{i}\hat{k}$, $\hat{i}\hat{j}\hat{k} = 1$. It follows from the defining relations that the set $(\pm 1, \pm i, \pm j, \pm k)$ is a group under split quaternion multiplication which is isomorphic to the dihedral group of a square. Following Table 1 encodes the multiplication of basis split quaternions.

Table 1. Split quaternion multiplication table.

.	**1**	$\hat{i}$	$\hat{j}$	$\hat{k}$
1	1	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	$\hat{i}$	-1	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$\hat{j}$	$-\hat{k}$	1	$-\hat{i}$
$\hat{k}$	$\hat{k}$	$\hat{j}$	$\hat{i}$	1

The split octonion is an eight-dimensional algebraic structure, which is non-associative algebra over some field with basis 1, $\acute{t}_1$, $\acute{t}_2$, $\acute{t}_3$, $\acute{t}_4$, $\acute{t}_5$, $\acute{t}_6$ and $\acute{t}_7$. The subtraction and addition in split octonions is computed by subtracting and adding corresponding terms and their coefficients. Their multiplication is given in this table. The product of each term can be given by multiplication of the coefficients and a multiplication table of the unit split octonions is given following Table 2.

Table 2. Split octonions' multiplication table.

.	$\acute{t}_1$	$\acute{t}_2$	$\acute{t}_3$	$\acute{t}_4$	$\acute{t}_5$	$\acute{t}_6$	$\acute{t}_7$
$\acute{t}_1$	-1	$\acute{t}_3$	$-\acute{t}_2$	$-\acute{t}_7$	$\acute{t}_6$	$-\acute{t}_5$	$\acute{t}_4$
$\acute{t}_2$	$-\acute{t}_3$	-1	$\acute{t}_1$	$-\acute{t}_6$	$-\acute{t}_7$	$\acute{t}_4$	$\acute{t}_5$
$\acute{t}_3$	$\acute{t}_2$	$-\acute{t}_1$	-1	$\acute{t}_5$	$-\acute{t}_4$	$-\acute{t}_7$	$\acute{t}_6$
$\acute{t}_4$	$\acute{t}_7$	$\acute{t}_6$	$-\acute{t}_5$	1	$-\acute{t}_3$	$\acute{t}_2$	$\acute{t}_1$
$\acute{t}_5$	$-\acute{t}_6$	$\acute{t}_7$	$\acute{t}_4$	$\acute{t}_3$	1	$-\acute{t}_1$	$\acute{t}_2$
$\acute{t}_6$	$\acute{t}_5$	$-\acute{t}_4$	$\acute{t}_7$	$-\acute{t}_2$	$\acute{t}_1$	1	$\acute{t}_3$
$\acute{t}_7$	$-\acute{t}_4$	$-\acute{t}_5$	$-\acute{t}_6$	$-\acute{t}_1$	$-\acute{t}_2$	$-\acute{t}_3$	1

From the table, we get very useful results:

$$\acute{t}_i^2 = -1, \forall i = 1, ..., 3,$$

$$\acute{t}_i^2 = 1, \forall i = 4, ..., 7,$$

$$\acute{t}_i \acute{t}_j = -\acute{t}_j \acute{t}_i, \forall i \neq j.$$

Brand in [19] computed the roots of a quaternion over $\mathbb{R}$. Strictly speaking, he proved mainly De Moivres theorem and then used it to find *nth* roots of a quaternion. His approach paved way for finding roots of a quaternion in an efficient and intelligent way. Ozdemir in [20] computed the roots of a split quaternion. In [21], authors discussed Euler's formula and De Moivres formula for quaternions. In [15], authors gave some geometrical applications of the split quaternion. It is important to mention that these two algebras can also be constructed for $\mathbb{Z}_p$ over prime finite fields of characteristic P. In this way, we obtain finite algebras with entirely different properties. Recently, the ring of quaternion over $\mathbb{Z}_p$ was studied by Michael Aristidou in [22,23], where they computed the idempotents and nilpotents in $\mathbb{H}/\mathbb{Z}_p$. In [18], authors computed the roots of a general quadratic polynomial in algebra of split quaternion over $\mathbb{R}$. They also computed fixed points of general quadratic polynomials in the same sittings. A natural question arises as to what happens with the same situations over $\mathbb{Z}_p$. Authors in [24] discussed split-quaternion over Z_p in algebraic settings.

In the present article, we first obtain the roots of a general quadratic polynomial in the algebra of split quaternion over $\mathbb{Z}_p$. Some characterizations of fixed points in terms of the coefficients of these polynomials are also given. As a consequence, we give some computations about algebraic properties of particular classes of elements in this settings. We also give examples as well as the codes that create these examples with ease. For a computer program, we refer to Appendix A at the end of the article. We hope that our results will be helpful in understanding the communication in machine language and cryptography.

Definition 1. *Let $x \in \mathbb{H}_s$, $x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where $a_i \in \mathbb{R}$. The* conjugate of x *is defined as $\bar{x} = a_0 - a_1\hat{i} - a_2\hat{j} - a_3\hat{k}$. The square of pseudo-norm of x is given by*

$$N(x) = x\bar{x} = a_0^2 + a_1^2 - a_3^2 - a_4^2. \quad (1)$$

Definition 2. *Let $x = a_0 + \sum_{i=1}^{7} a_i\acute{t_i} \in \mathbb{O}_s/\mathbb{Z}_p$. The* conjugate of x *is defined as*

$$\begin{aligned} \overline{x} &= \overline{a_0 + \sum_{i=1}^{7} a_i\acute{t_i}} \\ &= a_0 + \sum_{i=1}^{7} a_i\overline{\acute{t_i}} \\ &= a_0 - \sum_{i=1}^{7} a_i\acute{t_i} \\ &= a_0 + \sum_{i=1}^{7} \acute{a_i}\acute{t_i}, \end{aligned}$$

where $\acute{a_i} = -a_i$ where $i = 1, 2, ..., 7$. The square of pseudo-norm of x is given by

$$N(x) = x\overline{x} = \sum_{i=0}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2.$$

2. Main Results

In this section, we formulate our main results. At first, we give these results for split quaternions and then we move towards split octonions.

2.1. Some Fixed Points Results of Quadratic Functions in Split Quaternions over the Prime Field

We first solve a general quadratic polynomial in algebra of split quaternion. As a consequence, we find fixed points of associated functions in this algebra.

Theorem 1. *The quadratic equation $ax^2 + bx + c = 0$ $a, b, c \in \mathbb{Z}_p$, where p is an odd prime and $p \nmid a$, has root $x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p$ if and only if $a_0 = \frac{p-b}{2a}$ and $a_1^2 - a_2^2 - a_3^2 = (\frac{p^2-b^2}{4a^2}) + \frac{c}{a}$.*

Proof.

$$\begin{aligned} x &= a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, &(2)\\ x^2 &= (a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k})^2, &(3)\end{aligned}$$

$$\begin{aligned} &= a_0^2 - a_1^2 + a_2^2 + a_3^2 + 2a_0a_1\hat{i} + 2a_0a_2\hat{j} + 2a_0a_3\hat{k}\\ &= a_0^2 + a_0^2 - \|x\| + 2a_0a_1\hat{i} + 2a_0a_2\hat{j} + 2a_0a_3\hat{k}\\ &= 2a_0^2 - \|x\| + 2a_0a_1\hat{i} + 2a_0a_2\hat{j} + 2a_0a_3\hat{k}\\ &= 2a_0(a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - \|x\|\\ &= 2a_0x - \|x\|.\end{aligned}$$

Putting x and x^2 into $ax^2 + bx + c = 0$, we have

$$\begin{aligned} 2aa_0x - a\|x\| + bx + c &= 0,\\ (2aa_0 + b)x - a\|x\| + c &= 0,\\ (2aa_0 + b)(a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - a(a_0^2 + a_1^2 - a_2^2 - a_3^2) + c &= 0,\\ (2aa_0 + b)a_0 + (2aa_0 + b)(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - a(a_0^2 + a_1^2 - a_2^2 - a_3^2) + c &= 0.\end{aligned}$$

Comparing vector terms in the above equation, we get

$$\begin{aligned} 2aa_0 + b &= 0, &(4)\\ a_0 &= \frac{-b}{2a} = \frac{p-b}{2a}. &(5)\end{aligned}$$

Comparing constant terms, we get

$$\begin{aligned} (2aa_0 + b)a_0 - a(a_0^2 + a_1^2 - a_2^2 - a_3^2) + c &= 0, &(6)\\ (2aa_0 + b)a_0 - aa_0^2 + c &= a(a_1^2 - a_2^2 - a_3^2), &(7)\\ aa_0^2 + ba_0 + c &= a(a_1^2 - a_2^2 - a_3^2), &(8)\\ (aa_0 + b)a_0 + c &= a(a_1^2 - a_2^2 - a_3^2), &(9)\\ (a(\frac{p-b}{2a}) + b)\frac{p-b}{2a} + c &= a(a_1^2 - a_2^2 - a_3^2), &(10)\\ \frac{p^2-b^2}{4a^2} + \frac{c}{a} &= a_1^2 - a_2^2 - a_3^2. &(11)\end{aligned}$$

□

On the basis of the above results 2.1, we arrive at a new result given as

Theorem 2. *The fixed point of function $f(x) = x^2 + (b+1)x + c$ where $a, b, c \in \mathbb{Z}_p$, p is an odd prime and $p \nmid a$ is $x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p$ if and only if $a_0 = \frac{p-b}{2a}$ and $a_1^2 - a_2^2 - a_3^2 = (\frac{p^2-b^2}{4a^2}) + \frac{c}{a}$.*

Proof. It is enough to give a new relation $f(x) = g(x) + x$, where $g(x) = x^2 + bx + c$. Then, existence of fixed points for $f(x)$ is equivalent to the solutions of $g(x)$. Then, the required result is immediate from the above theorem. □

Theorem 3. *Let p be an odd prime, $p \nmid a$, if $x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p$ is a root of quadratic equation $x^2 + bx + c = 0$, where $a, b, c \in \mathbb{Z}_p$. Then, conjugate of x i.e., $\bar{x} = a_0 - a_1\hat{i} - a_2\hat{j} - a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p$ is also the root of quadratic equation $x^2 + bx + c = 0$.*

Proof. The proof follows simply by using condition of Theorem 1 applied on the conjugate of x. □

Theorem 4. *Let p be an odd prime, $p \nmid a$, if $x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p$ be the fixed point of function $f(x) = x^2 + (b+1)x + c$, where $a, b, c \in Z_p$. Then, the conjugate of x i.e., $\bar{x} = a_0 - a_1\hat{i} - a_2\hat{j} - a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p$ also be the fix point of function $f(x) = x^2 + (b+1)x + c$.*

Proof. Again, it is enough to use relation $f(x) = g(x) + x$ where $g(x) = x^2 + bx + c$. Then, the existence of fixed points for $f(x)$ is equivalent to the solutions of $g(x)$. Then, the required result is immediate from the above theorem. □

The following two theorems are new results about the number of fixed points of $f(x) = x^2 + (b+1)x + c$.

Theorem 5. $|Fix(f)| = \begin{cases} p^2, & b = 0, c = 0, \\ p^2 + p + 2, & c = 0, b \neq 0. \end{cases}$

Proof. We split the proof in cases.

Case 1: For $c = 0$ and $b = 0$, we obtain two $\mathbb{H}_s/\mathbb{Z}_p \cong \mathbb{M}_2(\mathbb{Z}_p)$,where p is prime. It is easy to see that $\mathbb{H}_s/\mathbb{Z}_p$ and $\mathbb{M}_2(\mathbb{Z}_p)$ are isomorphic as algebras, the map $\varphi : \mathbb{H}_s/\mathbb{Z}_p \longmapsto \mathbb{M}_2(\mathbb{Z}_p)$ is defined as $\varphi(a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = a_0\binom{10}{01} + a_1\binom{0p-1}{10} + a_2\binom{0p-1}{p-10} + a_3\binom{p-10}{01}$. As $\mathbb{H}_s/\mathbb{Z}_p \cong \mathbb{M}_2(\mathbb{Z}_p)$, so we find the number of nilpotent elements in $\mathbb{M}_2(\mathbb{Z}_p)$. It is well-known by Fine and Herstein that the probability that $n \times n$ matrix over a Galois field having p^α elements have $p^{\alpha.n}$ nilpotent elements. As in our case, $\alpha = 1$ and $n = 2$, thus the probability that the 2×2 matrix over $\mathbb{Z}_p$ has p^{-2} nilpotent elements:

$$\frac{|nil(\mathbb{M}_2(\mathbb{Z}_p))|}{|(\mathbb{M}_2(\mathbb{Z}_p|))} = p^{-2}, \tag{12}$$

$$\frac{|nil(\mathbb{M}_2(\mathbb{Z}_p))|}{p^4} = p^{-2}, \tag{13}$$

$$|nil(\mathbb{M}_2(\mathbb{Z}_p))| = p^2. \tag{14}$$

Case 2: For $c = 0$ and $b \neq 0$, we obtain as many points as there are matrices $M_2(\mathbb{Z}_p)$ because of the above isomorphism, and, using the argument given in 2, we arrive at the result. □

Theorem 6. *Let $b \neq 0$ and $c \neq 0$. Then,* $|Fix(f)| = \begin{cases} p^2 - p, & p \equiv 1(mod3), \\ p^2 + p, & p \equiv 2(mod3), \\ 3, & p = 3. \end{cases}$

Proof. Case 1: For $p = 3$, there is nothing to prove.
Case 2: For $p \equiv 1(mod3)$, we have two further cases:

case I: If $p \equiv 3(mod4)$,
$x^2 + y^2 = z$ has a unique solution for z = 0.
$x^2 + y^2 = z$ has $(p+1)$ options for $z \neq 0$, thus $(p+1)(p-1)$ options in all.
Thus, we get that $x^2 + y^2 = z$ has total number of solutions $(p+1)(p-1) + 1 = p^2 - 1 + 1 = p^2$.
Now, when $z = 0$, we get no solution for a_1:

$$\begin{aligned} \sharp &= 1(p+1) + 2(\frac{p-1}{2})(p+1) && (15)\\ &= p + 1 + p^2 - 1 && (16)\\ &= p^2 + p. && (17) \end{aligned}$$

case II: If $p \equiv 1(mod4)$,
$x^2 + y^2 = z$ has $(2p-1)$ solutions for z = 0.
$x^2 + y^2 = z$ has $(p-1)$ options for $z \neq 0$, thus $(p-1)(p-1)$ options in all.
Thus, we get that $x^2 + y^2 = z$ has total number of solutions
$(p-1)(p-1) + (2p-1) = p^2 - p - p + 1 + 2p - 1 = p^2$.
Now, when $z = 0$, we get two solutions for a_1:

$$\begin{aligned} \sharp &= 2(2p-1) + 2(\frac{p-3}{2})(p-1) + 1(p-1) && (18)\\ &= 4p - 2 + p^2 - p - 3p + 3 + p - 1 && (19)\\ &= p^2 + p. && (20) \end{aligned}$$

Case 3: For $p \equiv 2(mod3)$, we have two further cases:
case I: If $p \equiv 3(mod4)$
$x^2 + y^2 = z$ has a unique solution for z = 0.
$x^2 + y^2 = z$ has $(p+1)$ options for $z \neq 0$. So $(p+1)(p-1)$ options in all.
Thus we get, $x^2 + y^2 = z$ has total number of solutions $(p+1)(p-1) + 1 = p^2 - 1 + 1 = p^2$
Now, when $z = 0$, we get no solution for a_1:

$$\begin{aligned} \sharp &= 1(2) + 2(\frac{p-3}{2})(p+1) + 1(p+1) && (21)\\ &= 2 + p^2 + p - 3p - 3 + p + 1 && (22)\\ &= p^2 - p. && (23) \end{aligned}$$

case II: If $p \equiv 1(mod4)$,
$x^2 + y^2 = z$ has $(2p-1)$ solutions for z = 0.
$x^2 + y^2 = z$ has $(p-1)$ options for $z \neq 0$. So $(p-1)(p-1)$ options in all.
Thus we get, $x^2 + y^2 = z$ has total number of solutions
$(p-1)(p-1) + (2p-1) = p^2 - p - p + 1 + 2p - 1 = p^2$.
Now, when $z = 0$, we get two solutions for a_1.

$$\begin{aligned} \sharp &= 1(p-1) + 2(\frac{p-1}{2})(p-1) && (24)\\ &= p - 1 + p^2 - p - p + 1 && (25)\\ &= p^2 - p. && (26) \end{aligned}$$

□

2.2. Some Algebraic Consequences about $\mathbb{H}_s/\mathbb{Z}_p$

We can understand the algebraic structure of $\mathbb{H}_s/\mathbb{Z}_p$ with ease. The following results are simple facts obtained from the previous section.

Corollary 1. *Let p be an odd prime, an element*

$$x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p \tag{27}$$

is idempotent $\Leftrightarrow a_0 = \frac{p+1}{2}$ *and* $a_1^2 - a_2^2 - a_3^2 = \frac{p^2-1}{4}$.

Proof. Taking $a = 1$, $b = p-1$ and $c = p$ in the above theorem, we have

$$\begin{aligned} x^2 + (p-1)x + p &= 0, \\ x^2 - x &= 0, \\ x^2 &= x \end{aligned}$$

has root

$$x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k},$$

where

$$\begin{aligned} a_0 &= \frac{p-b}{2a} \\ &= \frac{p+1}{2}, \end{aligned}$$

and

$$\begin{aligned} a_1^2 - a_2^2 - a_3^2 &= \frac{p^2-b^2}{4a^2} + \frac{c}{a} = \frac{p^2-(-1)^2}{4(1)^2} + \frac{0}{1} \\ &= \frac{p^2-1}{4}. \end{aligned}$$

In other words, we can say x is idempotent. □

We also present similar results but without proof as they can be derived similarly.

Corollary 2. *Let p be an odd prime an element and*

$$x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p \tag{28}$$

is idempotent if and only if $a_0 = \frac{p+1}{2}$ *and* $\|x\| = 0$.

Corollary 3. *Let p be an odd prime and $x \in \mathbb{H}_s/\mathbb{Z}_p$. If x is an idempotent, then $\|x\| = 0$.*

Corollary 4. *Let p be an odd prime. If $x \in \mathbb{H}_s/\mathbb{Z}_p$ is idempotent, then $\bar{x}$ is also an idempotent.*

Corollary 5. *Let p be an odd prime. If $x \in \mathbb{H}_s/\mathbb{Z}_p$ and x is of the form $x = a_0$. If x is idempotent, then it is either* 0 *or* 1.

Corollary 6. *Let p be an odd prime and $x \in \mathbb{H}_s/\mathbb{Z}_p$ of the form*

$$x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \tag{29}$$

where at least one $a_i \neq 0$. Then, x is not an idempotent.

Corollary 7. *Let p be an odd prime, and the quadratic equation $x^2 = 0$ has root $x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \in \mathbb{H}_s/\mathbb{Z}_p$, where $a_0 = \frac{p}{2}$ and $a_1^2 - a_2^2 - a_3^2 = \frac{p^2}{4}$.*

Proof. Taking $a = 1$, $b = 0$ and $c = 0$ in the above theorem, we have that

$$\begin{aligned} x^2 + (p)x + o &= 0, \\ x^2 &= 0 \end{aligned}$$

has root

$$x = a_0 + a_1\hat{i} + a_2\hat{j} + a_3\hat{k},$$

where

$$\begin{aligned} a_0 &= \frac{p-b}{2a} = \frac{p-0}{2} \\ &= \frac{p}{2}, \end{aligned}$$

and

$$\begin{aligned} a_1^2 - a_2^2 - a_3^2 &= \frac{p^2 - b^2}{4a^2} + \frac{c}{a} = \frac{p^2 - (0)^2}{4(1)^2} + \frac{0}{1} \\ a_1^2 - a_2^2 - a_3^2 &= \frac{p^2}{4}. \end{aligned}$$

In other words, we can say x is nilpotent. □

2.3. Some Fixed Points Results of Quadratic Functions in Split Octonions over the Prime Field

Theorem 7. *The quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{Z}_p$, p is an odd prime and $p \nmid a$ has root $x = a_0 + \sum_{i=1}^{7} a_i\acute{t}_i \in \mathbb{O}_s/\mathbb{Z}_p$ if and only if $a_0 = \frac{p-b}{2a}$ and $\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 = (\frac{p^2-b^2}{4a^2}) + \frac{c}{a}$.*

Proof.

$$ax^2 + bx + c = 0.$$

Take $x = a_0 + \sum_{i=1}^{7} a_i \acute{t}_i$, we have

$$\begin{aligned}
x^2 &= (a_0 + \sum_{i=1}^{7} a_i \acute{t}_i)^2 \\
&= (a_0)^2 + (\sum_{i=1}^{7} a_i \acute{t}_i)^2 + 2a_0 \sum_{i=1}^{7} a_i \acute{t}_i \\
&= (a_0)^2 - \sum_{i=1}^{7} a_i^2 + 2a_0 \sum_{i=1}^{7} a_i \acute{t}_i where (\sum_{i=1}^{7} a_i \acute{t}_i)^2 = -\sum_{i=1}^{7} a_i^2 \\
&= (a_0)^2 + a_0^2 - ||x|| + 2a_0 \sum_{i=1}^{7} a_i \acute{t}_i where ||x|| = a_0 + \sum_{i=1}^{7} a_i^2 \\
&= 2(a_0)^2 - ||x|| + 2a_0 \sum_{i=1}^{7} a_i \acute{t}_i \\
&= 2(a_0)^2 + 2a_0 \sum_{i=1}^{7} a_i \acute{t}_i - ||x||, \\
&= 2a_0 x - ||x||.
\end{aligned}$$

Putting it in the above equation, we get

$$a(2a_0 x - ||x||) + bx + c = 0, \tag{30}$$

$$2aa_0 x - a||x|| + bx + c = 0, \tag{31}$$

$$(2aa_0 + b)x - a||x|| + c = 0. \tag{32}$$

Here, $x = a_0 + \sum_{i=1}^{7} a_i \acute{t}_i$ and $||x|| = a_0^2 + \sum_{i=1}^{3} a_i^2 - \sum_{i=3}^{7} a_i^2$, we have

$$\begin{aligned}
(2aa_0 + b)(a_0 + \sum_{i=1}^{7} a_i \acute{t}_i) - a[a_0^2 + \sum_{i=1}^{3} a_i^2 - \sum_{i=3}^{7} a_i^2] + c &= o, \\
(2aa_0 + b)a_0 + (2aa_0 + b) \sum_{i=1}^{7} a_i \acute{t}_i - aa_0^2 - a \sum_{i=1}^{3} a_i^2 + a \sum_{i=3}^{7} a_i^2 + c &= o,
\end{aligned}$$

Comparing vector terms on both sides, we have

$$\begin{aligned}
(2aa_0 + b)a_i &= 0, \\
2aa_0 + b &= 0, \\
a_0 &= \frac{-b}{2a}, \\
a_0 &= \frac{p - b}{2a}.
\end{aligned}$$

Comparing constant terms on both sides, we have

$$\begin{aligned}
(2aa_0 + b)a_0 - aa_0^2 - a \sum_{i=1}^{3} a_i^2 + a \sum_{i=4}^{7} a_i^2 + c &= o, \\
2aa_0^2 + ba_0 - aa_0^2 + c &= a \sum_{i=1}^{3} a_i^2 - a \sum_{i=4}^{7} a_i^2, \\
a_0(aa_0 + b) + c &= a[\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2],
\end{aligned}$$

where $a_0 = \frac{p-b}{2a}$.

$$\begin{aligned}
(\frac{p-b}{2a})(a(\frac{p-b}{2a})+b)+c &= a[\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2], \\
(\frac{p-b}{2a})(\frac{p+b}{2})+c &= a[\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2], \\
(\frac{p^2-b^2}{4a})+c &= a[\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2], \\
(\frac{p^2-b^2}{4a^2})+\frac{c}{a} &= \sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2.
\end{aligned}$$

□

Theorem 8. *The fixed points of function $f(x) = ax^2 + (b+1)x + c$ are $x = a_0 + \sum_{i=1}^{7} a_i \acute{t}_i \in \mathbb{O}_s/\mathbb{Z}_p$, where $a_0 = \frac{p-b}{2a}$ and $\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 = (\frac{p^2-b^2}{4a^2}) + \frac{c}{a}$.*

Proof. It is enough to use relation $f(x) = g(x) + x$ where $g(x) = ax^2 + bx + c$. Then, the existence of fixed points for $f(x)$ is equivalent to the solutions of $g(x)$. Then, the required result is immediate from the above theorem. □

Corollary 8. *The fixed point of function $f(x) = x^2 + x$ are $x = a_0 + \sum_{i=1}^{7} a_i \acute{t}_i \in \mathbb{O}_s/\mathbb{Z}_p$ where $a_0 = \frac{p}{2}$ and $\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 = \frac{p^2}{4}$.*

Proof. It is obvious from the above theorem, only by taking $a = 1$, $b = 0$ and $c = 0$. □

Theorem 9. *Let p be an odd prime. If $x = a_0 + \sum_{i=1}^{7} a_i \acute{t}_i \in \mathbb{O}_s/\mathbb{Z}_p$ is the root of the quadratic equation $ax^2 + bx + c = 0$ $a, b, c \in Z_p$, then $\overline{x} = a_0 + \sum_{i=1}^{7} \acute{a}_i \acute{t}_i \in \mathbb{O}_s/\mathbb{Z}_p$ is also the root of the quadratic equation $ax^2 + bx + c = 0$ $a, b, c \in Z_p$.*

Proof.

$$\begin{aligned}
\overline{x} &= \overline{a_0 + \sum_{i=1}^{7} a_i \acute{t}_i} = a_0 + \sum_{i=1}^{7} \acute{a}_i \overline{t_i} = a_0 - \sum_{i=1}^{7} a_i \acute{t}_i && (33) \\
&= a_0 + \sum_{i=1}^{7} \acute{a}_i \acute{t}_i, && (34)
\end{aligned}$$

where $\acute{a}_i = -a_i$ where $i = 1, 2, ..., 7$ as

$$a_0 = \frac{p-b}{2a}$$

and

$$\begin{aligned}
\sum_{i=1}^{3} \acute{a}_i^2 - \sum_{i=4}^{7} \acute{a}_i^2 &= \sum_{i=1}^{3} (-a_i)^2 - \sum_{i=4}^{7} (-a_i)^2 && (35) \\
&= \sum_{i=1}^{3} (a_i)^2 - \sum_{i=4}^{7} (a_i)^2 && (36) \\
&= \frac{p^2-b^2}{4a^2} + \frac{c}{a}. && (37)
\end{aligned}$$

It implies that $\overline{x}$ is the root of the quadratic equation $ax^2 + bx + c = 0$ $a, b, c \in Z_p$. □

Theorem 10. *If the function $f(x) = ax^2 + (b+1)x + c$ has fixed point $x = a_0 + \sum_{i=1}^{7} a_i t\acute{}_i \in \mathbb{O}_s/\mathbb{Z}_p$, then $\overline{x} = a_0 + \sum_{i=1}^{7} \acute{a}_i t\acute{}_i \in \mathbb{O}_s/\mathbb{Z}_p$ also is the fixed point of function $f(x) = ax^2 + (b+1)x + c$.*

Proof. It is enough to use relation $f(x) = g(x) + x$, where $g(x) = ax^2 + bx + c$. Then, the existence of fixed points for $f(x)$ is equivalent to the solutions of $g(x)$. Then, the required result is immediate from the above theorem. □

3. Some Algebraic Consequences about $\mathbb{O}_s/\mathbb{Z}_p$

Proposition 1. *Let p be an odd prime and an element*

$$x = a_0 + \sum_{i=1}^{7} a_i t\acute{}_i \in \mathbb{O}_s/\mathbb{Z}_p$$

is idempotent $\Leftrightarrow a_0 = \frac{p+1}{2}$ *and*

$$\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 = \frac{p^2-1}{4}.$$

Proof. Taking $a = 1$, $b = p - 1$ and $c = p$ in the above theorem, we have

$$\begin{aligned} x^2 + (p-1)x + p &= 0, & (38)\\ x^2 - x &= 0, & (39)\\ x^2 &= x & (40) \end{aligned}$$

has root

$$x = a_0 + \sum_{i=1}^{7} a_i t\acute{}_i, \tag{41}$$

where

$$\begin{aligned} a_0 &= \frac{p-b}{2a} = \frac{p-p+1}{2} & (42)\\ &= \frac{1}{2}, & (43) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 &= \frac{p^2-b^2}{4a^2} + \frac{c}{a} = \frac{p^2-(-1)^2}{4(1)^2} + \frac{0}{1} & (44)\\ &= \frac{p^2-1}{4}. & (45) \end{aligned}$$

In other words, we can say that x is idempotent. □

Proposition 2. *Let p be an odd prime and element*

$$x = a_0 + \sum_{i=1}^{7} a_i t\acute{}_i \in \mathbb{O}_s/\mathbb{Z}_p \tag{46}$$

is idempotent if and only if $a_0 = \frac{p+1}{2}$ and $\|x\| = 0$.

Proposition 3. *Let p be an odd prime and $x \in \mathbb{O}_s/\mathbb{Z}_p$. If x is an idempotent, then $\|x\| = 0$.*

Proposition 4. *Let p be an odd prime. If $x \in \mathbb{O}_s/\mathbb{Z}_p$ is idempotent, then $\bar{x}$ is also an idempotent.*

Proposition 5. *Let p be an odd prime. If $x = a_0 \in \mathbb{O}_s/\mathbb{Z}_p$ is idempotent, then it is either 0 or 1.*

Proposition 6. *Let p be an odd prime and $x \in \mathbb{O}_s/\mathbb{Z}_p$ be of the form*

$$x = \sum_{i=1}^{7} a_i \acute{t}_i, \tag{47}$$

where at least one $a_i \neq 0$. Then, x is not an idempotent.

Proposition 7. *Let p be an odd prime and the quadratic equation $x^2 = 0$ has root $x = a_0 + \sum_{i=1}^{7} a_i \acute{t}_i \in \mathbb{O}_s/\mathbb{Z}_p$, where $a_0 = \frac{p}{2}$ and $\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 = \frac{p^2}{4}$.*

Proof. Taking $a = 1$, $b = 0$ and $c = 0$ in the above theorem, we have

$$\begin{aligned} x^2 + (p)x + o &= 0, && (48) \\ x^2 &= 0 && (49) \end{aligned}$$

has root

$$x = a_0 + \sum_{i=1}^{7} a_i \acute{t}_i, \tag{50}$$

where

$$\begin{aligned} a_0 &= \frac{p-b}{2a} = \frac{p-0}{2} && (51) \\ &= \frac{p}{2} && (52) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 &= \frac{p^2 - b^2}{4a^2} + \frac{c}{a} = \frac{p^2 - (0)^2}{4(1)^2} + \frac{0}{1} && (53) \\ &= \frac{p^2}{4}. && (54) \end{aligned}$$

In other words, we can say that x is nilpotent. □

Using results of the previous section and programs mentioned in the Appendix A, we can give many examples.

4. Examples

In this section, we add examples relating to the previous section. These results are generated by the codes given in Appendix A. These along with other examples can be created using codes, and results can be applied to crypto systems and communication channel systems.

Example 1. *We find all solutions of $x^2 - x = 0$ over $\mathbb{H}_s/\mathbb{Z}_7$. As above, we see that, if $x \in \mathbb{H}_s/\mathbb{Z}_7$, then $a_0 = \frac{7-(-1)}{2} = 4$ and following are the values of a_1, a_2 and a_3, respectively, satisfying the equation $-a_1^2 + a_2^2 + a_3^2 = -5$ or $a_1^2 - a_2^2 - a_3^2 = 5$.*

$$
\begin{array}{cccccccc}
(0,1,1) & (0,1,6) & (0,3,0) & (0,4,0) & (0,1,1) & (0,6,1) & (0,6,6) & (1,1,3) \\
(1,1,4) & (1,3,1) & (1,3,6) & (1,4,1) & (1,4,6) & (1,6,4) & (2,2,3) & (2,2,4) \\
(2,3,2) & (2,3,3) & (2,3,5) & (2,4,2) & (2,4,3) & (2,4,5) & (2,5,3) & (2,5,4) \\
(3,0,2) & (3,0,5) & (3,2,0) & (3,3,3) & (3,3,4) & (3,4,3) & (3,4,4) & (3,5,0) \\
(4,0,2) & (4,0,5) & (4,2,0) & (4,3,3) & (4,3,4) & (4,4,3) & (4,4,4) & (4,5,0) \\
(5,0,6) & (5,2,3) & (5,2,4) & (5,3,2) & (5,3,5) & (5,4,2) & (5,4,5) & (5,5,3) \\
(5,5,4) & (5,6,0) & (6,1,4) & (6,3,6) & (6,4,1) & (6,4,6) & (6,6,3) & (6,6,4)
\end{array} .
$$

Example 2. *We compute all solutions of $2x^2 + x = 0$ over $\mathbb{H}_s/\mathbb{Z}_5$. As above, we see that, if $x \in \mathbb{H}_s/\mathbb{Z}_5$, then $a_0 = 1$ and following are the values of a_1, a_2 and a_3, respectively, satisfying the equation $-a_1^2 + a_2^2 + a_3^2 = -4$ or $a_1^2 - a_2^2 - a_3^2 = 4$.*

$$
\begin{array}{ccccc}
(0,0,1) & (1,4,1) & (2,2,4) & (3,1,2) & (3,4,2) \\
(0,0,4) & (1,4,4) & (2,3,1) & (3,1,3) & (3,4,3) \\
(0,1,0) & (2,0,0) & (2,3,4) & (3,2,1) & (4,1,1) \\
(0,4,0) & (2,1,2) & (2,4,2) & (3,2,4) & (4,1,4) \\
(1,1,1) & (2,1,3) & (2,4,3) & (3,3,1) & (4,4,1) \\
(1,1,4) & (2,2,1) & (3,0,0) & (3,3,4) & (4,4,4)
\end{array} .
$$

Example 3. *We compute all solutions of $x^2 + x + 1 = 0$ over $\mathbb{H}_s/\mathbb{Z}_7$. As above, we see that, if $x \in \mathbb{H}_s/\mathbb{Z}_7$, then $a_0 = 3$ and following are the values of a_1, a_2 and a_3, respectively satisfying the equation $-a_1^2 + a_2^2 + a_3^2 = -6$ or $a_1^2 - a_2^2 - a_3^2 = 6$.*

$$
\begin{array}{ccccccc}
(0,0,1) & (1,0,3) & (6,0,3) & (2,1,2) & (5,1,2) & (3,1,3) & (4,1,3) \\
(0,0,6) & (1,0,4) & (6,0,4) & (2,1,5) & (5,1,5) & (3,1,4) & (4,1,4) \\
(0,1,0) & (1,1,6) & (6,1,6) & (2,2,1) & (5,2,1) & (3,3,1) & (4,3,1) \\
(0,2,2) & (1,3,0) & (6,3,0) & (2,2,6) & (5,2,6) & (3,3,6) & (4,3,6) \\
(0,2,5) & (1,4,0) & (6,4,0) & (2,5,1) & (5,5,1) & (3,4,1) & (4,4,1) \\
(0,5,2) & (1,6,1) & (6,6,1) & (2,5,6) & (5,5,6) & (3,4,6) & (4,4,6) \\
(0,5,5) & (1,1,1) & (6,1,1) & (2,6,2) & (5,6,2) & (3,6,3) & (4,6,2) \\
(0,6,0) & (1,6,6) & (6,6,6) & (2,6,5) & (5,6,5) & (3,6,4) & (4,6,5)
\end{array} .
$$

Example 4. *We compute all solutions of $x^2 = 0$ over $\mathbb{H}_s/\mathbb{Z}_5$. As above, we see that, if $x \in \mathbb{H}_s/\mathbb{Z}_5$, then $a_0 = 0$ and following are the values of a_1, a_2 and a_3, respectively, satisfying the equation $-a_1^2 + a_2^2 + a_3^2 = 0$ or $a_1^2 - a_2^2 - a_3^2 = 0$.*

$$
\begin{array}{ccccc}
(0,0,0) & (0,3,4) & (1,4,0) & (2,0,3) & (3,3,0) \\
(0,1,2) & (0,4,2) & (4,0,1) & (2,2,0) & (0,3,1) \\
(0,1,3) & (0,4,3) & (4,0,4) & (2,3,0) & (1,1,0) \\
(0,2,1) & (1,0,1) & (4,1,0) & (3,0,2) & (2,0,2) \\
(0,2,4) & (1,0,4) & (4,4,0) & (3,0,3) & (3,2,0)
\end{array} .
$$

Example 5. *We compute all solutions of $x^2 - x = 0$ over $\mathbb{O}_s/\mathbb{Z}_3$ (idempotents in the split octonion algebra). As above we see that $x = a_0 + \sum_{i=1}^{7} a_i \acute{f}_i \in \mathbb{O}_s/\mathbb{Z}_3$ where $a_0 = \frac{3-(-1)}{2} = 2$ and following is the values of a_1, a_2, a_3, a_4, a_5, a_6 and a_7 respectively satisfying the equation $\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 = (\frac{p^2-b^2}{4a^2}) + \frac{c}{a} = 2$.*
We do so by putting values for p = 3, a = 1, b = −1, c = 0 in above given code.

$$(2,1,0,1,1,0,2);(2,1,0,1,1,1,0);(2,1,0,1,1,2,0);(2,1,0,1,2,0,1);$$
$$(2,1,0,1,2,0,2);(2,1,0,1,2,1,0);(2,1,0,1,2,2,0);(2,1,0,2,0,1,1);$$
$$(2,1,0,2,0,1,2);(2,1,0,2,0,2,1);(2,1,0,2,0,2,2);(2,1,0,2,1,0,1);$$
$$(2,1,0,2,1,0,2);(2,1,0,2,1,1,0);(2,1,0,2,1,2,0);(2,1,0,2,2,0,1);$$
$$(2,1,0,2,2,0,2);(2,1,0,2,2,1,0);(2,1,0,2,2,2,0);(2,1,1,0,0,0,1);$$
$$(2,1,1,0,0,0,2);(2,1,1,0,0,1,0);(2,1,1,0,0,2,0);(2,1,1,0,1,0,0);$$
$$(2,1,1,0,2,0,0);(2,1,1,1,0,0,0);(2,1,1,1,1,1,1);(2,1,1,1,1,1,2);$$
$$(2,1,1,1,1,2,1);(2,1,1,1,1,1,1);(2,1,1,1,1,1,2);(2,1,1,1,1,2,1);$$
$$(2,1,1,1,1,2,2);(2,1,1,1,2,1,1);(2,1,1,1,2,1,2);(2,1,1,1,2,2,1);$$
$$(2,1,1,1,2,2,2);(2,1,1,2,0,0,0);(2,1,1,2,1,1,1);(2,1,1,2,1,1,2);$$
$$(2,1,1,2,1,2,1);(2,1,1,2,1,2,2);(2,1,1,2,2,1,1);(2,1,1,2,2,1,2);$$
$$(2,1,1,2,2,2,1);(2,1,1,2,2,2,2);(2,1,2,0,0,0,1);(2,1,2,0,0,0,2);$$
$$(2,1,2,0,0,1,0);(2,1,2,0,0,2,0);(2,1,2,0,1,0,0);(2,1,2,0,2,0,0);$$
$$(2,1,2,1,0,0,0);(2,1,2,1,1,1,1);(2,1,2,1,1,1,2);(2,1,2,1,1,2,1);$$
$$(2,1,2,1,1,2,2);(2,1,2,1,2,1,1);(2,1,2,1,2,1,2);(2,1,2,1,2,2,1);$$
$$(2,1,2,1,2,2,2);(2,1,2,2,0,0,0);(2,1,2,2,1,1,1);(2,1,2,2,1,1,2);$$
$$(2,1,2,2,1,2,1);(2,1,2,2,1,2,2);(2,1,2,2,2,1,1);(2,1,2,2,2,1,2);$$
$$(2,1,2,2,2,2,1);(2,1,2,2,2,2,2);(2,2,0,0,0,0,0);(2,2,0,0,1,1,1);$$
$$(2,2,0,0,1,1,2);(2,2,0,0,1,2,1);(2,2,0,0,1,2,2);(2,2,0,0,1,2,2);$$
$$(2,2,0,0,2,1,1);(2,2,0,0,2,1,2);(2,2,0,0,2,2,1);(2,2,0,0,2,2,2);$$
$$(2,2,0,1,0,1,1);(2,2,0,1,0,1,2);(2,2,0,1,0,2,1);(2,2,0,1,0,2,2);$$
$$(2,2,0,1,1,0,1);(2,2,0,1,1,0,2);(2,2,0,1,1,1,0);(2,2,0,1,1,2,0);$$
$$(2,2,0,1,2,0,1);(2,2,0,1,2,0,2);(2,2,0,1,2,1,0);(2,2,0,1,2,2,0);$$
$$(2,2,0,2,0,1,1);(2,2,0,2,0,1,1);(2,2,0,2,0,1,2);(2,2,0,2,0,2,1);$$
$$(2,2,0,2,0,2,2);(2,2,0,2,1,0,1);(2,2,0,2,1,0,2);(2,2,0,2,1,1,0);$$
$$(2,2,0,2,1,2,0);(2,2,0,2,2,0,1);(2,2,0,2,2,0,2);(2,2,0,2,2,1,0);$$
$$(2,2,0,2,2,2,0);(2,2,1,0,0,0,1);(2,2,1,0,0,0,2);(2,2,1,0,0,1,0);$$
$$(2,2,1,0,0,2,0);(2,2,1,0,1,0,0);(2,2,1,0,2,0,0);(2,2,1,1,0,0,0);$$
$$(2,2,1,1,1,1,1);(2,2,1,1,1,1,1);(2,2,1,1,1,1,2);(2,2,1,1,1,2,1);$$
$$(2,2,1,1,1,2,2);(2,2,1,1,2,1,1);(2,2,1,1,2,1,2);(2,2,1,1,2,2,1);$$
$$(2,2,1,1,2,2,2);(2,2,1,2,0,0,0);(2,2,1,2,1,1,1);(2,2,1,2,1,1,2);$$
$$(2,2,1,2,1,2,1);(2,2,1,2,1,2,2);(2,2,1,2,2,1,1);(2,2,1,2,2,1,2);$$
$$(2,2,1,2,2,2,1);(2,2,1,2,2,2,2);(2,2,2,0,0,0,1);(2,2,2,0,0,0,2);$$
$$(2,2,2,0,0,1,0);(2,2,2,0,0,2,0);(2,2,2,0,1,0,0);(2,2,2,0,2,0,0);$$
$$(2,2,2,1,0,0,0);(2,2,2,1,1,1,1);(2,2,2,1,1,1,2);(2,2,2,1,1,2,1);$$
$$(2,2,2,1,1,2,2);(2,2,2,1,2,1,2);(2,2,2,1,2,2,1);(2,2,2,1,2,2,2);$$
$$(2,2,2,2,0,0,0);(2,2,2,2,1,1,1);(2,2,2,2,1,1,2);(2,2,2,2,1,2,1);$$
$$(2,2,2,2,1,2,2);(2,2,2,2,2,1,1);(2,2,2,2,2,1,2);(2,2,2,2,2,2,1);$$
$$(2,2,2,2,2,2,2).$$

5. Conclusions and Further Directions

In this article, we produced some general results about fixed points of a general quadratic polynomial in algebras of split quaternion and octonion over $\mathbb{Z}_p$. We not only characterized these points in terms of the coefficients of these polynomials but also gave the cardinality of these points and also the programs that produced fixed points. We arrived at the following new results for a general quadratic function.

Theorem 11. $|Fix(f)| = \begin{cases} p^2, & b=0, c=0, \\ p^2+p+2, & c=0, b\neq 0. \end{cases}$

Theorem 12. *Let $b \neq 0$ and $c \neq 0$. Then,* $|Fix(f)| = \begin{cases} p^2-p, & p\equiv 1(mod3); \\ p^2+p, & p\equiv 2(mod3); \\ 3, & p=3. \end{cases}$

We also give the following two new results for the fixed points of a general quadratic quaternionic equation without proofs. Proofs are left as an open problem.

Theorem 13. $|Fix(f)| = \begin{cases} p^6, & b=0, c=0; \\ p^6+p^3, & c=0, b\neq 0. \end{cases}$

Theorem 14. *Let $b \neq 0$ and $c \neq 0$. Then,* $|Fix(f)| = \begin{cases} p^6+p^3, & p\equiv 1(mod3); \\ p^6-p^3, & p\equiv 2(mod3); \\ p^6, & p=3. \end{cases}$

We like to remark that new results can be obtained for a general cubic polynomials in these algebras.

6. Data Availability Statement

No such data has been used to prove these results.

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Appendix A. Computer Codes

Here, we put together some programs to compute fixed points and roots easily.

Appendix A.1. Program for Finding Solutions of the Quadratic Equation in $\mathbb{H}_s/\mathbb{Z}_p$

Following codes, count and print the number of solutions of quadratic equation $ax^2+bx+c=0$ in $\mathbb{H}_s/\mathbb{Z}_p$. These codes print the string a_1, a_2, a_3 with the understanding that the co-efficient $a_0 = \frac{p-b}{2a}$ is fixed in $\mathbb{H}_s/\mathbb{Z}_p$ and satisfying the relation $a_1^2-a_2^2-a_3^2 = (\frac{p^2-b^2}{4a^2}) + \frac{c}{a}$ or $-a_1^2+a_2^2+a_3^2 = (\frac{-p^2+b^2}{4a^2}) - \frac{c}{a}$ for $\mathbb{H}_s/\mathbb{Z}_p$.

CODE: This code will give solutions of the quadratic equation only by putting values for p, a, b, c, where p is an odd prime and $a, b, c\varepsilon\mathbb{Z}_p$.

```
#include<iostream>
```

```
#include<conio.h>
using namespace std;

main()
{
int a1, a2, a3, p, n, a, b, c, count;
count=0;
cout<<"Enter value for p: ";
cin>>p;
cout<<"Enter value for a: ";
cin>>a;
cout<<"Enter value for b: ";
cin>>b;
cout<<"Enter value for c: ";
cin>>c;
n=((p*p-b*b)/(4*a*a))+(c/a);
while(n<0)
n=n+p;
for(int i=0; i<p; i++)
{
a3=i;
for(int j=0; j<p; j++)
{
a2=j;
for(int k=0; k<p; k++)
{
a1=k;
int sum=(a1*a1)-(a2*a2)-(a3*a3);
while(sum<0)
sum=sum+p;
if(sum%p==n)
{
count++;
cout<<a1<<" "<<a2<<" "<<a3<<endl;
}
}
}
}
cout<<"\nCount: "<<count;
getch();
}
```

Appendix A.2. Program for Finding Roots of the Quadratic Equation in $\mathbb{O}_s/\mathbb{Z}_p$

Following codes, count and print the number of solutions of quadratic equation $ax^2 + bx + c = 0$ in $\mathbb{O}_s/\mathbb{Z}_p$. These codes print the string a_1, a_2, a_3, a_4, a_5, a_6, a_7 with the understanding that the co-efficient $a_0 = \frac{p-b}{2a}$ is fixed in $\mathbb{O}_s/\mathbb{Z}_p$ and satisfying the relation $\sum_{i=1}^{3} a_i^2 - \sum_{i=4}^{7} a_i^2 = (\frac{p^2-b^2}{4a^2}) + \frac{c}{a}$ for $\mathbb{O}_s/\mathbb{Z}_p$.

CODE: This code will give solutions of the quadratic equation only by putting values for p, a, b, c, where p is an odd prime and $a, b, c \in \mathbb{Z}_p$.

```
#include <iostream>
```

```
#include <fstream>
using namespace std;
int main(){
  int a1,a2,a3,a4,a5,a6,a7;
  int sum=0;
  int p;
  int n=0;
  int count=2;
  int totalCount=0;
  cout<<"Enter value of p: ";
  cin>>p;
  n = ((p*p -1)/4)%p;
  for(int i=0;i<p;i++)
  {
   a1 = i;
   for(int j=0; j<p; j++)
   {
    a2 = j;
    for(int k=0; k<p; k++)
    {
     a3 = k;
      for(int l=0; l<p;l++)
      {
       a4 = l;
       for(int m=0; m<p; m++)
       {
        a5 = m;
        for(int q=0; q<p; q++)
        {
         a6 = q;
         for(int r=0; r<p;r++)
         {
          a7 = r;
          totalCount++;
          cout<<a1<<" "<<a2<<" "<<a3<<" "<<a4<<" "<<a5
              <<" "<<a6<<" "<<a7<<endl;
          //dataFile << a1 << endl;
          sum = a1*a1+a2*a2+a3*a3-a4*a4-a5*a5-a6*a6-a7*a7;
          if(sum%p == n)
           count++;
          }
         }
        }
       }
      }
     }
    }
    cout<<"Total Count is: "<<totalCount<<endl;
    cout<<"Count is: "<<count<<endl;
    system("pause");
```

```
return 0;
}
```

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