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Generalized Random α-ψ-Contractive Mappings with Applications to Stochastic Differential Equation

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Abstract: The main purpose in this paper is to define the modification form of random α-admissible and random α-ψ-contractive maps. We establish new random fixed point theorems in complete separable metric spaces. The interpretation of our results provide the main theorems of Tchier and Vetro (2017) as directed corollaries. In addition, some applications to second order random differential equations are presenred here to interpret the usability of the obtained results.

Keywords: random fixed point; random α-admissible with respect to η; generalized random α-ψ-contractive mapping

1. Introduction

It is well known that random fixed point theorems are stochastic generalizations of a classical (or deterministic) fixed point theorem. In a separable complete metric space (Polish space), the existence of a random fixed point for contraction mapping has been considered recently by Špaček [1] and Hanš [2,3]. In atomic probability measure spaces, Mukherjea [4] introduced the random fixed point theorem versions of Schauder's fixed point theorems. Random ordinary differential equations are ordinary differential equations that include a stochastic process in their vector field. The recenly interest for the random version of some ordinary differential equations can see by Tchier and Vetro in [5]. It is more realistic to consider such equations as random operator equations, which are much more difficult to handle mathematically than deterministic equations. In 1976, Bharucha-Reid [6] presented the random fixed point results to verify the unique and measurable solutions of random operator equations. In 1977, the generalized random fixed point theorems of [1] for multivalued contraction in Polish spaces and their applications for solving some random differential equation results in Banach spaces were introduced by Itoh [7]. In 1984, Sehgal and Waters [8] proved the random fixed point theorem versions of the well-known Rothe's fixed point theorem. Since then, many improvements of random fixed point theorems have been established in the literature in several ways; see for example [9–15]. It is the most widely-applied random fixed point result in different areas of mathematics, statistics, engineering, and physics, among other. Recently, we received an enormous number of applications with considerable attention in various areas such as probability theory, nonlinear analysis, and for the study of random integral and random differential equations arising in various sciences (see, [16–22]).

In 2012, Samet et al. [23] investigated a new concept of α-admissible and α-ψ-contractive maps and also demonstrated some fixed point results in complete metric spaces. In the same way, Karapinar and Samet [24] initiated the definition of the concepts of generalized α-ψ-contractive-type mappings. In 2013, Salimi et al. [25] generalized these notions of α-admissible and α-ψ-contractive mappings and obtained certain fixed point results. Our results are proper extensions of the recent results in [26,27].

Recently, Tchier and Vetro [5] investigated random fixed point theorems for modified random α-admissible and random α-ψ-contractive maps.

Our goal in this work is to prove some random fixed point theorems for obtaining the generalization of random α-ψ-contractive maps in Polish spaces. By using our main results, we can assure the existence of random solutions of a second order random differential equation.

2. Preliminaries

We denote the Borel σ-algebra on a metric space M by $B(M)$. Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of Ω. By $\Sigma \times B(M)$, we mean the smallest σ-algebra on $\Omega \times M$ containing all the sets $A \times B$ (with $A \in \Sigma$ and $B \in B(M)$).

Definition 1. *Let M and N be two metric spaces and (Ω, Σ) be a measurable space. A mapping $f : \Omega \times M \to N$ is called Carathéodory if,*

(i) the mapping $\omega \to f(\omega, m)$ is $(\Sigma, B(N))$-measurable (Σ-measurable for short) for all $m \in M$,
(ii) the mapping; $m \to f(\omega, m)$ is continuous for all $\omega \in \Omega$.

We need the following results from Denkowski–Migórski–Papageorgiou [28].

Theorem 1. *[28]. Let (Ω, Σ) be a measurable space, M be a separable metric space, and N be a metric space. If $f : \Omega \times M \to N$ is a Carathéodory mapping, then f is $\Sigma \times B(M)$-measurable.*

Corollary 1. *[28]. Let (Ω, Σ) be a measurable space, M be a separable metric space, and N be a metric space. If $f : \Omega \times M \to N$ is a Carathéodory mapping and $u : \Omega \to M$ is Σ-measurable, then mapping $\omega \to f(\omega, u(\omega))$ is a Σ-measurable mapping from Ω into N.*

Definition 2. *[28]. Let (Ω, Σ) be a measurable space, M be a separable metric space, and N be a metric space. A function $f : \Omega \times M \to N$ is said to be superpositionally measurable (for short; sup-measurable), if for all $u : \Omega \to M$ being Σ-measurable, the function $\omega \to f(\omega, u(\omega))$ is Σ-measurable from Ω into N.*

Remark 1. *[28]. Corollary 1 says that a Carathéodory function is sup-measurable. Furthermore, every $\Sigma \times B(M)$-measurable function $f : \Omega \times M \to N$ is sup-measurable.*

Definition 3. *A mapping $f : \Omega \times M \to M$ is called a random operator whenever, for any $x \in M$, $f(\cdot, x)$ is Σ-measurable, so a random fixed point of f is Σ-measurable mapping $z : \Omega \times M$ such that $z(\omega) = f(\omega, z(\omega))$ for all $\omega \in \Omega$.*

Lemma 1. *[28]. Given that M and N are two locally-compact metric spaces, a mapping $f : \Omega \times M \to N$ is Carathéodory if and only if the mapping $\omega \to r(\omega)(\cdot) = f(\omega, \cdot)$ is Σ-measurable from Ω to $C(M, N)$ (i.e., the space of all continuous functions from M into N endowed with the compact-open topology).*

Denote with Ψ the family of all nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n denotes the n^{th} iteration of ψ.

Lemma 2. *[24]. Given nondecreasing function $\psi : [0,+\infty) \to [0,+\infty)$, then the following implication holds:*

$$\forall t > 0, \quad \lim_{n\to+\infty} \psi^n(t) = 0 \Longrightarrow \psi(t) < t.$$

Definition 4. *[5]. Given $T : \Omega \times M \to M$ and $\alpha : \Omega \times M \times M \to [0,+\infty)$, the map T is called random α-admissible if:*

$$u, v \in M,\ \omega \in \Omega,\ \ \alpha(\omega, u, v) \geq 1 \Longrightarrow \alpha(\omega, T(\omega, u), T(\omega, v)) \geq 1.$$

Definition 5. *[5]. Suppose that (Ω, Σ) is a measurable space, (M, d) is a separable metric space, and $T : \Omega \times M \to M$ is a given mapping. The map T is called a random α-ψ-contractive mapping if there is a function $\alpha : \Omega \times M \times M \to [0,+\infty)$ and $\psi_\omega \in \Psi$, $\omega \in \Omega$, such that:*

$$d(T(\omega, u), T(\omega, v)) \leq \psi_\omega(d(u, v)),$$

for all $u, v \in M$ and $\omega \in \Omega$ where $\alpha(\omega, u, v) \geq 1$.

The examples of random α-admissible and random α-ψ-contractive maps are shown in [5].

3. Main Results

Firstly, we will start this section by introducing the concept of mapping as the following definitions.

Definition 6. *Given $T : \Omega \times M \to M$ and $\alpha, \eta : \Omega \times M \times M \to [0,+\infty)$, the mapping T is called a random α-admissible with respect to η if:*

$$u, v \in M,\ \omega \in \Omega,\ \ \alpha(\omega, u, v) \geq \eta(\omega, u, v) \Rightarrow \alpha(\omega, T(\omega, u), T(\omega, v)) \geq \eta(\omega, T(\omega, u), T(\omega, v)).$$

It is easy to see that if we take $\eta(\omega, u, v) = 1$ in the above definition, then it can be reduced to Definition 4.

Definition 7. *Suppose that (Ω, Σ) is a measurable space, (M, d) is a separable space, and $T : \Omega \times M \to M$ is a given mapping. The mapping T is called a generalized random α-ψ-contractive map if there exists a function $\alpha, \eta : \Omega \times M \times M \to [0,+\infty)$ and $\psi_\omega \in \Psi$, $\omega \in \Omega$, such that:*

$$\alpha(\omega, u, v) \geq \eta(\omega, u, v) \Rightarrow d(T(\omega, u), T(\omega, v)) \leq \psi_\omega(O(\omega, (u, v))), \tag{1}$$

where:

$$O(\omega, (u, v)) = \max\left\{ d(u, v), \frac{d(u, T(\omega, u)) + d(v, T(\omega, v))}{2}, \frac{d(u, T(\omega, v)) + d(v, T(\omega, u))}{2} \right\}$$

for all $u, v \in M$ and $\omega \in \Omega$.

The following are our main results.

Theorem 2. *Suppose that (Ω, Σ) is a measurable space and (M, d) is a Polish space. Given $T : \Omega \times M \to M$ and $\alpha, \eta : \Omega \times M \times M \to [0,+\infty)$, the hypotheses are the following:*

(H1) *T is random α-admissible with respect to η.*

(H2) *there is a measurable mapping $u_0 : \Omega \to M$ such that:*

$$\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq \eta(\omega, u_0(\omega), T(\omega, u_0(\omega))),$$

for every $\omega \in \Omega$.

(H3) *T is a Carathéodory mapping.*

(H4) *T is a generalized random α-ψ-contractive map.*

Then, T has a random fixed point; this means that there is measurable $\zeta : \Omega \to M$ such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$.

Proof. The hypothesis (H2) ensures that there is a measurable mapping $u_0 : \Omega \to M$ such that:

$$\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq \eta(\omega, u_0(\omega), T(\omega, u_0(\omega))),$$

for all $\omega \in \Omega$. Define the sequence $\{u_n(\omega)\}$ in M by:

$$u_n(\omega) = T^n(\omega, u_0(\omega)) = T(\omega, u_{n-1}(\omega)), \ \forall n \in \mathbb{N} \cup \{0\} \text{ and } \omega \in \Omega.$$

If $u_n(\omega) = u_{n+1}(\omega)$ $\forall$ $n \in \mathbb{N} \cup \{0\}$ and $\omega \in \Omega$, then $\zeta(\omega) = u_n(\omega)$ is a random fixed point of T.

Assume that there exists $\omega \in \Omega$ such that $u_n(\omega) \neq u_{n+1}(\omega)$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is random α-admissible with respect to η, (H1) and $\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) = \eta(\omega, u_0(\omega), T(\omega, u_0(\omega)))$, we have:

$$\begin{aligned} &\alpha(\omega, u_1(\omega), u_2(\omega)) = \alpha(\omega, T(\omega, u_0(\omega)), T^2(\omega, u_0(\omega))) \\ &\geq \eta(\omega, T(\omega, u_0(\omega)), T^2(\omega, u_0(\omega))) = \eta(\omega, u_1(\omega), u_2(\omega)). \end{aligned}$$

Continuing this process, we obtain that:

$$\alpha(\omega, u_n(\omega), u_{n+1}(\omega)) \geq \eta(\omega, u_n(\omega), u_{n+1}(\omega)), \ \forall n \in \mathbb{N} \cup \{0\} \text{ and } \omega \in \Omega. \tag{2}$$

Therefore, by (2) and the hypothesis (H4) with $u = u_{n-1}(\omega)$, $v = u_n(\omega)$, we get:

$$d(T(\omega, u_{n-1}(\omega)), T(\omega, u_n(\omega))) \leq \psi_\omega(O(\omega, (u_{n-1}(\omega), u_n(\omega)))).$$

On the other hand,

$$\begin{aligned} O(\omega, (u_{n-1}(\omega), u_n(\omega))) &= \max\Big\{d(u_{n-1}(\omega), u_n(\omega)), \\ &\quad \tfrac{d(u_{n-1}(\omega), T(\omega, u_{n-1}(\omega))) + d(u_n(\omega), T(\omega, u_n(\omega)))}{2}, \\ &\quad \tfrac{d(u_{n-1}(\omega), T(\omega, u_n(\omega))) + d(u_n(\omega), T(\omega, u_{n-1}(\omega)))}{2}\Big\} \\ &= \max\Big\{d(u_{n-1}(\omega), u_n(\omega)), \\ &\quad \tfrac{d(u_{n-1}(\omega), u_n(\omega)) + d(u_n(\omega), u_{n+1}(\omega))}{2}, \\ &\quad \tfrac{d(u_{n-1}(\omega), u_{n+1}(\omega))}{2}\Big\} \\ &\leq \max\Big\{d(u_{n-1}(\omega), u_n(\omega)), \\ &\quad \tfrac{d(u_{n-1}(\omega), u_n(\omega)) + d(u_n(\omega), u_{n+1}(\omega))}{2}\Big\} \\ &\leq \max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\}, \end{aligned}$$

which imply that:

$$d(u_n(\omega), u_{n+1}(\omega)) \leq \psi_\omega(\max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\}).$$

Now, if $\max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\} = d(u_n(\omega), u_{n+1}(\omega))$ for all $n \in \mathbb{N}$, then:

$$\begin{aligned} d(u_n(\omega), u_{n+1}(\omega)) &\leq \psi_\omega(\max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\}) \\ &= \psi_\omega(d(u_n(\omega), u_{n+1}(\omega))) \\ &< d(u_n(\omega), u_{n+1}(\omega)). \end{aligned}$$

This is a contradiction. Therefore, we obtain that:

$$d(u_n(\omega), u_{n+1}(\omega)) \leq \psi_\omega d(u_{n-1}(\omega), u_n(\omega)),$$

for all $n \in \mathbb{N}$. By mathematical induction, we get that:

$$d(u_n(\omega), u_{n+1}(\omega)) \leq \psi_\omega^n d(u_0(\omega), u_1(\omega)).$$

Fix $\epsilon > 0$, and let $n(\epsilon) \in \mathbb{N}$ such that:

$$\sum_{n \geq n(\epsilon)} \psi_\omega d(u_n(\omega), u_{n+1}(\omega)) < \epsilon \ \forall \, n \in \mathbb{N}.$$

Putting $n, m \in \mathbb{N}$ with $m > n \geq n(\epsilon)$, so, by applying the triangular inequality, we obtain that:

$$\begin{aligned} d(u_n(\omega), u_m(\omega)) &\leq \sum_{k=n}^{m-1} d(u_k(\omega), u_{k+1}(\omega)) \\ &\leq \sum_{k=n}^{m-1} \psi_\omega^k(d(u_0(\omega), u_1(\omega))) \\ &\leq \sum_{n \geq n(\epsilon)} \psi_\omega^n(d(u_0(\omega), u_1(\omega))) \\ &< \epsilon. \end{aligned}$$

The argument implies that the sequence $\{u_n(\omega)\}$ is Cauchy. Since (M, d) is a complete space, there is $\zeta : \Omega \to M$ such that $u_n(\omega) \to \zeta(\omega)$ as $n \to +\infty$ for all $\omega \in \Omega$. Since T is a Carathéodory mapping (Hypothesis (H3)), then, for all $n \in \mathbb{N}$, u_n is measurable, and also, $u_{n+1}(\omega) = T(\omega, u_n(\omega)) \to T(\omega, \zeta(\omega))$ as $n \to +\infty$ for all $\omega \in \Omega$. By the uniqueness of the limit, we have $\zeta(\omega) = T(\omega, \zeta(\omega))$, that is $\zeta(\omega)$ is a random fixed point of T. Note that ζ is measurable since it is a limit of a sequence of measurable space. □

By taking $\eta(\omega, u, v) = 1, \forall \omega \in \Omega, u, v \in M$ in Theorem 2, we obtain the following result.

Corollary 2. *Suppose that (Ω, Σ) is a measurable space and (M, d) is a Polish space. Given $T : \Omega \times M \to M$ and $\alpha : \Omega \times M \times M \to [0, +\infty)$, the hypotheses are the following:*

(H1) *T is random α-admissible.*

(H2) *there is a measurable mapping $u_0 : \Omega \to M$ such that:*

$$\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq 1,$$

for all $\omega \in \Omega$.

(H3) *T is a Carathéodory mapping.*

(H4) *T is a generalized random α-ψ-contractive mapping.*

Then, T has a random fixed point; this means that there exists measurable $\zeta : \Omega \to M$ such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$.

Theorem 3. *Given (Ω, Σ) is a measurable space, (M, d) is a Polish space, $T : \Omega \times M \to M$, and $\alpha : \Omega \times M \times M \to [0, +\infty)$, the hypotheses are the following:*

(G1) *T is random α-admissible with respect to η.*

(G2) *There exists a measurable mapping $u_0 : \Omega \to M$ such that, for all $\omega \in \Omega$.*

$$\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq \eta(\omega, u_0(\omega), T(\omega, u_0(\omega))).$$

(G3) *T is sup-measurable.*

(G4) *T is a generalized random α-ψ-contractive mapping.*

(G5) *If a sequence $\{u_n(\omega)\} \in M$ such that:*

$$\alpha(\omega, u_n(\omega), u_{n+1}(\omega)) \geq \eta(\omega, u_n(\omega), u_{n+1}(\omega))$$

for $\omega \in \Omega$, $n \in \mathbb{N} \cup \{0\}$, and $u_n(\omega) \to u(\omega)$ as $n \to +\infty$, then:

$$\alpha(\omega, u_n(\omega), u(\omega)) \geq \eta(\omega, u_n(\omega), u(\omega)),$$

for all $\omega \in \Omega$ and for all $n \in \mathbb{N} \cup \{0\}$.

Then, T has a random fixed point; this means that $\zeta : \Omega \to M$ is measurable such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for $\omega \in \Omega$.

Proof. A similar reason comes from the proof of the above theorem. Then, the sequence $\{u_n(\omega)\}$ is a Cauchy sequence for all $\omega \in \Omega$. This means that there exists $\zeta : \Omega \to M$ such that $u_n(\omega) \to \zeta(\omega)$ as $n \to +\infty$ for all $\omega \in \Omega$. On the other hand, from (2) and Hypothesis (G5), we have:

$$\alpha(\omega, u_n(\omega), \zeta(\omega)) \geq \eta(\omega, u_n(\omega), \zeta(\omega)) \;\; for \;\; all \;\; n \in \mathbb{N} \cup \{0\}, \omega \in \Omega. \tag{3}$$

Now, using the triangle inequality (3) and (G4), we get:

$$d(u_{n+1}(\omega), T(\omega, \zeta(\omega))) \leq \psi_\omega(O(\omega, u_n(\omega), \zeta(\omega))),$$

where:

$$\begin{aligned} O(\omega, (u_{n-1}(\omega), u_n(\omega))) \;\; = \;\; & \max\Big\{ d(u_n(\omega), \zeta(\omega)), \\ & \frac{d(u_n(\omega), u_{n+1}(\omega)) + d(\zeta(\omega), T(\omega, \zeta(\omega)))}{2}, \\ & \frac{d(u_n(\omega), T(\omega, \zeta(\omega))) + d(\zeta(\omega), u_{n+1}(\omega))}{2} \Big\}. \end{aligned}$$

Since $O(\omega, (u_{n-1}(\omega), u_n(\omega))) > 0$, then:

$$\begin{aligned} d(u_{n+1}(\omega), T(\omega, \zeta(\omega))) & \leq \psi_\omega(O(\omega, u_n(\omega), \zeta(\omega))) \\ & < O(\omega, u_n(\omega), \zeta(\omega)). \end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality, we have:

$$\begin{aligned} d(\zeta(\omega), T(\omega, \zeta(\omega))) & = \lim_{n\to\infty} d(u_{n+1}(\omega), T(\omega, \zeta(\omega))) \\ & \leq \lim_{n\to\infty} O(\omega, u_n(\omega), \zeta(\omega)) \\ & = \frac{d(\zeta(\omega), T(\omega, \zeta(\omega)))}{2}, \end{aligned}$$

which implies

$$d(T(\omega,\zeta(\omega)),\zeta(\omega)) = 0,$$

that is $T(\omega,\zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$. By the hypothesis that T is sup-measurable, we see that u_n for all $n \in \mathbb{N}$ is measurable and also ζ is measurable. Thus, ζ is a random fixed point of T. □

Example 1. *Given $\Omega = M = [0,1]$ is endowed with the usual metric and letting (Ω,Σ) be a measurable space, where Σ is the σ-algebra of Lebesgue's measurable subset of Ω, define $T : \Omega \times M \to M$ for all $u, v \in M$ and $\omega \in \Omega$ by:*

$$T(\omega,u) = \begin{cases} \frac{\omega}{4}\ , & u \in [0,1), \\ 0\ , & u = 1. \end{cases}$$

Also, define $\alpha, \eta : \Omega \times M \times M \to [0,+\infty)$ by:

$$\alpha(\omega,u,v) = \begin{cases} \frac{1}{\omega}\ , & \text{if } (u,v) \in \left([0,\frac{1}{4}] \times [\frac{1}{4},1]\right) \cup \left([\frac{1}{4},1] \times [0,\frac{1}{4}]\right),\ \omega > 0, \\ 0\ , & \text{otherwise}. \end{cases}$$

and:

$$\eta(\omega,u,v) = \begin{cases} 0\ , & \text{if } \omega = 0, \\ 1\ , & \text{otherwise}. \end{cases}$$

Let $\psi_\omega : [0,+\infty) \to [0,+\infty)$ be defined by $\psi_\omega(t) = \frac{1}{2\omega}t$.

Clearly, T is random α-admissible with respect to η and sup-measurable. Now, if $u_0 : \Omega \to M$ is defined by $u_0(\omega) = 0$, then Hypothesis (G2) holds true for all $\omega \in \Omega$.

Let $u \in [0,1)$ and $v = 1$. Hence,

$$\begin{aligned} d(T(\omega,u),T(\omega,v)) &= |\frac{\omega}{4} - 0| \\ &\leq \frac{1}{4}d(v,T(\omega,v)) \leq \psi_\omega(O(\omega,(u,v))). \end{aligned}$$

The other cases are trivial, and Hypothesis (G4) holds.

Finally, let $\{u_n(\omega)\} \in M$ be a sequence such that:

$$\alpha(\omega,u_n(\omega),u_{n+1}(\omega)) \geq \eta(\omega,u_n(\omega),u_{n+1}(\omega))$$

for $\omega \in \Omega$, $n \in \mathbb{N} \cup \{0\}$ and $u_n(\omega) \to u(\omega)$ as $n \to +\infty$. From the definition of α and η,

$$(u_n(\omega),u_{n+1}(\omega)) \in \left([0,\frac{1}{4}] \times [\frac{1}{4},1]\right) \cup \left([\frac{1}{4},1] \times [0,\frac{1}{4}]\right),$$

for every $\omega > 0$. Since $\left([0,\frac{1}{4}] \times [\frac{1}{4},1]\right) \cup \left([\frac{1}{4},1] \times [0,\frac{1}{4}]\right)$ is a closed set with respect to the Euclidean metric, we get that:

$$(u(\omega),u(\omega)) \in \left([0,\frac{1}{4}] \times [\frac{1}{4},1]\right) \cup \left([\frac{1}{4},1] \times [0,\frac{1}{4}]\right),$$

which implies that $u(\omega) = \frac{1}{4}$, for all $\omega > 0$. Thus, we have:

$$\alpha(\omega,u_n(\omega),u(\omega)) = \frac{1}{\omega} \geq \eta(\omega,u_n(\omega),u(\omega)) = 1,$$

for all $\omega > 0$. Case $\omega = 0$ is trivial. Hypothesis (G5) holds. Then, all of the hypotheses of Theorem 3 hold. Therefore, T has a random fixed point.

4. Application to Ordinary Random Differential Equations

Let (Ω, Σ) be a measurable space. Let $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which means that $\omega \mapsto f(\omega, t, u)$ is measurable for all $(t, u) \in [0,1] \times \mathbb{R}$ and $(t, u) \mapsto f(\omega, t, u)$ is continuous for all $\omega \in \Omega$. In this section, we apply Corollary 2 to prove the existence of a random solution of a second order random differential equation of the form:

$$-\frac{d^2u}{dt^2}(\omega, t) = f(\omega, t, u(\omega, t)), \quad t \in [0,1], \tag{4}$$

$$u(\omega, 0) = u(\omega, 1) = 0$$

For all $\omega \in \Omega$, we have that $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ has certain regularities and Ω is nonempty. From (4), in absence of ω, we retrieve the system:

$$-\frac{d^2u}{dt^2}(t) = f(t, u(t)), \quad t \in [0,1], \tag{5}$$
$$u(0) = u(1) = 0.$$

Recall that the Green's function associated with (5) is given by:

$$G(t,s) = \begin{cases} t(1-s) \ , & 0 \le t \le s \le 1, \\ s(1-t) \ , & 0 \le s \le t \le 1. \end{cases}$$

By a random solution of System (4), we mean a measurable mapping $u : \Omega \to C([0,1], \mathbb{R})$ satisfying (4), where $C([0,1], \mathbb{R})$ denotes the space of all continuous functions defined on $[0,1]$ endowed with the metric:

$$d_\infty(x, y) = \|x - y\|_\infty.$$

For short, we will write $u(\omega, t) := (u(\omega))(t)$.

The boundary value problem (4) can be written as the random integral equation:

$$u(\omega, t) = \int_0^1 G(t,s) f(\omega, s, u(s)) ds,$$

for all $t \in [0,1]$ and $\omega \in \Omega$.

Define the random integral operator $F : \Omega \times C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ by:

$$F(\omega, u)(t) = \int_0^1 G(t,s) f(\omega, s, u(s)) ds, \tag{6}$$

for all $u \in C([0,1], \mathbb{R})$ and $\omega \in \Omega$. Then, Problem (4) is equivalent to finding a random fixed point of F.

Remark 2. *[5]. F is a random operator from $\Omega \times C([0,1], \mathbb{R})$ into $C([0,1], \mathbb{R})$. In fact, given $u \in C([0,1], \mathbb{R})$, since f is a Carathéodory function for fixed $s \in [0,1]$, the function $h : \Omega \times [0,1] \to \mathbb{R}$, defined by $h(\omega, t) = G(t,s) f(\omega, s, u(s))$, is Carathéodory. By Lemma 1, the integral in (6) is the limit of a finite sum of measurable functions. Therefore, the mapping $\omega \to F(\omega, u)$ is measurable, and hence, F is a random operator.*

Remark 3. *[5]. Given $h : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $u \in C([0,1], \mathbb{R})$, and letting a sequence $\{u_n\} \subset C([0,1], \mathbb{R})$ converge to u, then there exists an interval $[a,b] \subset \mathbb{R}$ such that $u_n(s), u(s) \in [a,b]$ for all $s \in [0,1]$. The continuity of the function $h(\omega, \cdot, \cdot)$ in $[0,1] \times \mathbb{R}$ for fixed $\omega \in \Omega$ ensures that the function $h(\omega, \cdot, \cdot)$ is uniformly continuous in $[0,1] \times [a,b]$.*

Now, we prove a random fixed point of random integral operator F.

Theorem 4. *Suppose that for each $\omega \in \Omega$, there exist $\psi_\omega \in \Psi$ and $\theta : \Omega \times C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \to \mathbb{R}$ such that the following hypotheses hold:*

(i) *If $\theta(\omega, u, v) \geq 0$ for all $u, v \in C([0,1],\mathbb{R})$, then for every $t \in [0,1]$, we have:*

$$\begin{aligned}&|f(\omega,t,u(t)) - f(\omega,t,v(t))| \\ &\leq \psi_\omega\Bigg(\max\Bigg\{ |u(t)-v(t)|, \frac{1}{2}[|u(t) - F(\omega,u(t))| + |v(t) - F(\omega,v(t))|], \\ &\frac{1}{2}[|u(t) - F(\omega,v(t))| + |v(t) - F(\omega,u(t))|] \Bigg\}\Bigg).\end{aligned}$$

(ii) *There exists a measurable mapping $u_0 \colon \Omega \to C([0,1],\mathbb{R})$ such that, for all $\omega \in \Omega$, we have:*

$$\theta(\omega, u_0(\omega), F(\omega, u_0(\omega))) \geq 0.$$

(iii) *For each $\omega \in \Omega$ and $u, v \in C([0,1],\mathbb{R})$, we have:*

$$\theta(\omega,u,v) \geq 0 \Rightarrow \theta(\omega, F(\omega,u), F(\omega,v)) \geq 0.$$

Then, the random integral operator F has a random fixed point.

Proof. For fixed $\omega \in \Omega$, we show that $F(\omega,\cdot)$ is continuous. Indeed, consider a sequence $\{u_n\} \in C([0,1],\mathbb{R})$ with $u_n \to u \in C([0,1],\mathbb{R})$ as $n \to +\infty$. By Remark 3, there exists an interval $[a,b] \subset \mathbb{R}$ such that $u_n(s), u(s) \in [a,b]$ for all $s \in [0,1]$. In addition, the functions $f(\omega,\cdot,\cdot)$ are uniformly continuous in $[0,1] \times [a,b]$. Thus, for fixed $\epsilon > 0$, there exists $\delta > 0$ such that:

$$|f(\omega,s_1,u_1(s_1)) - f(\omega,s_2,u_2(s_2))| < \epsilon,$$

for all $s_1, s_2 \in [0,1]$ and $u_1, u_2 \in [a,b]$ such that $|s_1 - s_2| + |u_1 - u_2| < \delta$.

Now, let $n(\delta) \in \mathbb{N}$ such that $\|u_n - u\|_\infty < \delta$ whenever $n \geq n(\delta)$. Then, for every $n \geq n(\delta)$, we have:

$$|f(\omega,s,u_n(s)) - f(\omega,s,u(s))| < \epsilon.$$

Consequently, for $t \in [0,1]$ and $n \geq n(\delta)$, using that $\int_0^1 G(t,s)ds = \frac{1}{8}$, we have:

$$\begin{aligned}|F(\omega,u_n)(t) - F(\omega,u)(t)| &\leq \int_0^1 |G(t,s)||f(\omega,s,u_n(s)) - f(\omega,s,u(s))|ds \\ &\leq \epsilon\end{aligned}$$

$$\Rightarrow \|F(\omega,u_n) - F(\omega,u)\|_\infty \leq \epsilon.$$

Therefore, $d_\infty(F(\omega,u_n), F(\omega,u)) \to 0$ as $n \to +\infty \Rightarrow F(\omega,\cdot)$ is a continuous operator for each fixed $\omega \in \Omega$.

Thus, by Remark 2, $F : \Omega \times C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is a Carathéodory function.

The next step is to show that the random integral operator F satisfies a generalized random α-ψ-contractive type condition as in (H4). Therefore, for each $\omega \in \Omega$ and all $u, v \in C([0,1],\mathbb{R})$ such that $\theta(\omega,u,v) \geq 0$, we prove that:

$$d_\infty(F(\omega,u), F(\omega,v)) \leq \psi_\omega(O(\omega,(u,v))),$$

where:

$$O(\omega,(u,v)) = \max\left\{ d(u,v), \frac{d(u,F(\omega,u)) + d(v,F(\omega,v))}{2}, \frac{d(u,F(\omega,v)) + d(v,F(\omega,u))}{2} \right\}.$$

Indeed, let $\omega \in \Omega$ be fixed and $u, v \in C([0,1], \mathbb{R})$ be such that $\theta(\omega, u, v) \geq 0$, then:

$$\begin{aligned}
&|F(\omega,u)(t) - F(\omega,v)(t)| \\
&= \left| \int_0^1 G(t,s)[f(\omega,s,u(s)) - f(\omega,s,v(s))]ds \right| \\
&\leq \int_0^1 G(t,s)|f(\omega,s,u(s)) - f(\omega,s,v(s))|ds \\
&\leq \int_0^1 G(t,s)\Bigg[\psi_\omega\Bigg(\max\Bigg\{ |u(s)-v(s)|, \frac{1}{2}[|u(s) - F(\omega,u(s))| + |v(s) - F(\omega,v(s))|], \\
&\quad \frac{1}{2}[|u(s) - F(\omega,v(s))| + |v(s) - F(\omega,u(s))|] \Bigg\}\Bigg)\Bigg]ds \\
&\leq \int_0^1 G(t,s)\Bigg[\psi_\omega\Bigg(\max\Bigg\{ |u(s)-v(s)\|, \frac{1}{2}[\|u(s) - F(\omega,u(s))\| + \|v(s) - F(\omega,v(s))\|], \\
&\quad \frac{1}{2}[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|] \Bigg\}\Bigg)\Bigg]ds \\
&= \left(\int_0^1 G(t,s)ds \right)\psi_\omega\Bigg(\max\Bigg\{ |u(s)-v(s)\|, \frac{1}{2}[\|u(s) - F(\omega,u(s))\| \\
&\quad + \|v(s) - F(\omega,v(s))\|], \frac{1}{2}[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|] \Bigg\}\Bigg) \\
&\leq \psi_\omega\Bigg(\max\Bigg\{ |u(s)-v(s)\|, \frac{1}{2}[\|u(s) - F(\omega,u(s))\| + \|v(s) - F(\omega,v(s))\|], \\
&\quad \frac{1}{2}[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|] \Bigg\}\Bigg).
\end{aligned}$$

Then:

$$\begin{aligned}
&\|F(\omega,u) - F(\omega,v)\| \\
&\leq \psi_\omega\Bigg(\max\Bigg\{ |u(s)-v(s)\|, \frac{1}{2}[\|u(s) - F(\omega,u(s))\| + \|v(s) - F(\omega,v(s))\|], \\
&\quad \frac{1}{2}[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|] \Bigg\}\Bigg).
\end{aligned}$$

Let $\alpha : \Omega \times C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R}) \to [0, +\infty)$ be a function given as:

$$\alpha(\omega,u,v) = \begin{cases} 1 & if \quad \theta(\omega,u,v) \geq 0, \\ 0 & otherwise, \end{cases}$$

for all $\omega \in \Omega$. Therefore, for all $u, v \in C([0,1], \mathbb{R})$ with $\alpha(\omega, u, v) \geq 1$, we get:

$$\begin{aligned}
&\|F(\omega,u) - F(\omega,v)\|_\infty \\
&\leq \psi_\omega\Bigg(\max\Bigg\{ |u(s)-v(s)\|_\infty, \frac{1}{2}[\|u(s) - F(\omega,u(s))\|_\infty + \|v(s) - F(\omega,v(s))\|_\infty], \\
&\quad \frac{1}{2}[\|u(s) - F(\omega,v(s))\|_\infty + \|v(s) - F(\omega,u(s))\|_\infty] \Bigg\}\Bigg),
\end{aligned}$$

which means that F is a generalized random α-ψ-contractive integral operator.

Note that, for any $\omega \in \Omega$ and $u, v \in C([0,1], \mathbb{R})$, we have:

$$\alpha(\omega,u,v) \geq 1 \Rightarrow \theta(\omega,u,v) \geq 0 \Rightarrow \theta(\omega, F(\omega,u), F(\omega,v)) \geq 0 \Rightarrow \alpha(\omega, F(\omega,u), F(\omega,v)) \geq 1,$$

which means that F is a random α-admissible integral operator. Moreover, Hypothesis (ii) ensures that there exists a measurable mapping $u_0 : \Omega \to C([0,1], \mathbb{R})$ such that $\alpha(\omega, u_0(\omega), F(\omega, u_0(\omega))) \geq 1$ for all $\omega \in \Omega$. All of the hypotheses of Corollary 2 are satisfied, and hence, the random integral operator F has a random fixed point. □

By an application of Theorem 4, we deduce that Problem (4) admits a random solution.

Example 2. *Let (Ω, Σ) be a measurable space, where $\Omega = [0, +\infty)$ and Σ is the σ-algebra of Borel on $[0, +\infty)$. Consider the two-point boundary value problem:*

$$-\frac{d^2u}{dt^2}(\omega, t) = \frac{1}{7e^{\omega^2 t+1}(1 + |u(\omega, t)|)}, \quad t \in [0,1], \tag{7}$$

$$u(\omega, 0) = u(\omega, 1) = 0$$

for all $u \in C([0,1], \mathbb{R})$ and $\omega \in \Omega$. Solving this problem is equivalent to finding a random fixed point of the integral operator $F : \Omega \times C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ given by:

$$F(\omega, u)(t) = \int_0^1 \frac{G(t,s)}{7e^{\omega^2 t+1}(1 + |u(\omega, t)|)} ds, \tag{8}$$

for all $u \in C([0,1], \mathbb{R})$ and $\omega \in \Omega$. Then, Problem (4) is equivalent to finding a random fixed point of F.

Clearly, $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ given by $f(\omega, t, u) = \frac{1}{7e^{\omega^2 t+1}(1+|u|)}$ is a Carathéodory function. Hypotheses (ii) and (iii) of Theorem 4 hold true by defining $\theta(\omega, u, v) = 1$, for all $\omega \in \Omega$ and $u, v \in C([0,1], \mathbb{R})$. Consequently, Hypothesis (i) is satisfied with $\psi_\omega(t) = \frac{t}{7e}$. Therefore, by Theorem 4, the integral operator F has a random fixed point, and the two-point boundary value problem (7) has at least one random solution.

Example 3. *Let (Ω, Σ) be a measurable space, where $\Omega = [0, +\infty)$ and Σ is the σ-algebra of Borel on $[0, +\infty)$. Consider the two-point boundary value problem:*

$$-\frac{d^2u}{dt^2}(\omega, t) = \omega t + \frac{u(t)}{2 + \sin t\frac{\pi}{2}}, \quad t \in [0,1], \tag{9}$$

$$u(\omega, 0) = u(\omega, 1) = 0$$

for all $u \in C([0,1], \mathbb{R})$ and $\omega \in \Omega$. Clearly, $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ given by $f(\omega, t, u) = \omega t + \frac{u}{2+\sin t\frac{\pi}{2}}$ is a Carathéodory function. Hypotheses (ii) and (iii) of Theorem 4 hold true by defining $\theta(\omega, u, v) = 1$, for all $\omega \in \Omega$ and $u, v \in C([0,1], \mathbb{R})$. Since,

$$|f(\omega, t, u(t)) - f(\omega, t, v(t))| = \frac{1}{2 + \sin t\frac{\pi}{2}} |v(t) - u(t)|$$

Hypothesis (i) is satisfied with $\psi_\omega(t) = \frac{1}{2+\sin t\frac{\pi}{2}}$. Therefore, by Theorem 4, the two-point boundary value problem (9) has at least one random solution.

5. Conclusions

We present the random version of α-ψ-contractive mappings with respect to η, previously known in complete separable metric spaces. We proved random fixed point theorems in complete separable metric spaces and proved random solutions of second order random differential equations. The presented theorems extend and improve the corresponding results given in the literature such as Tchier and Vetro, in [5].

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