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Chen Inequalities for Warped Product Pointwise Bi-Slant Submanifolds of Complex Space Forms and Its Applications

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Abstract: In this paper, by using new-concept pointwise bi-slant immersions, we derive a fundamental inequality theorem for the squared norm of the mean curvature via isometric warped-product pointwise bi-slant immersions into complex space forms, involving the constant holomorphic sectional curvature c, the Laplacian of the well-defined warping function, the squared norm of the warping function, and pointwise slant functions. Some applications are also given.

Keywords: mean curvature; warped products; compact Riemannian manifolds; pointwise bi-slant immersions; inequalities

1. Introduction

In the submanifolds theory, creating a relationship between extrinsic and intrinsic invariants is considered to be one of the most basic problems. Most of these relations play a notable role in submanifolds geometry. The role of immersibility and non-immersibility in studying the submanifolds geometry of a Riemannian manifold was affected by the pioneering work of the Nash embedding theorem [1], where every Riemannian manifold realizes an isometric immersion into a Euclidean space of sufficiently high codimension. This becomes a very useful object for the submanifolds theory, and was taken up by several authors (for instance, see [2–15]). Its main purpose was considered to be how Riemannian manifolds could always be treated as Riemannian submanifolds of Euclidean spaces. Inspired by this fact, Nolker [16] classified the isometric immersions of a warped product decomposition of standard spaces. Motivated by these approaches, Chen started one of his programs of research in order to study the impressibility and non-immersibility of Riemannian warped products into Riemannian manifolds, especially in Riemannian space forms (see [11,17–19]). Recently, a lot of solutions have been provided to his problems by many geometers (see [18] and references therein).

The field of study which includes the inequalities for warped products in contact metric manifolds and the Hermitian manifold is gaining importance. In particular, in [17], Chen observed the strong isometrically immersed relationship between the warping function f of a warped product $M_1 \times_f M_2$ and the norm of the mean curvature, which isometrically immersed into a real space form.

Theorem 1. *Let $\widetilde{M}(c)$ be a m-dimensional real space form and let $\varphi : M = M_1 \times_f M_2$ be an isometric immersion of an n-dimensional warped product into $\widetilde{M}(c)$. Then:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}||H||^2 + n_1 c, \tag{1}$$

where $n_i = dim M_i$, $i = 1, 2$, and Δ is the Laplacian operator of M_1 and H is the mean curvature vector of M^n. Moreover, the equality holds in (1) if, and only if, φ is mixed and totally geodesic and $n_1 H_1 = n_2 H_2$ such that H_1 and H_2 are partially mean curvatures of M_1 and M_2, respectively.

In [2,5,20–31], the authors discuss the study of Einstein, contact metrics, and warped product manifolds for the above-mentioned problems. Furthermore, in regard to the collections of such inequalities, we referred to [12] and references therein. The motivation came from the study of Chen and Uddin [32], which proved the non-triviality of warped-product pointwise bi-slant submanifolds of a Kaehler manifold with supporting examples. If the sectional curvature is constant with a Kaehler metric, then it is called complex space forms. In this paper, we consider the warped-product pointwise bi-slant submanifolds which isometrically immerse into a complex space form, where we then obtain a relationship between the squared norm of the mean curvature, constant sectional curvature, the warping function, and pointwise bi-slant functions. We will announce the main result of this paper in the following.

Theorem 2. *Let $\widetilde{M}^{2m}(c)$ be the complex space form and let $\varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \to \widetilde{M}^{2m}(c)$ be an isometric immersion from warped product pointwise bi-slant submanifolds into $\widetilde{M}^{2m}(c)$. Then, the following inequality is satisfied:*

$$\Delta(ln f) \leq ||\nabla \ln f||^2 + \frac{n^2}{4n_2}||H||^2 + \frac{n_1 c}{4} - \frac{3c}{4n_2}\Big(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2\Big), \tag{2}$$

where θ_1 and θ_2 are pointwise slant functions along M_1 and M_2, respectively. Furthermore, ∇ and Δ are the gradient and the Laplacian operator on $M_1^{n_1}$, respectively, and H is the mean curvature vector of M^n. The equality case holds in (2) if and only if φ is a mixed totally geodesic isometric immersion and the following satisfies

$$\frac{H_1}{H_2} = \frac{n_2}{n_1}$$

where H_1 and H_2 are the mean curvature vectors along $M_1^{n_1}$ and $M_2^{n_2}$, respectively.

As an application of Theorem 2 in a compact orientated Riemannian manifold with a free boundary condition, we prove that:

Theorem 3. *Let $M^n = M_1^{n_1} \times_f M_2^{n_2}$ be a compact, orientate warped product pointwise bi-slant submanifold in a complex space form $\widetilde{M}^{2m}(c)$ such that $M_1^{n_1}$ is a n_1-dimensional and $M_2^{n_2}$ is a n_2-dimensional pointwise slant submanifold $\widetilde{M}^{2m}(c)$. Then, M^n is simply a Riemannian product if, and only if:*

$$||H||^2 \geq \frac{c}{n^2}\Big(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2 - n_1 n_2\Big), \tag{3}$$

where H is the mean curvature vector of M^n. Moreover, θ_1 and θ_2 are pointwise slant functions.

By using classifications of pointwise bi-slant submanifolds which were defined in [32], we derived similar inequalities for warped product pointwise pseudo-slant submanifolds [33], warped product pointwise semi-slant submanifolds [34], and CR-warped product submanifolds [17] in a complex space form as well.

2. Preliminaries and Notations

An almost complex structure J and a Riemannian metric g, such that $J^2 = -I$ and $g(JX, JY) = g(X, Y)$, for $X, Y \in \mathfrak{X}(\widetilde{M})$, where I denotes the identity map and $\mathfrak{X}(\widetilde{M})$ is the space containing vector fields tangent to $\widetilde{M}$, then (M, J, g) is an almost Hermitian manifold. If the almost complex structure satisfied $(\widetilde{\nabla}_U J)V = 0$, for any $U, V \in \mathfrak{X}(\widetilde{M})$ and $\widetilde{\nabla}$ is a Levi-Cevita connection $\widetilde{M}$. In this case, $\widetilde{M}$ is called the Kaehler manifold. A complex space form of constant holomorphic sectional curvature c is denoted by $\widetilde{M}^{2m}(c)$, and its curvature tensor $\widetilde{R}$ can be expressed as:

$$\widetilde{R}(U,V,Z,W) = \frac{c}{4}\bigg(g(U,Z)g(V,W) - g(V,Z)g(U,W) + g(U,JZ)g(JV,W) \\ - g(V,JZ)g(U,JW) + 2g(U,JV)g(JZ,W)\bigg), \tag{4}$$

for every $U, V, Z, W \in \mathfrak{X}(\widetilde{M}^{2m}(c))$. A Riemannian manifold $\widetilde{M}^m$ and its submanifold M, the Gauss and Weingarten formulas are defined by $\widetilde{\nabla}_U V = \nabla_U V + h(U,V),\ and\ \widetilde{\nabla}_U \xi = -A_\xi U + \nabla^\perp_U \xi$, respectively for each $U, V \in \mathfrak{X}(M)$ and for the normal vector field ξ of M, where h and A_ξ are denoted as the second fundamental form and shape operator. They are related as $g(h(U,V),N) = g(A_N U, V)$. Now, for any $U \in \mathfrak{X}(M)$ and for the normal vector field ξ of M, we have:

$$(i)\ JU = PU + FU, \quad (ii)\ \ J\xi = t\xi + f\xi, \tag{5}$$

where $PU(t\xi)$ and $FU(f\xi)$ are tangential to M and normal to M, respectively. Similarly, the equations of Gauss are given by:

$$R(U,V,Z,W) = \widetilde{R}(U,V,Z,W) + g\big(h(U,W),h(V,Z)\big) - g\big(h(U,Z),h(V,W)\big). \tag{6}$$

for all U, V, Z, W are tangent M, where R and $\widetilde{R}$ are defined as the curvature tensor of $\widetilde{M}^m$ and M^n, respectively.

The mean curvature H of Riemannian submanifold M^n is given by

$$H = \frac{1}{n} trace(h).$$

A submanifold M^n of Riemannian manifold $\widetilde{M}^m$ is said to be totally umbilical and totally geodesic if $h(U,V) = g(U,V)H$ and $h(U,V) = 0$, for any $U, V \in \mathfrak{X}(M)$, respectively, where H is the mean curvature vector of M^n. Furthermore, if $H = 0$, them M^n is minimal in $\widetilde{M}^m$.

A new class called a "pointwise slant submanifold" has been studied in almost Hermitian manifolds by Chen-Gray [35]. They provided the following definitions of these submanifolds:

Definition 1. *[35] A submanifold M^n of an almost Hermitian manifold $\widetilde{M}^{2m}$ is a pointwise slant if, for any non-zero vector $X \in \mathfrak{X}(T_x M)$ and each given point $x \in M^n$, the angle $\theta(X)$ between JX and tangent space $T_x M$ is free from the choice of the nonzero vector X. In this case, the Wirtinger angle become a real-valued function and it is non-constant along M^n, which is defined on T^*M such that $\theta : T^*M \to \mathbb{R}$.*

Chen-Gray in [35] derived a characterization for the pointwise slant submanifold, where M^n is a pointwise slant submanifold if, and only if, there exists a constant $\lambda \in [0,1]$ such that $P^2 = -\cos^2\theta I$, where P is a (1,1) tensor field and I is an identity map. For more classifications, we referred to [35].

Following the above concept, a pointwise bi-slant immersion was defined by Chen-Uddin in [18], where they defined it as follows:

Definition 2. *A submanifold M^n of an almost Hermitian manifold $\widetilde{M}^{2m}$ is said to be a pointwise bi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}_{\theta_1}$ and $\mathcal{D}_{\theta_2}$, such that:*

(i) $TM^n = \mathcal{D}_{\theta_1} \oplus \mathcal{D}_{\theta_2}$;

(ii) $J\mathcal{D}_{\theta_1} \perp \mathcal{D}_{\theta_2}$ *and* $J\mathcal{D}_{\theta_2} \perp \mathcal{D}_{\theta_1}$;

(iii) Each distribution $\mathcal{D}_{\theta_i}$ is a pointwise slant with a slant function $\theta_i : T^*M \to \mathbb{R}\;\; for\, i = 1,2$.

Remark 1. *A pointwise bi-slant submanifold is a bi-slant submanifold if each slant functions $\theta_i : T^*M \to \mathbb{R}\;\; for\, i = 1,2$. are constant along M^n (see [13]).*

Remark 2. *If $\theta_1 = \frac{\pi}{2}$ or $\theta_2 = \frac{\pi}{2}$, then M^n is called a pointwise pseudo-slant submanifold (see [33]).*

Remark 3. *If $\theta_1 = 0$ or $\theta_2 = 0$, in this case, M^n is a coinciding pointwise semi-slant submanifold (see [14,34]).*

Remark 4. *If $\theta_2 = \frac{\pi}{2}$ and $\theta_1 = 0$, then M^n is CR-submanifold of the almost Hermitian manifold.*

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\widetilde{M}^m$, and denoted at $\widetilde{\tau}(T_x\widetilde{M}^m)$, which, at some x in $\widetilde{M}^m$, is given:

$$\widetilde{\tau}(T_x\widetilde{M}^m) = \sum_{1\leq\alpha<\beta\leq m} \widetilde{K}_{\alpha\beta}, \tag{7}$$

where $\widetilde{K}_{\alpha\beta} = \widetilde{K}\big(e_\alpha \wedge e_\beta\big)$. It is clear that the first equality (7) is congruent to the following equation, which will be frequently used in subsequent proof:

$$2\widetilde{\tau}(T_x\widetilde{M}^m) = \sum_{1\leq\alpha<\beta\leq m} \widetilde{K}_{\alpha\beta},\;\; 1 \leq \alpha, \beta \leq n. \tag{8}$$

Similarly, scalar curvature $\widetilde{\tau}(L_x)$ of L-plan is given by:

$$\widetilde{\tau}(L_x) = \sum_{1\leq\alpha<\beta\leq m} \widetilde{K}_{\alpha\beta}, \tag{9}$$

An orthonormal basis of the tangent space T_xM is $\{e_1, \cdots e_n\}$ such that $e_r = (e_{n+1}, \cdots e_m)$ belong to the normal space $T^\perp M$. Then, we have:

$$\begin{aligned} h^r_{\alpha\beta} &= g(h(e_\alpha, e_\beta), e_r), \\ ||h||^2 &= \sum_{\alpha,\beta=1}^{n} g\big(h(e_\alpha, e_\beta), h(e_\alpha, e_\beta)\big). \end{aligned} \tag{10}$$

Let $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ be the sectional curvatures of the plane section spanned by e_α and e_β at x in a submanifold M^n and a Riemannian manifold $\widetilde{M}^m$, respectively. Thus, $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ are the intrinsic and extrinsic sectional curvatures of the span $\{e_\alpha, e_\beta\}$ at x. Thus, from the Gauss Equation (6)(i), we have:

$$\begin{aligned} 2\tau(T_xM^n) = K_{\alpha\beta} &= 2\widetilde{\tau}(T_xM^n) + \sum_{r=n+1}^{m} \Big(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2\Big) \\ &= \widetilde{K}_{\alpha\beta} + \sum_{r=n+1}^{m} \Big(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2\Big). \end{aligned} \tag{11}$$

The following consequences come from (6) and (11), as:

$$\tau(T_xM_1^{n_1}) = \sum_{r=n+1}^{m} \sum_{1\leq i<j\leq n_1} \Big(h^r_{ii}h^r_{jj} - (h^r_{ij})^2\Big) + \widetilde{\tau}(T_xM_1^{n_1}). \tag{12}$$

Similarly, we have:

$$\tau(T_xM_2^{n_2}) = \sum_{r=n+1}^{m} \sum_{n_1+1\leq a<b\leq n} \left(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\right) + \widetilde{\tau}(T_xM_2^{n_2}). \tag{13}$$

Assume that $M_1^{n_1}$ and $M_2^{n_2}$ are two Riemannian manifolds with their Riemannian metrics g_1 and g_2, respectively. Let f be a smooth function defined on $M_1^{n_1}$. Then, the warped product manifold $M^n = M_1^{n_1} \times_f M_2^{n_2}$ is the manifold $M_1^{n_1} \times M_2^{n_2}$ furnished by the Riemannian metric $g = g_1 + f^2 g_2$, which defined in [36]. When considering that the $M^n = M_1^{n_1} \times_f M_2^{n_2}$ is the warped product manifold, then for any $X \in \mathfrak{X}(M_1)$ and $Z \in \mathfrak{X}(M_2)$, we find that:

$$\nabla_Z X = \nabla_X Z = (X \ln f)Z. \tag{14}$$

Let $\{e_1, \cdots e_n\}$ be an orthonormal frame for M^n; then, summing up the vector fields such that:

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} K(e_\alpha \wedge e_\beta) = \sum_{\alpha=1}^{n_1}\sum_{\beta=1}^{n_2}\left((\nabla_{e_\alpha} e_\alpha)\ln f - e_\alpha(e_\beta \ln f) - (e_\alpha \ln f)^2\right).$$

From (Equation (3.3) in [11]), the above equation implies that:

$$\sum_{\alpha=1}^{n_1}\sum_{\beta=1}^{n_2} K(e_\alpha \wedge e_\beta) = n_2\Big(\Delta(\ln f) - ||\nabla(\ln f)||^2\Big) = \frac{n_2 \Delta f}{f}. \tag{15}$$

Remark 5. *A warped product manifold* $M^n = M_1^{n_1} \times_f M_2^{n_2}$ *is said to be trivial or a simple Riemannian product manifold if the warping function* f *is constant.*

3. Main Inequality for Warped Product Pointwise Bi-Slant Submanifolds

To obtain similar inequalities like Theorem 1, for warped product pointwise bi-slant submanifolds of complex space forms, we need to recall the following lemma.

Lemma 1. *[10] Let* $a_1, a_2, \dots a_n, a_{n+1}$ *be* $n+1$ *be real numbers with*

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + a_{n+1}), n \geq 2.$$

Then $2a_1.a_2 \geq a_3$ *holds if and only if* $a_1 + a_2 = a_3 = \cdots = a_k$.

Proof of Theorem 2. If substitute $X = Z = e_\alpha$ and $Y = W = e_\beta$ for $1 \leq \alpha, \beta \leq n$ in (4), and (6), taking summing up then

$$\sum_{\alpha,\beta=1}^{n} \widetilde{R}(e_\alpha, e_\beta, e_\alpha, e_\beta) = \frac{c}{4}\left(n(n-1) + 3\sum_{\alpha,\beta=1}^{n} g^2(Je_\alpha, e_\beta)\right). \tag{16}$$

As M^n is a pointwise bi-slant submanifold, we defined an adapted orthonormal frame as $n = 2d_1 + 2d_2$ follows $\{e_1, e_2 = \sec\theta_1 Pe_1, \dots, e_{2d_1-1}, e_{2d_1} = \sec\theta_1 Pe_{2d_1-1}, \dots, e_{2d_1+1}, e_{2d_1+2} = \sec\theta_2 Pe_{2d_1+1}, \dots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta_2 Pe_{2d_1+2d_2-1}\}$. Thus, we defined it such that $g(e_1, Je_2) = -g(Je_1, e_2) = g(Je_1, \sec\theta_1 Pe_1)$, which implies that $g(e_1, Je_2) = -\sec\theta_1 g(Pe_1, Pe_1)$.

□

Following ((2.8) in [32]), we get $g(e_1, Je_2) = \cos\theta_1 g(e_1, e_2)$. Therefore, we easily obtained the following relation:

$$g^2(e_\alpha, Je_\beta) = \begin{cases} \cos^2\theta_1, \ for\ each\ \alpha = 1, \ldots, 2d_1 - 1, \\ \cos^2\theta_2, \ for\ each\ \beta = 2d_1 + 1, \ldots, 2d_1 + 2d_1 - 1. \end{cases}$$

Hence, we have:

$$\sum_{\alpha,\beta=1}^{n} g^2(Je_\alpha, e_\beta) = \big(n_1 \cos^2\theta + n_2 \cos^2\theta\big). \tag{17}$$

Following from (17), (16), and (6), we find that:

$$2\tau = \frac{c}{4} n(n-1) + \frac{c}{4}\Bigg(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2\Bigg) + n^2||H||^2 - ||h||^2. \tag{18}$$

Let us assume that:

$$\delta = 2\tau - \frac{c}{4} n(n-1) - \frac{c}{4}\Bigg(3n_1 \cos^2\theta_1 + 3n_2 \cos^2\theta_2\Bigg) - \frac{n^2}{2}||H||^2. \tag{19}$$

Then, from (19), and (18), we get:

$$n^2||H||^2 = 2\big(\delta + ||h||^2\big). \tag{20}$$

Thus, from an orthogonal frame $\{e_1, e_2, \cdots e_n\}$, the proceeding equation takes the new form:

$$\begin{aligned}\Bigg(\sum_{r=n+1}^{2m}\sum_{i=1}^{n} h^r_{AA}\Bigg)^2 = 2\Bigg(\delta + \sum_{r=n+1}^{2m}\sum_{i=1}^{n}(h^r_{AA})^2 + \sum_{r=n+1}^{2m+1}\sum_{i<j=1}^{n}(h^r_{AB})^2 \\ + \sum_{r=n+1}^{2m}\sum_{A,B=1}^{n}(h^r_{AB})^2\Bigg).\end{aligned} \tag{21}$$

This can be expressed in more detail, such as:

$$\begin{aligned}\frac{1}{2}\Bigg(h^{n+1}_{11} + \sum_{A=2}^{n_1} h^{n+1}_{AA} + \sum_{l=n_1+1}^{n} h^{n+1}_{ll}\Bigg)^2 &= \delta + (h^{n+1}_{11})^2 + \sum_{A=2}^{n_1}(h^{n+1}_{AA})^2 + \sum_{l=n_1+1}^{n}(h^{n+1}_{ll})^2 \\ &- \sum_{2\le B\ne q\le n_1} h^{n+1}_{BB}h^{n+1}_{qq} - \sum_{n_1+1\le l\ne s\le n} h^{n+1}_{ll}h^{n+1}_{ss} \\ &+ \sum_{A<B=1}^{n}(h^{n+1}_{AB})^2 + \sum_{r=n+1}^{2m}\sum_{A,B=1}^{n}(h^r_{AB})^2.\end{aligned} \tag{22}$$

Assume that $a_1 = h^{n+1}_{11}$, $a_2 = \sum_{A=2}^{n_1} h^{n+1}_{AA}$, and $a_3 = \sum_{l=n_1+1}^{n} h^{n+1}_{ll}$. Then, applying Lemma 1 in (22), we derive:

$$\begin{aligned}\frac{\delta}{2} + \sum_{A<B=1}^{n}(h^{n+1}_{AB})^2 + \frac{1}{2}\sum_{r=n+1}^{2m}\sum_{A,B=1}^{n}(h^r_{AB})^2 \le \sum_{2\le B\ne q\le n_1} h^{n+1}_{BB}h^{n+1}_{qq} \\ + \sum_{n_1+1\le l\ne s\le n} h^{n+1}_{ll}h^{n+1}_{ss}.\end{aligned} \tag{23}$$

with equality holds in (23) if and only if

$$\sum_{A=2}^{n_1} h_{AA}^{n+1} = \sum_{B=n_1+1}^{n} h_{BB}^{n+1}. \tag{24}$$

On the other hand, from (15), we have:

$$\frac{n_2 \Delta f}{f} = \tau - \sum_{1 \leq A < B \leq n_1} K(e_A \wedge e_B) - \sum_{n_1+1 \leq l < q \leq n} K(e_l \wedge e_q). \tag{25}$$

Then from (6) and the scalar curvature for the complex space form (11), we get:

$$\begin{aligned} n_2 \frac{\Delta f}{f} &= \tau - \frac{n_1(n_1-1)c}{8} - \frac{3n_1 c}{4}\cos^2\theta_1 - \sum_{r=n+1}^{2m} \sum_{1 \leq A \neq B \leq n_1} (h_{AA}^r h_{BB}^r - (h_{AB}^r)^2) \\ &\quad - \frac{n_2(n_2-1)c}{8} - \frac{3n_2 c}{4}\cos^2\theta_2 - \sum_{r=n+1}^{2m} \sum_{n_1+1 \leq l \neq q \leq n} (h_{ll}^r h_{qq}^r - (h_{lq}^r)^2). \end{aligned} \tag{26}$$

Now from (23) and (26), we have:

$$n_2 \frac{\Delta f}{f} \leq \rho - \frac{n(n-1)c}{8} + \frac{n_1 n_2 c}{4} - \frac{3n_1 c}{4}\cos^2\theta_1 - \frac{\delta}{2} - \frac{3n_2 c}{4}\cos^2\theta_2. \tag{27}$$

Using (19) in the above equation and relation $\frac{\Delta f}{f} = \Delta(\ln f) - ||\nabla \ln f||^2$, we derive:

$$n_2\Big(\Delta(\ln f) - ||\nabla \ln f||^2\Big) \leq \frac{n^2}{4}||H||^2 + \frac{c}{4}\Bigg(n_1 n_2 + 3n_1\cos^2\theta_1 + 3n_2\cos^2\theta_2\Bigg). \tag{28}$$

which implies inequality. The equality sign holds in (2) if, and only if, the leaving terms in (23) and (24) imply that:

$$\sum_{r=n+2}^{2m} \sum_{B=1}^{n_1} h_{BB}^r = \sum_{r=n+2}^{2m} \sum_{A=n_1+1}^{n_1} h_{AA}^r = 0, \tag{29}$$

and $n_1 H_1 = n_2 H_2$, where H_1 and H_2 are partially mean curvature vectors on $M_1^{n_1}$ and $M_2^{n_2}$, respectively. Moreover, also from (23), we find that

$$\begin{aligned} h_{AB}^r = 0, \quad for\ each\ \ 1 \leq A &\leq n_1 \\ n_1 + 1 \leq B &\leq n \\ n + 1 \leq r &\leq 2m. \end{aligned} \tag{30}$$

This shows that φ is a mixed, totally geodesic immersion. The converse part of (30) is true in a warped product pointwise bi-slant into the complex space form. Thus, we reached our promised result.

Consequences of Theorem 2

Inspired by the research in [6,34] and using the Remark 3 in Theorem 2 for pointwise semi-slant warped product submanifolds, we obtained:

Corollary 1. *Let $\varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \to \widetilde{M}^{2m}(c)$ be an isometric immersion from the warped product pointwise semi-slant submanifold into a complex space form $\widetilde{M}^{2m}(c)$, where $M_1^{n_1}$ is the holomorphic and $M_2^{n_2}$ is the pointwise slant submanifolds of $\widetilde{M}^{2m}(c)$. Then, we have the following inequality:*

$$\Delta(lnf) \leq ||\nabla \ln f||^2 + \frac{n^2}{4n_2}||H||^2 + \frac{n_1 c}{4} - \frac{3c}{4n_2}\left(n_1 + n_2 \cos^2\theta\right), \tag{31}$$

where $n_i = dim M_i$, $i = 1, 2$. Furthermore, ∇ and Δ are the gradient and the Laplacian operator on $M_1^{n_1}$, respectively, and H is the mean curvature vector of M^n. The equality sign holds in (31) if, and only if, $n_1 H_1 = n_2 H_2$, where H_1 and H_2 are the mean curvature vectors along $M_1^{n_1}$ and $M_2^{n_2}$, respectively, and φ is a mixed, totally geodesic immersion.

From the motivation studied in [14,34], we present the following consequence of Theorem 2 by using the Remark 2 for a nontrivial warped product pointwise pseudo-slant submanifold of a complex space, such that:

Corollary 2. *Let $\varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \to \widetilde{M}^{2m}(c)$ be an isometric immersion from a warped product pointwise pseudo-slant submanifold into a complex space form $\widetilde{M}^{2m}(c)$, such that $M_1^{n_1}$ is a totally real and $M_2^{n_2}$ is a pointwise slant submanifold of $\widetilde{M}^{2m}(c)$. Then, we have the following inequality:*

$$\Delta(lnf) \leq ||\nabla \ln f||^2 + \frac{n^2}{4n_2}||H||^2 + \frac{n_1 c}{4} - \frac{3c}{4}\cos^2\theta, \tag{32}$$

where $n_i = dim M_i$, $i = 1, 2$. Furthermore, ∇ and Δ are the gradient and the Laplacian operator on $M_1^{n_1}$, respectively, and H is the mean curvature vector of M^n. The equality condition holds in (32) if, and only if, the following satisfies

$$\frac{H_1}{H_2} = \frac{n_2}{n_1}$$

: where H_1 and H_2 are the mean curvature vectors along $M_1^{n_1}$ and $M_2^{n_2}$, respectively, and φ is a mixed, totally geodesic isometric immersion.

Corollary 3. *Let $\varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \to \widetilde{M}^{2m}(c)$ be an isometric immersion from a warped product pointwise pseudo-slant submanifold into a complex space form $\widetilde{M}^{2m}(c)$, such that $M_1^{n_1}$ is a pointwise slant and $M_2^{n_2}$ is a totally real submanifold of $\widetilde{M}^{2m}(c)$. Then, we have the following:*

$$\Delta(lnf) \leq ||\nabla \ln f||^2 + \frac{n^2}{4n_2}||H||^2 + \frac{n_1 c}{4} - \frac{3n_1 c}{4n_2}\cos^2\theta, \tag{33}$$

where $n_i = dim M_i$, $i = 1, 2$. Furthermore, ∇ and Δ are the gradient and the Laplacian operator on $M_1^{n_1}$, respectively, and H is the mean curvature vector of M^n. This equally holds in (33) if, and only if, φ is a mixed, totally geodesic isometric immersion and the following satisfies

$$\frac{H_1}{H_2} = \frac{n_2}{n_1},$$

, where H_1 and H_2 are the mean curvature vectors along $M_1^{n_1}$ and $M_2^{n_2}$, respectively.

Similarly, using Remark 4 and from [17], we got the following result from Theorem 2:

Corollary 4. *Let $\varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \to \widetilde{M}^{2m}(c)$ be an isometric immersion from a CR-warped product into a complex space form $\widetilde{M}^{2m}(c)$, such that $M_1^{n_1}$ is a holomorphic submanifold and $M_2^{n_2}$ is a totally real submanifold of $\widetilde{M}^{2m}(c)$. Then, we get the following:*

$$\Delta(lnf) \leq ||\nabla \ln f||^2 + \frac{n^2}{4n_2}||H||^2 + \frac{n_1 c}{4} - \frac{3n_1 c}{4n_2}, \tag{34}$$

where $n_i = dimM_i$, $i = 1, 2$. Furthermore, ∇ and Δ are the gradient and the Laplacian operator on $M_1^{n_1}$, respectively, and H is the mean curvature vector of M^n. The same holds in (34) if, and only if, φ is mixed and totally geodesic, and $n_1 H_1 = n_2 H_2$, where H_1 and H_2 are the mean curvature vectors on $M_1^{n_1}$ and $M_2^{n_2}$, respectively.

In particular, if both pointwise slant functions $\theta_1 = \theta_2 = \frac{\pi}{2}$, then M^n is becomes a totally real warped product submanifold—thus, we obtain:

Corollary 5. *Let $\varphi : M^n = M_1^{n_1} \times_f M_2^{n_2} \to \widetilde{M}^{2m}(c)$ be an isometric immersion from an n-dimensional, totally real warped product submanifold into a 2m-dimensional complex space form $\widetilde{M}^{2m}(c)$, where $M_1^{n_1}$ and $M_2^{n_2}$ are totally real submanifolds of $\widetilde{M}^{2m}(c)$. Then, we have the following:*

$$\Delta(lnf) \leq ||\nabla \ln f||^2 + \frac{n^2}{4n_2}||H||^2 + \frac{n_1 c}{4}, \tag{35}$$

where $n_i = dimM_i$, $i = 1, 2$ and Δ is the Laplacian operator on $M_1^{n_1}$. The same holds in (35) if, and only if, φ is mixed and totally geodesic, and the following satisfies

$$\frac{H_1}{H_2} = \frac{n_2}{n_1},$$

where H_1 and H_2 are the mean curvature vectors on $M_1^{n_1}$ and $M_2^{n_2}$, respectively.

Proof of Theorem 3. In this direction, we consider the warped product pointwise bi-slant submanifolds as a compact oriented Riemannian manifold without boundary. If the inequality (2) holds:

$$\Delta(lnf) - ||\nabla \ln f||^2 \leq \frac{n^2}{4n_2}||H||^2 + \frac{c}{4n_2}\left(n_1 n_2 - 3n_1 \cos^2\theta_1 - 3n_2 \cos^2\theta_2\right). \tag{36}$$

Since M^n is a compact oriented Riemannian submanifold without boundary, then we have following formula with respect to the volume element:

$$\int_{M^n} \Delta f dV = 0. \tag{37}$$

From the hypothesis of the theorem, M^n is a compact warped product submanifold; then from (37), we derive:

$$\int_M \left(\frac{c}{4n_2}\left(3n_1\cos^2\theta_1 + 3n_2\cos^2\theta_2 - n_1 n_2\right) - \frac{1}{4n_2}\sum_{i=1}^{n}(h_{ii}^{n+1})^2\right)dV \leq \int_M (||\nabla \ln f||^2)dV. \tag{38}$$

Now, we assume that M^n is a Riemannian product, and the warping function f must be constant on M^n. Then, from (38), we get the inequality (3).
□

Conversely, let the inequality (3) hold; then from (38), we derive:

$$0 \leq \int_{M^n} (||\nabla \ln f||^2) \leq 0.$$

The above condition implies that $||\nabla \ln f||^2 = 0$, where this means that f is a constant function on M^n. Hence, M^n is simply a Riemannian product of $M_1^{n_1}$ and $M_2^{n_2}$, respectively. Thus, the theorem is proved. We give some other important corollaries as consequences of Theorem 2, as follows:

Corollary 6. *Let $M^n = M_1^{n_1} \times_f M_2^{n_2}$ be a warped product pointwise bi-slant submanifold of a complex space form $\widetilde{M}^{2m}(c)$ with warping function f, such that $n_1 = dim M_1$ and $n_2 = dim M_2$. If φ is an isometrically minimal immersion from warped product M^n into $\widetilde{M}^{2m}(c)$, then we obtain:*

$$\Delta(lnf) \leq ||\nabla \ln f||^2 + \frac{c}{4n_2}\bigg(n_1 n_2 - 3n_1 \cos^2\theta_1 - 3n_2 \cos^2\theta_2\bigg). \tag{39}$$

Corollary 7. *Let $M^n = M_1^{n_1} \times_f M_2^{n_2}$ be a warped product pointwise bi-slant submanifold of a complex space form $\widetilde{M}^{2m}(c)$ with warping function f, such that $n_1 = dim M_1$ and $n_2 = dim M_\theta$. Then, there is no existing minimal isometric immersion φ from warped product M^n into $\widetilde{M}^{2m}(c)$ with:*

$$\Delta(lnf) > ||\nabla \ln f||^2 + \frac{c}{4n_2}\bigg(n_1 n_2 - 3n_1 \cos^2\theta_1 - 3n_2 \cos^2\theta_2\bigg). \tag{40}$$

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