Generalized Symmetries and mCK Method Analysis of the (2+1)-Dimensional Coupled Burgers Equations

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Abstract: In this paper, generalized symmetries and mCK method are employed to analyze the (2+1)-dimensional coupled Burgers equations. Firstly, based on the generalized symmetries method, the corresponding symmetries of the (2+1)-dimensional coupled Burgers equations are derived. And then, using the mCK method, symmetry transformation group theorem is presented. From symmetry transformation group theorem, a great many of new solutions can be derived. Lastly, Lie algebra for given symmetry group are considered.

Keywords: (2+1)-dimensional coupled Burgers equations; generalized symmetries; mCK method; new exact solutions

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1. Introduction

In this paper, by using the generalized symmetries and mCK methods, we will study the following (2+1)-dimensional coupled Burgers equations

$$\begin{cases} u_t - 2uu_x - 2u_y v - u_{xx} - u_{yy} = 0, \\ v_t - 2vv_y - 2uv_x - v_{xx} - v_{yy} = 0. \end{cases} \tag{1}$$

It is well known that Burger's types of equations [1,2] can be encountered in many fields, such as fluid mechanics, shock wave formation, boundary layer behavior, continuum traffic simulation, convection dominated diffusion phenomena and so on. As more physical description of the (2+1)-dimensional coupled Burgers equations see ([1–11]). There are many authors have been employed various methods to study these types of equations ([1,3–11]). In paper ([8]), one of the authors and his collaborators considered Equation (1) with the Lie symmetry method and suitable Painleve truncated expansions, they derived some exact solutions and conservation laws for Equation (1). In paper ([9]), one of the authors and his collaborators studied a new coupled (2+1)-dimensional Burgers equations, they obtained the symmetry, soliton solutions for the equations. The authors ([12]), based on the Darboux transformation, studied the explicit solutions of a modified 2+1-dimensional coupled Burgers equation. In paper ([13]), they considered finite symmetry transformation group, they also considered localized structures of the (2+1)-dimensional coupled Burgers equation. We generalized their results and give another results. The authors ([14]) give some new exact solutions of Burgers type equations

with conformable derivative. The authors ([15]) studied time fractional simplified modified Kawahara equation using symmetry method. Motivated by theses papers, it is necessary consider them again in different methods.

This article is a continuation of the previous ones ([8,9]). In order to provide more information about the (2+1) dimensional Burgers equation, it is necessary to consider the coupled (2+1) dimensional Burgers equation in different ways. Perhaps in this way, we can find the connections and differences between the different methods.

Our aim in this work is to consider the (2+1)-dimensional coupled Burgers equations using the generalized symmetries method and mCK method. This article is divided into the following parts: Firstly, we consider the (2+1)-dimensional coupled Burgers equations using the generalized symmetry method. Secondly, mCK method is employed to derive new exact solutions for the (2+1)-dimensional coupled Burgers equations. Lie algebra for given symmetry group also derived. Lastly, the conclusions are presented.

2. Generalized Symmetries for the (2+1)-Dimensional Coupled Burgers Equations

Now we studied the symmetries of Equation (1.1) by using the generalized symmetry method ([16–18]).

If nonlinear evolution equation are given, that is to say,

$$F(t, x, y, u, u_t, u_x, u_y, v, v_t \cdots) = 0, \tag{2}$$

for all solutions of u, we say a function σ is a symmetry of Equation (2), if it satisfies the following equation

$$F'(u, v)\sigma = 0, \tag{3}$$

where

$$F'(u, v)\sigma = \frac{\partial F}{\partial u}\sigma + \frac{\partial F}{\partial u_t}\sigma_t + \frac{\partial F}{\partial u_x}\sigma_x + \frac{\partial F}{\partial v_t}\sigma_t + \frac{\partial F}{\partial v_x}\sigma_x + \frac{\partial F}{\partial u_y}\sigma_y + \frac{\partial F}{\partial u_{tx}}\sigma_{tx} + \frac{\partial F}{\partial u_{xx}}\sigma_{xx} + \cdots \tag{4}$$

It is immediately generate the symmetry equations of Equation (1) as follows

$$\begin{aligned}
\sigma_{1t} - 2\sigma_1 u_x - 2u\sigma_{1x} - 2\sigma_{1y}v - 2u_y\theta - \sigma_{1xx} - \sigma_{1yy} &= 0, \\
\theta_t - 2\theta_x u - 2\sigma_1 v_x - 2v_y\theta - 2\theta_y v - \theta_{xx} - \theta_{yy} &= 0.
\end{aligned} \tag{5}$$

In order to get our goal, we set the following functions,

$$\begin{aligned}
\sigma_1 &= au_x + bu_t + cu_y + eu + gv + l, \\
\theta &= av_x + bv_t + cv_y + mv + hu + k,
\end{aligned} \tag{6}$$

where a, b, c, e, g, l, m, h and k are functions of t, x and y to be determined.

Substituting (6) into (5) and using (1), and replace u_t by $2uu_x + 2u_yv + u_{xx} + u_{yy}$, and v_t by $2vv_y + 2uv_x + v_{xx} + v_{yy}$. And then we get the determining equations, it is generate the following results

$$\begin{aligned}
a &= (c_1t + c_2)x + 2c_6t - c_3y + c_7, \quad b = c_1t^2 + 2c_2t + c_4, \\
c &= (c_1t + c_2)y + 2c_5t + c_3x + c_8, e = m = c_1t + c_2, g = c_3, \\
l &= \frac{c_1}{2}x + c_6, h = -c_3, k = \frac{c_1}{2}y + c_5,
\end{aligned} \tag{7}$$

here $c_i(i = 1, 2 \cdots 8)$ are arbitrary constants. Putting (7) into (6), we have

$$
\begin{aligned}
\sigma_1 = {} & \left((c_1 t + c_2)x + 2c_6 t - c_3 y + c_7\right)u_x + (c_1 t^2 + 2c_2 t + c_4)u_t + \Big((c_1 t \\
& + c_2)y + 2c_5 t + c_3 x + c_8\Big)u_y + (c_1 t + c_2)u + c_3 v + \frac{c_1}{2}x + c_6, \\
\theta = {} & \left((c_1 t + c_2)x + 2c_6 t - c_3 y + c_7\right)u_x + (c_1 t^2 + 2c_2 t + c_4)u_t + \Big((c_1 t \\
& + c_2)y + 2c_5 t + c_3 x + c_8\Big)u_y + (c_1 t + c_2)v - c_3 u + \frac{c_1}{2}y + c_5.
\end{aligned}
\tag{8}
$$

In other words, we get

$$
\begin{aligned}
\sigma = {} & c_1\left[xtu_x + t^2 u_t + tyu_y + (tu + \tfrac{1}{2}x)u + (tv + \tfrac{1}{2}y)v\right] \\
& + c_2\left[xu_x + 2tu_t + yu_y + u + v\right] + c_3\left[-yu_x + xu_y + v - u\right] \\
& + c_4 u_t + c_5(2tu_y + 1) + c_6(2tu_x + 1) + c_7 u_x + c_8 u_y.
\end{aligned}
\tag{9}
$$

So we derived the equivalent vector expression is follows

$$
\begin{aligned}
V = {} & \left((c_1 t + c_2)x + 2c_6 t - c_3 y + c_7\right)\tfrac{\partial}{\partial x} + (c_1 t^2 + 2c_2 t + c_4)\tfrac{\partial}{\partial t} + \left((c_1 t + c_2)y + 2c_5 t + c_3 x + c_8\right)\tfrac{\partial}{\partial y} \\
& - \left((c_1 t + c_2)u + c_3 v + \tfrac{c_1}{2}x + c_6\right)\tfrac{\partial}{\partial u} - \left((c_1 t + c_2)v - c_3 u + \tfrac{c_1}{2}y + c_5\right)\tfrac{\partial}{\partial v}.
\end{aligned}
\tag{10}
$$

In fact, it is coincide with the obtained vector field $V_i(i = 1, 2 \cdots 8)$ of paper [8]. That is to say, for this assumption, it is the same results for the Lie symmetry method. However, for other assumption, we do not know what happened, we will studied them in future work. In paper [3], the authors derived the similarity reductions in detail. Here, we do not list all of them.

3. Non-Lie Symmetry Groups for the (2+1)-Dimensional Coupled Burgers Equations

In this section, we use the mCK method [19] to analyze the (2+1)-dimensional coupled Burgers equations. Lou et al. proposed a simple method (called the mCK method) to get the non-Lie point symmetry group [19] of the given nonlinear evolution equations. Firstly, we consider the following transformation

$$
\begin{aligned}
u(x, y, t) &= \alpha_1 + \beta_1 U(\xi, \eta, \tau) + \gamma_1 V(\xi, \eta, \tau), \\
v(x, y, t) &= \alpha_2 + \beta_2 V(\xi, \eta, \tau) + \gamma_2 U(\xi, \eta, \tau),
\end{aligned}
\tag{11}
$$

where $\alpha_1 = \alpha_1(x, y, t), \beta_1 = \beta_1(x, y, t), \alpha_2 = \alpha_2(x, y, t), \beta_2 = \beta_2(x, y, t), \gamma_1 = \gamma_1(x, y, t), \gamma_2 = \gamma_2(x, y, t), \xi = \xi(x, y, t), \eta = \eta(x, y, t)$ and $\tau = \tau(x, y, t)$ are functions of $\{x, y, t\}$ to be further determined. In this case, based on the mCK method, it is requiring $U(\xi, \eta, \tau)$ and $V(\xi, \eta, \tau)$ are satisfy the same as $u(x, y, t)$ and $v(x, y, t)$ under the transformation $\{u, v, x, y, t\} \to \{U, V, \xi, \eta, \tau\}$.

Putting Equation (11) into Equation (1) and requiring that $U(\xi, \eta, \tau)$ and $V(\xi, \eta, \tau)$ satisfy the same equations as Equation (1)

$$
\begin{cases}
U_\tau - 2UU_\xi - 2U_\eta V - U_{\xi\xi} - U_{\eta\eta} = 0, \\
V_\tau - 2VV_\eta - 2V_\xi U - V_{\xi\xi} - V_{\eta\eta} = 0.
\end{cases}
\tag{12}
$$

Replacing U_τ by $2UU_\xi + 2U_\eta V + U_{\xi\xi} + U_{\eta\eta}$, and V_τ by $2VV_\eta + 2V_\xi U + V_{\xi\xi} + V_{\eta\eta}$, we can arrive at complicated differential equations with U, V and their different derivatives. After that, we ruling out the coefficients of U, V and their derivatives, one obtains over-determined equations for the unknown functions of $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2, \xi, \eta, \tau\}$.

$$\tau_x = 0, \tau_y = 0,$$
$$\gamma_{1t} - \gamma_{1xx} - \gamma_{1yy} - 2\alpha_1\gamma_{1x} - 2\gamma_1\alpha_{1x} - 2\alpha_2\gamma_{1y} - 2\beta_2\alpha_{1y} = 0,$$
$$\beta_{1t} - \beta_{1xx} - \beta_{1yy} - 2\alpha_1\beta_{1x} - 2\beta_1\alpha_{1x} - 2\alpha_2\beta_{1y} - 2\gamma_2\alpha_{1y} = 0,$$
$$\alpha_{1t} - 2\alpha_1\alpha_{1x} - 2\alpha_2\alpha_{1y} - \alpha_{1xx} - \alpha_{1yy} = 0,$$
$$\beta_1(\tau_t - \xi_x^2 - \xi_y^2) = 0, \gamma_1(\tau_t - \xi_x^2 - \xi_y^2) = 0,$$
$$\beta_1\xi_t - \beta_1\xi_{yy} - \beta_1\xi_{xx} - 2\beta_{1x}\xi_x - 2\beta_{1y}\xi_y - 2\alpha_1\beta_1\xi_x - 2\alpha_2\beta_1\xi_y = 0,$$
$$\gamma_1\xi_t - \gamma_1\xi_{yy} - \gamma_1\xi_{xx} - 2\gamma_{1x}\xi_x - 2\gamma_{1y}\xi_y - 2\alpha_1\gamma_1\xi_x - 2\alpha_2\gamma_1\xi_y = 0,$$
$$\beta_1(\tau_t - \eta_x^2 - \eta_y^2) = 0, \gamma_1(\tau_t - \eta_x^2 - \eta_y^2) = 0,$$
$$\beta_1\eta_t - \beta_1\eta_{yy} - \beta_1\eta_{xx} - 2\beta_{1x}\eta_x - 2\beta_{1y}\eta_y - 2\alpha_1\beta_1\eta_x - 2\alpha_2\beta_1\eta_y = 0,$$
$$\gamma_1\eta_t - \gamma_1\eta_{yy} - \gamma_1\eta_{xx} - 2\gamma_{1x}\eta_x - 2\gamma_{1y}\eta_y - 2\alpha_1\gamma_1\eta_x - 2\alpha_2\gamma_1\eta_y = 0,$$
$$-2\beta_{1y}\gamma_2 - 2\beta_{1x}\beta_1 = 0, -2\gamma_{1x}\gamma_1 - 2\gamma_{1y}\beta_2 = 0, \tag{13}$$

$$2\gamma_1(\tau_t - \gamma_2\xi_y - \beta_1\xi_x) = 0, 2\beta_1(\tau_t - \beta_2\eta_y - \gamma_1\eta_x) = 0,$$
$$-2\beta_1\gamma_1\eta_x - 2\gamma_1\gamma_2\eta_y = 0, -2\beta_1\gamma_1\xi_x - 2\beta_1\beta_2\xi_y = 0,$$
$$2\beta_1(\tau_t - \beta_1\xi_x - \gamma_2\xi_y) = 0, 2\gamma_1(\tau_t - \gamma_1\eta_x - \beta_2\eta_y) = 0,$$
$$-2\beta_1(\gamma_2\eta_y + \beta_1\eta_x) = 0, -2\gamma_1(\gamma_1\xi_x + \beta_2\xi_y) = 0,$$
$$-2\gamma_1(\eta_x\xi_x + \eta_y\xi_y) = 0, -2\beta_1(\eta_x\xi_x + \eta_y\xi_y) = 0, \tag{14}$$
$$-2\beta_1\gamma_{1x} - 2\gamma_1\beta_{1x} - 2\gamma_{1y}\gamma_2 - 2\beta_{1y}\beta_2 = 0,$$
$$\alpha_{2t} - 2\alpha_1\alpha_{2x} - 2\alpha_2\alpha_{2y} - \alpha_{2xx} - \alpha_{2yy} = 0,$$
$$\gamma_2\eta_t - \gamma_2\eta_{yy} - \gamma_2\eta_{xx} - 2\gamma_{2x}\eta_x - 2\gamma_{2y}\eta_y - 2\alpha_1\gamma_2\eta_x - 2\alpha_2\gamma_2\eta_y = 0,$$
$$\gamma_2\xi_t - \gamma_2\xi_{yy} - \gamma_2\xi_{xx} - 2\gamma_{2x}\xi_x - 2\gamma_{2y}\xi_y - 2\alpha_1\gamma_2\xi_x - 2\alpha_2\gamma_2\xi_y = 0,$$

$$\beta_2\xi_t - \beta_2\xi_{yy} - \beta_2\xi_{xx} - 2\beta_{2x}\xi_x - 2\beta_{2y}\xi_y - 2\alpha_1\beta_2\xi_x - 2\alpha_2\beta_2\xi_y = 0,$$
$$\beta_2\eta_t - \beta_2\eta_{yy} - \beta_2\eta_{xx} - 2\beta_{2x}\eta_x - 2\beta_{2y}\eta_y - 2\alpha_1\beta_2\eta_x - 2\alpha_2\beta_2\eta_y = 0,$$
$$\beta_2(\tau_t - \eta_x^2 - \eta_y^2) = 0, \gamma_2(\tau_t - \eta_x^2 - \eta_y^2) = 0,$$
$$\beta_2(\tau_t - \xi_x^2 - \xi_y^2) = 0, \gamma_2(\tau_t - \xi_x^2 - \xi_y^2) = 0, \tag{15}$$
$$\beta_{2t} - \beta_{2xx} - \beta_{2yy} - 2\alpha_1\beta_{2x} - 2\beta_2\alpha_{2y} - 2\alpha_2\beta_{2y} - 2\gamma_1\alpha_{2x} = 0,$$
$$\gamma_{2t} - \gamma_{2xx} - \gamma_{2yy} - 2\alpha_1\gamma_{2x} - 2\gamma_2\alpha_{2y} - 2\alpha_2\gamma_{2y} - 2\beta_1\alpha_{2x} = 0,$$

$$-2\beta_{2y}\beta_2 - 2\beta_{2x}\gamma_1 = 0, -2\gamma_{2x}\beta_1 - 2\gamma_{2y}\gamma_2 = 0,$$
$$2\beta_2(\tau_t - \gamma_2\xi_y - \beta_1\xi_x) = 0, 2\gamma_2(\tau_t - \beta_2\eta_y - \gamma_1\eta_x) = 0,$$
$$-2\beta_1\beta_2\eta_x - 2\beta_2\gamma_2\eta_y = 0, -2\gamma_2\gamma_1\xi_x - 2\beta_2\gamma_2\xi_y = 0,$$
$$2\gamma_2(\tau_t - \beta_1\xi_x - \gamma_2\xi_y) = 0, 2\beta_2(\tau_t - \gamma_1\eta_x - \beta_2\eta_y) = 0,$$
$$-2\gamma_2(\gamma_2\eta_y + \beta_1\eta_x) = 0, -2\beta_2(\gamma_1\xi_x + \beta_2\xi_y) = 0, \tag{16}$$
$$-2\gamma_2(\eta_x\xi_x + \eta_y\xi_y) = 0, -2\beta_2(\eta_x\xi_x + \eta_y\xi_y) = 0,$$
$$-2\beta_2\gamma_{2y} - 2\gamma_2\beta_{2y} - 2\gamma_{2x}\gamma_1 - 2\beta_{1x}\beta_1 = 0.$$

It is clear that it is rather difficult to solve the over determined equations. So we consider only $\gamma_1 = \gamma_2 = 0$ on these equations. Then solving the above equations, we get

$$\tau = \tau(t), \xi = \tau_t^{\frac{1}{2}}x + b(t), \eta = \tau_t^{\frac{1}{2}}y + c(t), \beta_1 = \tau_t^{\frac{1}{2}}, \alpha_1 = \frac{\frac{1}{2}\tau_{tt}x + b_t\tau_t^{\frac{1}{2}}}{2\tau_t}, \beta_2 = \tau_t^{\frac{1}{2}}, \alpha_2 = \frac{\frac{1}{2}\tau_{tt}y + c_t\tau_t^{\frac{1}{2}}}{2\tau_t}, \tag{17}$$

where $b(t)$ and $c(t)$ are arbitrary functions. The results of paper ([13]) is the special cases for our results.

On the other side, if we let $\beta_1 = \beta_2 = 0$, after some calculations, we get

$$\tau = \tau(t), \xi = \tau_t^{\frac{1}{2}}y + b(t), \eta = \tau_t^{\frac{1}{2}}x + c(t), \gamma_1 = \tau_t^{\frac{1}{2}}, \alpha_1 = \frac{\frac{1}{2}\tau_{tt}x + b_t\tau_t^{\frac{1}{2}}}{2\tau_t}, \gamma_2 = \tau_t^{\frac{1}{2}}, \alpha_2 = \frac{\frac{1}{2}\tau_{tt}y + c_t\tau_t^{\frac{1}{2}}}{2\tau_t}, \tag{18}$$

where $b(t)$ and $c(t)$ are arbitrary functions. It is easy to see that only ξ and η is different.

In summary, based on the above analysis, for the (2+1)-dimensional coupled Burgers equations, the following symmetry transformation group theorem are obtained:

Theorem 1. *If* $\{U = U(\xi, \eta, \tau), V = V(\xi, \eta, \tau)\}$ *are solutions of Equation (1), then the solutions of Equation (1)* u, v *can be expressed by*

$$u = \frac{\frac{1}{2}\tau_{tt}x + b_t\tau_t^{\frac{1}{2}}}{2\tau_t} + \tau_t^{\frac{1}{2}}U(\xi, \eta, \tau), v = \frac{\frac{1}{2}\tau_{tt}y + c_t\tau_t^{\frac{1}{2}}}{2\tau_t} + \tau_t^{\frac{1}{2}}V(\xi, \eta, \tau), \tag{19}$$

where $\xi, \eta, \tau, b(t)$ *and* $c(t)$ *are decided by Equation (17) or Equation (18).*

In this case, based on Theorem 1, we can derive abundant of new solutions of the (2+1)-dimensional coupled Burgers equations from its given solutions, for example, we select the following the single kink solutions [6] as seed solution.

$$u = \frac{k_1 e^{k_1(x+y)+2k_1^2 t}}{1 + k_1 e^{k_1(x+y)+2k_1^2 t}}, v = \frac{k_1 e^{k_1(x+y)+2k_1^2 t}}{1 + k_1 e^{k_1(x+y)+2k_1^2 t}}. \tag{20}$$

From Theorem 1, one can present new solutions of Equation (1) as follows

$$u = \frac{\frac{1}{2}\tau_{tt}x + b_t\tau_t^{\frac{1}{2}}}{2\tau_t} + \tau_t^{\frac{1}{2}}\frac{k_1 e^{k_1(\xi+\eta)+2k_1^2\tau}}{1 + k_1 e^{k_1(\xi+\eta)+2k_1^2\tau}}, v = \frac{\frac{1}{2}\tau_{tt}y + c_t\tau_t^{\frac{1}{2}}}{2\tau_t} + \tau_t^{\frac{1}{2}}\frac{k_1 e^{k_1(\xi+\eta)+2k_1^2\tau}}{1 + k_1 e^{k_1(\xi+\eta)+2k_1^2\tau}}, \tag{21}$$

where $\xi, \eta, \tau, b(t)$ *and* $c(t)$ *are decided by Equation (17) or Equation (18). When taking* $\tau = t, b(t) = c(t) = 0$, *the solutions determined by Equation (21) are the results in Ref. [6]. Thus, it is clear that, we generalize the corresponding results in Ref. [6].*

Remark 1. *Following the same process, a great many of new solutions can be derived through given solutions [3,6,8] for the coupled Burgers equations. Thus we generalize the corresponding results in [3,6,8].*

4. Lie Algebra for Given Symmetry Group

Now, from Theorem 1, it can be found that the Lie point symmetry group, in fact, is a special case of the given symmetry group. If we let [13]

$$\tau(t) = t + \varepsilon f(t), \ b(t) = \varepsilon h(t), \ c(t) = \varepsilon m(t), \tag{22}$$

where ε is an infinitesimal parameter, f, h and m are arbitrary functions of t.

We can derive the Lie point symmetries

$$\begin{aligned} \sigma_1 &= \left(\frac{1}{2}xf_t + h(t)\right)u_x + \left(\frac{1}{2}yf_t + m(t)\right)u_y + f(t)u_t + \left(\frac{1}{4}xf_{tt} + \frac{1}{2}h_t + \frac{1}{2}uf_t\right), \\ \sigma_2 &= \left(\frac{1}{2}xf_t + h(t)\right)u_x + \left(\frac{1}{2}yf_t + m(t)\right)u_y + f(t)u_t + \left(\frac{1}{4}yf_{tt} + \frac{1}{2}m_t + \frac{1}{2}vf_t\right). \end{aligned} \tag{23}$$

The corresponding equivalent vectors expression as follows

$$V = \left(\frac{1}{2}xf_t + h(t)\right)\frac{\partial}{\partial x} + f(t)\frac{\partial}{\partial t} + \left(\frac{1}{2}yf_t + m(t)\right)\frac{\partial}{\partial y}$$
$$- \left(\frac{1}{4}xf_{tt} + \frac{1}{2}h_t + \frac{1}{2}uf_t\right)\frac{\partial}{\partial u} - \left(\frac{1}{4}yf_{tt} + \frac{1}{2}m_t + \frac{1}{2}vf_t\right)\frac{\partial}{\partial v}. \tag{24}$$

As the functions f, g and m are arbitrary, thus the corresponding Lie algebra is an infinite-dimensional Lie algebra.

If we set $f = 2c_1t + c_2, m = 2c_3t + c_4, h = 2c_5t + c_6$, one obtains

$$V = \left(c_1x + 2c_5t + c_6\right)\frac{\partial}{\partial x} + (2c_1t + c_2)\frac{\partial}{\partial t} + \left(c_1y + 2c_3t + c_4\right)\frac{\partial}{\partial y} - \left(c_5 + c_1u\right)\frac{\partial}{\partial u} - \left(c_3 + c_1v\right)\frac{\partial}{\partial v}. \tag{25}$$

From (25), we can derive the corresponding Lie algebra is spanned by the following six vector fields

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial y}, V_3 = \frac{\partial}{\partial t}, V_4 = 2t\frac{\partial}{\partial x} - \frac{\partial}{\partial u},$$
$$V_5 = 2t\frac{\partial}{\partial y} - \frac{\partial}{\partial v}, V_6 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{26}$$

It is clear that the symmetry generators form a closed Lie algebra.

5. Conclusions

In this letter, according to the generalized symmetries analysis method, we find out that the corresponding symmetries. In fact, for our assumption ($\sigma_1 = au_x + bu_t + cu_y + eu + gv + l$, $\theta = av_x + bv_t + cv_y + mv + hu + k$,), that is to say, they are linear functions, it is coincided with the classical Lie symmetry method. However, for other setting, if they are nonlinear functions, what will be happened? It is worth studying them in future works. Meanwhile, based on the mCK method, we also give the non-Lie symmetry groups for the (2+1)-dimensional coupled Burgers equations, which have important application in many fields. These results are helpful for explaining complex physical phenomena. Here, we only considered the special cases, it is also need to be investigated them for the general cases. Until now, the classical symmetry, the local conservation laws are presented. There are many issues need to be considered, such as, nonlocal symmetries, and (2+1)-dimensional coupled Burgers equations with variable coefficients as well as their discrete versions, and so on.

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