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The Non-Relativistic Limit of the DKP Equation in Non-Commutative Phase-Space

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Abstract: The non-relativistic limit of the relativistic DKP equation for both of zero and unity spin particles is studied through the canonical transformation known as the Foldy–Wouthuysen transformation, similar to that of the case of the Dirac equation for spin-1/2 particles. By considering only the non-commutativity in phases with a non-interacting fields case leads to the non-commutative Schrödinger equation; thereafter, considering the non-commutativity in phase and space with an external electromagnetic field thus leads to extract a phase-space non-commutative Schrödinger–Pauli equation; there, we examined the effect of the non-commutativity in phase-space on the non-relativistic limit of the DKP equation. However, with both Bopp–Shift linear transformation through the Heisenberg-like commutation relations, and the Moyal–Weyl product, we introduced the non-commutativity in phase and space.

Keywords: DKP equation; noncommutative DKP equation; Schrödinger equation; noncommutative SchrödingerPauli equation; phase-space noncommutativity; Foldy-Wouthuysen transformation; non-relativistic limit

1. Introduction

Lately, it has been very interesting to investigate the theoretical basis of the modern physics to explain the nature and the behavior of the matter and energy on the subatomic scale, sometimes referred to as quantum theories, such as the quantum gravity [1,2] or quantum general relativity (QGR) [3], quantum optics and information, the standard model and the gauge theories [4]. This investigation sometimes can be represented in terms of the low-energy regime through the examination of the non-relativistic properties in miscellaneous interactions such as the external electromagnetic fields (EMF), Dirac or DKP oscillator interaction [5,6], Lennard–Jones potential, Coulomb potential, square and step potential; the non-relativistic limit is about low speeds in front of the speed of the light, in more detail, it is for the regime of weak-energy in front of the mass-energy $\frac{pc}{mc^2} \ll 1$ [7], where the non-relativistic limit can be realized through numerous methods, among them the Foldy–Wouthuysen (FW) transformation [8], and Eriksen's method [9] proposed in 1958. Based on the methodical derivation of the unitary transformation that makes the Hamiltonian a diagonal operator, it can be also used when an electromagnetic field is present. There is also the Cayley transformation, and the Cini–Touschek transformation, and the method of development in power of $\hbar$ [10] and the Douglas–Kroll–Hess (DKH) approach [11,12], which was used mostly as part of relativistic quantum chemistry, and it depends on separating (block–diagonalize) relativistic Hamiltonians into two parts. One part describes electrons in the case of Dirac Hamiltonian, for example, while the other gives rise to the negative energy states. The non-relativistic limit of the relativistic equations was essentially investigated by converting the Hamiltonian from an odd form to an even form.

In this work, we investigated the non-relativistic limit of the DKP equation according to the Foldy–Wouthuysen transformation which occupies an extraordinary position in quantum physics because of its unprecedented properties that drive us to obtain totally block-diagonalized operators (even Hamiltonians), keeping the properties of the operators in the FW representation like those of the classical representation. The fantastic benefit of the FW representation is the manageable form of operators and quantities; with these advantages, the FW representation gives the best chance to obtain a significant non-relativistic limit of the relativistic quantum mechanics. The transition to the non-relativistic limit generally is due to a replacement of the operators in our quantum-mechanical systems to its corresponding classical quantities. This implicit or explicit replacement used in all calculations was devoted to the Foldy–Wouthuysen transformation. Take into consideration that the Foldy–Wouthuysen transformation takes the original Hamiltonian to an even form in the case of presence (or absence) of the electromagnetic field, and the diagonalization of a Hamiltonian does not perforce drive to the FW representation.

The Duffin–Kemmer–Petiau (DKP) equation introduced by R.J. Duffin, Nicholas Kemmer, and G. Petiau and it is a 1st-order relativistic wave equation provide a relativistic description of spin-0 and spin-1 particles in a single relativistic equation, the DKP equation has been considered in connection with various aspects including the nucleus elastic scattering, the Aharonov–Bohm (AB) effect [13], in cosmic string background, with meson–nuclear interaction, quantum chromodyn amics (QCD), five-dimensional Galilean invariance [14]. From the corresponding DKP equation, we can derive the DKP oscillator and investigate the energy eigenvalues (spectrum), eigenfunctions, the influence of the topological defect on the equation of motion.

More precisely, in this work, we investigated the non-relativistic limit of the DKP equation using the FW transformation in a non-commutative phase space (NCPS). This study was presented and considered for the importance and benefits of non-commutative geometry (NCG) in both quantum mechanics and quantum fields, where the idea behind the non-commutativity in spacetime is highly motivated by quantum mechanics (QM) and the origin of the non-commutative geometry pertaining to the research of topological spaces (when commutative C^*–algebras of functions are replaced by non-commutative algebras). The concept of NCG was rekindled by Connes and others [15–18], who theorized the idea of a differential structure in the non-commutative setting. The non-commutative theory replaces the noncommutativity of the operators related to the space-time coordinates by a deformation of the algebra of the functions defined on the space-time. On the other hand, a non-commutative version of a field theory is obtained by replacing ordinary theory to a non-commutative one by replacing ordinary fields with non-commutative fields and ordinary products with Moyal–Weyl products. Precisely, Nathan Seiberg and Edward Witten in the past few years released their famous article [19], which was from the most cited article according to Spires (Stanford Physics Information Retrieval System), it prompted and encouraged a wide amount of interest in NCG, which became mainstream for a couple years. Taking into consideration that the notions of non-commutativity in phase space based principally on the Seiberg–Witten map, the star product, and the Bopp–Shift linear transformation.

Throughout this article, the fundamental properties of the DKP Hamiltonian in the Foldy–Wouthuysen representation in the non-commutative phase-space are studied and the Schrödinger and the Schrödinger–Pauli equations [20,21] are found. Knowing that the former extracted in case of only considering the noncommutativity in phase with the absence of the electromagnetic field, and the latter extracted in the case of presence of the external electromagnetic field and the non-commutativity considered in phase and in the space. Taking into account that we obtained the phase-space non-commutative DKP equation using both Bopp–Shift linear transformation through the Heisenberg- like commutation relations, and the Gronewold–Moyal product ($\star$-product).

2. Review of the Non-Commutative Geometry

Firstly, let us review the basic formulas of the NC algebra. At string scales, space does not commute as shown in the theory of NCG, so that we admit the operators of coordinates and kinetic momentum in the (2+1)d non-commutative phase-space x_i^{nc} and p_i^{nc}, respectively, and the Heisenberg-like non-commutative commutation relations [22] appear as follows:

$$\left[x_i^{nc}, x_j^{nc}\right] = i\Theta_{ij}, \left[p_i^{nc}, p_j^{nc}\right] = i\eta_{ij}, \left[x_i^{nc}, p_j^{nc}\right] = i\hbar^{eff}\delta_{ij}, \ (i,j=1,2), \tag{1}$$

the effective Plank constant can be written as

$$\hbar^{eff} = \hbar(1 + \frac{\Theta\eta}{4\hbar^2}), \tag{2}$$

with

$$\Theta_{ij} = \epsilon_{ijk}\Theta_k, \ \Theta_k = (0,0,\Theta), \ \eta_{ij} = \epsilon_{ijk}\eta_k, \ \eta_k = (0,0,\eta). \tag{3}$$

Θ_{ij} and η_{ij} are antisymmetric constant tensors, Θ, η are real-valued non-commutativity parameters, they are supposed to be very small, with the dimension of $lenght^2$, $momentum^2$, respectively.

For some investigations about non-commutative systems concerning the NC parameters, the experimental limit of about 100 nHz on possible sidereal variations (the highest energy variations supported by the experiment) gives estimated limits at about $\Theta \simeq 4.10^{-40}$ m^2, $\eta \simeq 1.76 \times 10^{-61}$ Kg^{2}m^{2}s^{-2}, and $\hbar^{eff} \simeq 10^{-67}$ (SI) [23]. These values agree with the higher limits on the basic scales of coordinate and momentum, and these bounds will be suppressed if the magnetic field used in the experiment is weak ($B \simeq 5$ mG).

In the (2+1)d commutative phase-space, the canonical variables x_i and p_i satisfy the following commutative algebra

$$\left[x_i, x_j\right] = \left[p_i, p_j\right] = 0, \ \left[x_i, p_j\right] = i\hbar\delta_{ij}(i,j=1,2). \tag{4}$$

The non-commutative geometry Equation (1) in turn is described at the level of fields and functions, by the Gronewold–Moyal product ($\star$-product) [24–26] defined as

$$\begin{aligned}(f \star g)(x,p) &= \exp[\tfrac{i}{2}\Theta_{ab}\partial_{x_a}\partial_{x_b} + \tfrac{i}{2}\eta_{ab}\partial_{p_a}\partial_{p_b}]f(x_a,p_a)\,g(x_b,p_b) = f(x,p)g(x,p) \\ &+ \textstyle\sum_{n=1}\left(\frac{1}{n!}\right)\left(\frac{i}{2}\right)^n \Theta^{a_1b_1}...\Theta^{a_nb_n}\partial_{a_1}...\partial_{a_k}f(x,p)\partial_{b_1}...\partial_{b_k}g(x,p) \\ &+ \textstyle\sum_{n=1}\left(\frac{1}{n!}\right)\left(\frac{i}{2}\right)^n \eta^{a_1b_1}...\eta^{a_nb_n}\partial_{a_1}...\partial_{a_k}f(x,p)\partial_{b_1}...\partial_{b_k}g(x,p).\end{aligned} \tag{5}$$

Because of the nature of the $\star$-product, the non-commutative field theories for the slowly varying fields or low energies ($\Theta E^2 < 1$) completely reduce to their commutative version.

The NCPS operators are linked to the commutative operators through the Heisenberg–Weyl algebra in terms of the aka Bopp-shift translation which was introduced from Equation (5) [27,28], and it is given by

$$x_i^{nc} = x_i - \frac{1}{2\hbar}\Theta_{ij}p_j, \quad p_i^{nc} = p_i + \frac{1}{2\hbar}\eta_{ij}x_j. \tag{6}$$

When also $\Theta = \eta = 0$, the NCPS algebra reduces to the commutative algebra.

3. Schrödinger Equation from the DKP Equation in the Non-Commutative Phase

The DKP equation (Kemmer equation) is a 1st-order relativistic wave equation provides a relativistic description of a free boson with nonzero mass m, and it is given by [29–31]

$$(i\beta^\mu\partial_\mu - m)\psi = 0, \tag{7}$$

where $\hbar = c = 1$ (natural units), and ψ is the boson wave function, and $\beta^\mu = (\beta^0, \vec{\beta})$ are the DKP square matrices being used to define the so-called DKP algebra which satisfy the following algebraic relation

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\nu\lambda}\beta^\mu, \tag{8}$$

the following relations can be implied

$$\begin{gathered}\beta_0\beta_k\beta_0 = 0\,, k = 1,2,3,\\ \beta_0^3 = \beta_0,\\ \beta_\mu a^\mu \beta_\nu \beta_\lambda a^\lambda = \beta_\mu a^\mu a_\nu,\\ (\vec{\beta}\vec{a})\beta_0(\vec{\beta}\vec{a}) = 0,\end{gathered} \tag{9}$$

where the Greek letters μ, ν, λ being $0, 1, 2, 3$ and $g^{\mu\nu} = diag\,(1, -1, -1, -1)$ being the metric tensor in Minkowski space-time, and repeated indices are automatically summed over, is employed. For example, $a_\mu b^\mu = a_0 b^0 - \vec{a}\vec{b}$, and Equation (8) has three irreducible representations: a 10d representation provides a description of spin-1 bosons, a 5d representation provides a description of spin-0 bosons (spinless particles), and a 1d representation which is a trivial representation.

In a small part of this paper (subsection III-A), we use the representation of order 5 which represents the particles with 0-spin. Therefore, β^μ are 5×5 matrices defined as:

$$\beta^0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \beta^1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \beta^2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \beta^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

Then, the stationary state ψ is a vector with a 5-component wave function, and it can be given by

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^T. \tag{11}$$

For spin-1 bosons, β^μ are 10×10 matrices, and the state ψ is a vector with a 10-component wave function, which can be given as

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9, \psi_{10})^T. \tag{12}$$

The Kemmer equation for spin-0 is almost related to the Klein–Gordon equation [32], and for spin-1 is associated with the Proca equations [33].

We have

$$\partial_\mu \psi = \beta_\mu \partial^\nu \beta_\nu \psi, \tag{13}$$

multiplying Equation (7) by β_0 and getting the zero component of Equation (13), we can denote

$$H_0(\vec{P}, \vec{x}) = \vec{\alpha}\vec{P} + \beta_0 m, \tag{14}$$

with

$$\alpha_k = \beta_0\beta_k - \beta_k\beta_0. \tag{15}$$

Equation (7) can be also written in the form

$$\left(\vec{\beta}\vec{P} + m\right) = i\beta_0 \frac{\partial}{\partial t}\psi. \tag{16}$$

Substituting Equation (6) into Equation (14), the DKP Hamiltonian in a non-commutative phase becomes

$$H_0^{nc}(\vec{P}, \vec{x}) = \alpha^i(P^i + \frac{1}{2}\eta^{ij}x^j) + \beta_0 m, \tag{17}$$

with $\eta^{ij} = 2\eta^k \epsilon^{kij}$, and according to the definition of the vectorial product $(A \times B)_\mu = \epsilon_{\mu\nu\lambda} A_\nu . B_\lambda$, we shall denote the DKP Hamiltonian in a non-commutative phase by

$$H_0^{nc}(\vec{P}, \vec{x}) = \vec{\alpha}\vec{P} + (\vec{\alpha} \times \vec{x})\vec{\eta} + \beta_0 m. \tag{18}$$

The non-commutative phase parameter forms a tetrahedron with the position $\vec{x}$ and the DKP vector $\vec{\beta}$.

The Foldy–Wouthuysen Transformation for a Free Boson in a Non-Commutative Phase

The Foldy–Wouthuysen transformation eliminates the odd part entirely from the wave equation Hamiltonian, and reduces it to an even part (diagonal form). The unitary FW transformation is presented by the following transformations:

$$\psi_{FW} = U_{FW}\psi = e^{iS}\psi, \tag{19}$$

with U_{FW} being a unitary operator, and S being the time-independent Hermitian operator

$$S = -i\frac{\vec{\beta}\vec{P}}{|\vec{P}|}\theta, \tag{20}$$

where θ is a function, taking into account that $tan(2|\vec{P}|\theta) = \frac{|\vec{P}|}{m}$. The transformed Hamiltonian should contain no odd operators

$$\tilde{H}_0^{nc} = U_{FW} H_0^{nc} U_{FW}^\dagger, \tag{21}$$

by applying the transformation (21) to Equation (18), knowing that S is the non-explicitly time-dependent operator,

$$\tilde{H}_0^{nc} = e^{\frac{\vec{\beta}\vec{P}}{|\vec{P}|}\theta} \left(\vec{\alpha}\vec{P} + \left(\vec{\alpha} \times \vec{x}\right)\vec{\eta} + \beta_0 m \right) e^{-\frac{\vec{\beta}\vec{P}}{|\vec{P}|}\theta}, \tag{22}$$

as $U_{FW}U_{FW}^\dagger = 1$. In this case, Equation (22) is written as

$$\tilde{H}_0^{nc} = e^{2\frac{\vec{\beta}\vec{P}}{|\vec{P}|}\theta} \left(\vec{\alpha}\vec{P} + \left(\vec{\alpha} \times \vec{x}\right)\vec{\eta} + \beta_0 m \right), \tag{23}$$

knowing that, from Equation (8),

$$\left(\vec{\beta}\vec{P}\right)^3 = \sum_{i,j,k} P_i P_j P_k \frac{(\beta_i\beta_j\beta_k + \beta_k\beta_j\beta_i)}{2}, \tag{24}$$

$$2\left(\vec{\beta}\vec{P}\right)^3 = -\sum_{i,j,k} P_i P_j P_k \left(\beta_i\delta_{ij} + \beta_k\delta_{ji}\right), \tag{25}$$

so that

$$\left(\vec{\beta}\vec{P}\right)^3 = -|\vec{P}|^2\left(\vec{\beta}\vec{P}\right), \tag{26}$$

noting that

$$\left(\frac{\vec{\beta}\vec{P}}{|\vec{P}|}\right)^2 = i^2, \tag{27}$$

where the unitary operator is

$$e^{\frac{\vec{\beta}\vec{P}}{|\vec{P}|}\theta} = cos(|\vec{P}|\theta) + \frac{\vec{\beta}\vec{P}}{|\vec{P}|} sin(|\vec{P}|\theta), \tag{28}$$

Equation (23) becomes

$$\tilde{H}_0^{nc} = \left(cos(2|\vec{P}|\theta) + \frac{\vec{\beta}\vec{P}}{|\vec{P}|} sin(2|\vec{P}|\theta) \right) \left(\vec{\alpha}\vec{P} + \left(\vec{\alpha} \times \vec{x} \right) \vec{\eta} + \beta_0 m \right), \tag{29}$$

using the property $(\vec{\alpha} \times \vec{x})\vec{\eta} = -(\vec{x} \times \vec{\alpha})\vec{\eta} = (\vec{x} \times \vec{\eta})\vec{\alpha}$ (mixed product property). Then, Equation (29) changes to

$$\tilde{H}_0^{nc} = \vec{\alpha}(\vec{P} + \vec{x} \times \vec{\eta})(cos(2|\vec{P}|\theta) - \frac{m}{|\vec{P}|} sin(2|\vec{P}|\theta)) + \beta_0(m\, cos(2|\vec{P}|\theta) + |\vec{P}| sin(2|\vec{P}|\theta)). \tag{30}$$

In order to eliminate the odd part, we chose

$$sin(2|\vec{P}|\theta) = \frac{|\vec{P}|}{E}, cos(2|\vec{P}|\theta) = \frac{m}{E}, \tag{31}$$

one arrives at

$$\tilde{H}_0^{nc} = \frac{\beta_0}{E}(\vec{P}^2 + m^2). \tag{32}$$

Last but not least, this satisfies the Schrödinger equation, and Equation (32) is similar to the case of commutative phase and space, so that we find that the effect of the non-commutativity in phase on the non-relativistic limit of the DKP Hamiltonian vanished, due to the fact that the non-commutativity parameter entangled explicitly into the non-diagonal part of the non-commutative Hamiltonian. In another way, the non-commutativity in phase affects the odd part of the DKP Hamiltonian. This is for the case of no interaction with potentials.

For the transformed wave function, we merely take the case of the spin-0 representation. We choose the wave function

$$\psi(\vec{x}, t) = \psi(\vec{x})e^{-iEt}. \tag{33}$$

From Equation (16), we rewrite the DKP equation in (2+1)d as follows:

$$\left(\beta^1 P_x^{nc} + \beta^2 P_y^{nc} + m \right) \psi = \beta_0 E \psi, \tag{34}$$

Substituting ψ into Equation (34) gives us

$$-m\psi_1 + E\psi_2 + P_x^{nc}\psi_3 + P_y^{nc}\psi_4 = 0, \tag{35}$$

$$E\psi_1 - m\psi_2 = 0, \quad P_y^{nc}\psi_1 + m\psi_4 = 0 \quad P_x^{nc}\psi_1 + m\psi_3 = 0, \quad m\psi_5 = 0. \tag{36}$$

It is clear that the five components ($\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$) are not independent from each other. With ($\psi_2 = \frac{E}{m}\psi_1, \psi_3 = \frac{-P_x^{nc}}{m}\psi_1, \psi_4 = \frac{-P_y^{nc}}{m}\psi_1$), and combining the above equations, we get the dynamical equation of component ψ_1

$$\left(-m^2 + E^2 - (P_x^{nc})^2 - \left(P_y^{nc} \right)^2 \right) \psi_1 = 0. \tag{37}$$

This leads us to

$$E = \pm\sqrt{m^2 + \left(\overrightarrow{P}^{nc}\right)^2}, \tag{38}$$

where see the Appendix A

$$\left(\overrightarrow{P}^{nc}\right)^2 = \overrightarrow{P}^2 - 2\overrightarrow{L}\overrightarrow{\eta} + 0\left(\eta^2\right). \tag{39}$$

Then, we find

$$E = \pm\sqrt{m^2 + \overrightarrow{P}^2 - 2\overrightarrow{L}\overrightarrow{\eta}}, \tag{40}$$

so that, substituting Equation (40) into Equation (33), we obtain

$$\psi(\overrightarrow{x}, t) = \psi(\overrightarrow{x})e^{-i\sqrt{m^2+\overrightarrow{P}^2-2\overrightarrow{L}\overrightarrow{\eta}}\,t}. \tag{41}$$

Then, our transformed wave function Equation (19) becomes

$$\psi_{FW} = \psi(\overrightarrow{x})e^{\frac{\overrightarrow{\beta}\overrightarrow{P}}{|\overrightarrow{P}|}\theta - i\sqrt{m^2+\overrightarrow{P}^2-2\overrightarrow{L}\overrightarrow{\eta}}\,t}, \tag{42}$$

unlike what happened with the Hamiltonian (18). Here, in the wavefunction Equation (42), the effect of the phase non-commutativity does not vanish because, in the calculations of the energy (Hamiltonian eigenvalue), in order to obtain the wavefunction, η was not entangled with an odd term for that it remains in the equation. In this part of the work, we say that FW transformation did not eliminate the effect of the phase non-commutativity from the Schrödinger equation as was expected.

4. Schrödinger–Pauli Equation from the DKP Equation in Non-Commutative Phase-Space

At first, defining the electromagnetic field $A_\mu = (A_0, \overrightarrow{A})$ by inserting the following covariant derivative in the DKP equation

$$D_\mu = \partial_\mu - ieA_\mu, \tag{43}$$

satisfies the commutation relation

$$\left[D_\mu, D_\nu\right] = -ieF_{\mu\nu},\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{44}$$

then, the DKP equation in the presence of an electromagnetic interaction (EMI) is

$$\left(i\not{D} - m\right)\psi = 0, \tag{45}$$

with $\not{D} = \beta_\mu D^\mu$. Then, the suitable physical form for ψ (when there is EMI) can be written as

$$\psi = \begin{pmatrix} \frac{i}{\sqrt{m}} D_\mu \varphi \\ \sqrt{m}\varphi \end{pmatrix}. \tag{46}$$

We will explain the presence of an apparently abnormal term devoid of physical interpretation in the DKP Hamiltonian. This term is generated because of the consideration of the minimal coupling Equation (43) in the Kemmer equation, so that contracting Equation (45) on the left by $D_\mu \beta^\mu \beta^\nu$ leads to

$$i\beta^\rho\beta^\nu\beta^\mu D_\rho D_\mu \psi = mD_\rho\beta^\rho\beta^\nu\psi. \tag{47}$$

After some algebraic considerations and simplifications, the above relation becomes

$$D^{\mu}\psi = \beta^{\mu}\beta^{\nu}D_{\mu}\psi + \frac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{\nu}\beta^{\rho} + \beta^{\mu}g^{\nu\rho}\right)\psi. \tag{48}$$

Then, we multiply Equation (45) on the left by $-i\beta^{\rho}$ and making $\nu = 0$ in Equation (48), we obtain the Hamiltonian form of the Kemmer equation

$$i\partial_0\psi = H\psi, \tag{49}$$

where

$$H = i\left[\beta^{i},\beta^{0}\right]D_i + i\frac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right) - eA_0 + m\beta_0. \tag{50}$$

Finally, the above equation becomes

$$H = \vec{\alpha}(\vec{P} - e\vec{A}) - eA_0 + m\beta_0 + i\frac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right). \tag{51}$$

The term proportional to $\frac{e}{2m}$ in Equations (48)–(51) is previously regarded by Kemmer himself in his own original research [34]. This term has no clear physical interpretation unlike the other terms in these equations which have physical interpretations similar to similar terms obtained in the Dirac equation interacting with the electromagnetic field.

Using the $\star$-product, we find the DKP equation in the non-commutative phase-space

$$H(\vec{P},\vec{x})\star\psi(\vec{x}) = i\beta^{0}\partial_0\psi. \tag{52}$$

Firstly, using Equation (5), we link the non-commutative coordinates x_i^{nc} to the commutative one x_i so that we achieve the non-commutativity in space

$$H(\vec{P},\vec{x})\star\psi(\vec{x}) = \{\vec{\alpha}(\vec{P} - e\vec{A}(\vec{x})) - eA_0(\vec{x}) + m\beta_0 + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right)\}\star\psi(\vec{x}). \tag{53}$$

Considering $\vec{A}(\vec{x}) = k\vec{x}$, with k being a real constant so that the derivation in Equation (5) turns off in the first order $0(\Theta^2)$. Then, Equation (53)

$$\begin{aligned} H(\vec{P},\vec{x})\star\psi(\vec{x}) = H(\vec{P},\vec{x})\psi(\vec{x}) + \tfrac{i}{2}\Theta_{ab}\partial_a\left[-e\vec{\alpha}(\vec{A}(\vec{x})) - eA_0(\vec{x}) + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right)\right]\partial_b\psi(\vec{x}) \\ +\mathcal{O}(\Theta^2)\psi(\vec{x}) = i\beta^{0}\partial_0\psi(\vec{x}). \end{aligned} \tag{54}$$

where $\partial_a(m\beta_0) = \partial_a(\vec{\alpha}\vec{P}) = 0$, we obtain

$$\begin{aligned} H(\vec{P},\vec{x})\star\psi(\vec{x}) = \vec{\alpha}(\vec{P} - e\vec{A}(\vec{x})) - eA_0(\vec{x}) + m\beta_0 + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right) \\ +\tfrac{i}{2}\Theta_{ab}\partial_a\left[-e\vec{\alpha}\vec{A}(\vec{x}) - eA_0(\vec{x}) + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right)\right]\partial_b\psi(\vec{x}) = i\beta^{0}\partial_0\psi(\vec{x}). \end{aligned} \tag{55}$$

Secondly, using Equation (6), we link the non-commutative kinetic momentum P_i^{nc} to the commutative one P_i so that we achieve the the non-commutativity in phase into Equation (55), to find the DKP equation in the complete non-commutative phase-space

$$\begin{aligned} \alpha_i(P_i + \tfrac{1}{2\hbar}\eta_{ij}x_j - eA_i(\vec{x})) - eA_0(\vec{x}) + m\beta_0 + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right) \\ +\tfrac{i}{2}\Theta_{ab}\partial_a\left[-e\alpha_i A_i(\vec{x}) - eA_0(\vec{x}) + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^{\mu}\beta^{0}\beta^{\rho} + \beta^{\mu}g^{0\rho}\right)\right]\partial_b\psi(\vec{x}) = i\beta^{0}\partial_0\psi(\vec{x}), \end{aligned} \tag{56}$$

with $\eta^{ij} = 2\eta^k \epsilon^{kij}$, and, according to the definition of the vectorial product $(A \times B)_\mu = \epsilon_{\mu\nu\lambda} A_\nu . B_\lambda$, and, after minor simplification

$$\begin{aligned} H(\vec{P}, \vec{x}) \star \psi(\vec{x}) &= \vec{\alpha}(\vec{P} - e\vec{\alpha}\vec{A}) + \left(\vec{\alpha} \times \vec{x}\right)\vec{\eta} - eA_0 + m\beta_0 + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right) \\ &+ e\left(\vec{\nabla}\left(\vec{\alpha}\vec{A} + A_0 - \tfrac{i}{2m}F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right)\right) \times \vec{p}\right)\vec{\Theta}\psi(\vec{x}) = i\beta^0\partial_0\psi(\vec{x}). \end{aligned} \tag{57}$$

Foldy–Wouthuysen Transformation in Non-Commutative Phase-Space

We determine the Schrödinger–Pauli equation in NCPS, which means obtaining the non-relativistic limit from the DKP equation through the Foldy–Wouthuysen transformation, knowing that the FW transformation is suitable for weak fields. The DKP Hamiltonian in NCPS can be written in the form

$$\begin{aligned} H^{nc} &= \vec{\alpha}\vec{\pi} + \left(\vec{\alpha} \times \vec{x}\right).\vec{\eta} - eA_0 + m\beta_0 + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right) \\ &+ e\left(\vec{\nabla}\left(\vec{\alpha}\vec{A} + A_0 - \tfrac{i}{2m}F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right)\right) \times \vec{p}\right)\vec{\Theta}, \end{aligned} \tag{58}$$

with

$$\vec{\pi} = \vec{P} - e\vec{A}(\vec{x}). \tag{59}$$

For performing the Foldy–Wouthuysen transformation, we split our non-commutative DKP Hamiltonian Equation (58) to a block diagonal part (even operator ξ) and an off-diagonal part (odd operator $\mathcal{O}$), (Odd operators (off-diagonal matrices): α^i, β^i, β^0, ..., even operators (diagonal matrices): I_i, δ_{ij}, ...)

$$H^{nc} = \mathcal{O} + \xi, \tag{60}$$

with

$$\xi = -eA_0 + e((gradA_0) \times \vec{p})\vec{\Theta}, \tag{61}$$

$$\begin{aligned} \mathcal{O} &= \vec{\alpha}\vec{\pi} + \left(\vec{\alpha} \times \vec{x}\right)\vec{\eta} + m\beta_0 + i\tfrac{e}{2m}F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right) \\ &+ e\left(div\left(\vec{\alpha}\vec{A} - \tfrac{i}{2m}F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right)\right) \times \vec{p}\right)\vec{\Theta}. \end{aligned} \tag{62}$$

These are defined to satisfy $\xi\mathcal{O} = \mathcal{O}\xi$.

We consider that, if we multiply two odd operators (or even operators), we find an even operator, and, if multiplying an even operator with an odd operator, we obtain an odd operator. Then, using the Foldy–Wouthuysen transformation, we remove all odd operators. We may successively eliminate these odd terms from the DKP Hamiltonian in NCPS. Later, we will obtain a Hamiltonian completely free of odd operators. We further assume that ξ and $\mathcal{O}$ can not be less in order than $\left(\frac{1}{m}\right)^0$.

From Equations (19)–(21) and satisfying Equation (49), in the case of time-dependent Hamiltonian and time-dependent operator S, we consider the canonical transformation

$$\begin{aligned} \psi_{FW} &= e^{iS}\psi, \\ \tilde{H}^{nc} &= e^{iS}H^{nc}e^{-iS} - ie^{iS}\tfrac{\partial}{\partial t}e^{-iS}, \end{aligned} \tag{63}$$

while the Hermitian operator S may be considered small, and it is given by

$$S = -i\frac{\vec{\beta}\vec{\pi}}{m}, \tag{64}$$

so that we may perform an expansion in powers of $\frac{1}{m}$ of the DKP Hamiltonian (using the Baker–Campbell–Hausdorf formula [35–37])

$$\tilde{H}^{nc} = H^{nc} + \tfrac{\partial S}{\partial t} + i\left[S, H^{nc} + \tfrac{1}{2}\tfrac{\partial S}{\partial t}\right] - \tfrac{1}{2!}\left[S, \left[S, H^{nc} + \tfrac{1}{3}\tfrac{\partial S}{\partial t}\right]\right] - \tfrac{i}{3!}\left[S, \left[S, \left[S, H^{nc} + \tfrac{1}{4}\tfrac{\partial S}{\partial t}\right]\right]\right] + \ldots \tag{65}$$

and

$$\psi_{FW}^{nc} = \begin{pmatrix} \frac{i}{\sqrt{m}} D_\mu^{nc} \varphi^{nc} \\ \sqrt{m}\varphi^{nc} \end{pmatrix}. \tag{66}$$

We calculate our transformed Hamiltonian $\tilde{H}^{nc}$ of Equation (65), where we will retain only the terms in the approximation $(\frac{1}{m})^4$. For this, we first calculate the following switches (see the Appendix B):

$$\begin{array}{c} i\left[S, H^{nc} + \frac{1}{2}\frac{\partial S}{\partial t}\right] = i\left[S, \mathcal{O} + \xi + \frac{1}{2}\frac{\partial S}{\partial t}\right] = -\frac{e}{m}\vec{\beta}\, grad A_0 - \frac{1}{m}\beta_0 \vec{\pi}^2 \\ +\frac{1}{m}\left[\vec{\beta}\vec{\pi}, \left(\vec{\alpha}\times\vec{x}\right)\vec{\eta}\right] - \vec{\alpha}\vec{\pi} + \frac{e}{2m^2}\left[\vec{\beta}\vec{\pi}, \beta^0 \vec{\Sigma}\vec{H} + (\vec{\beta}\times\vec{\alpha})\vec{H}\right] + \frac{e}{m^2}\vec{\Sigma}\left(\vec{\beta}\times\vec{E}\right)(\beta^0)^2 \\ -\frac{ie}{m^2}(\vec{\beta}\vec{E})\{2(\vec{\beta}\vec{\pi})(\beta^0)^2 - \vec{\beta}\vec{\pi}\} + \frac{e}{m}\left[\vec{\beta}\vec{\pi}, (div(\vec{\alpha}\vec{A} - \frac{i}{2m}F_{\rho\mu}(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}))\times\vec{p})\vec{\Theta}\right] - \frac{e}{2m^2}\vec{\Sigma}(\vec{\pi}\times\frac{\partial\vec{A}}{\partial t}) \end{array}, \tag{67}$$

$$\begin{array}{c} \frac{-i}{2!}\left[S, \left[iS, H^{nc} + \frac{1}{3}\frac{\partial S}{\partial t}\right]\right] = \frac{e}{2m^2}\vec{\Sigma}(\vec{\pi}\times grad A_0) - \frac{1}{2m^2}\vec{\pi}^2(\vec{\alpha}\vec{\pi}) - \frac{1}{2m^2}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left(\vec{\alpha}\times\vec{x}\right)\vec{\eta}\right]\right] \\ -\frac{1}{2m}\beta_0\vec{\pi}^2 - \frac{ie}{4m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, -2(\vec{E}\vec{\beta})(\beta^0)^2 - i\beta^0\vec{S}\vec{H} - i(\vec{\beta}\times\vec{\alpha})\vec{H}\right]\right] \\ -\frac{e}{2m^2}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, (div(\vec{\alpha}\vec{A} - \frac{i}{2m}F_{\rho\mu}(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}))\times\vec{p})\vec{\Theta}\right]\right] + \frac{e}{6m^3}\left[\vec{\beta}\vec{\pi}, \vec{\Sigma}(\vec{\pi}\times\frac{\partial\vec{A}}{\partial t})\right] \end{array}, \tag{68}$$

$$\begin{array}{c} \frac{i}{3!}\left[S, \left[S, \left[S, H^{nc} + \frac{1}{4}\frac{\partial S}{\partial t}\right]\right]\right] = \frac{e}{12m^3}\left[\vec{\beta}\vec{\pi}, \vec{\Sigma}(\vec{\pi}\times grad A_0)\right] - \frac{1}{12m^3}\left[\vec{\beta}\vec{\pi}, \vec{\pi}^2(\vec{\alpha}\vec{\pi})\right] - \frac{1}{12m^2}\left[\vec{\beta}\vec{\pi}, \beta_0\vec{\pi}^2\right] \\ -\frac{1}{12m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left(\vec{\alpha}\times\vec{x}\right)\vec{\eta}\right]\right]\right] - \frac{ie}{24m^4}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, -2(\vec{E}\vec{\beta})(\beta^0)^2 - i\beta^0\vec{S}\vec{H} - i(\vec{\beta}\times\vec{\alpha})\vec{H}\right]\right]\right] \\ -\frac{e}{12m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, (div(\vec{\alpha}\vec{A} - \frac{i}{2m}F_{\rho\mu}(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}))\times\vec{p})\vec{\Theta}\right]\right]\right] + \frac{e}{48m^4}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \vec{\Sigma}(\vec{\pi}\times\frac{\partial\vec{A}}{\partial t})\right]\right] \end{array}, \tag{69}$$

the terms containing large powers of $\frac{1}{m}$ (up of $\frac{1}{m^4}$) in Equation (69) may be ignored. To be specific, we only consider terms of the order that we limit ourselves in the development. Hence, Equation (69) becomes

$$\begin{array}{c} \frac{i}{3!}\left[S, \left[S, \left[S, H^{nc} + \frac{1}{4}\frac{\partial S}{\partial t}\right]\right]\right] = \frac{e}{12m^3}\left[\vec{\beta}\vec{\pi}, \vec{\Sigma}(\vec{\pi}\times grad A_0)\right] \\ -\frac{1}{12m^3}\left[\vec{\beta}\vec{\pi}, \vec{\pi}^2(\vec{\alpha}\vec{\pi})\right] - \frac{1}{12m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left(\vec{\alpha}\times\vec{x}\right)\vec{\eta}\right]\right]\right] - \frac{1}{12m^2}\left[\vec{\beta}\vec{\pi}, \beta_0\vec{\pi}^2\right] \\ -\frac{e}{12m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, (div(\vec{\alpha}\vec{A} - \frac{i}{2m}F_{\rho\mu}(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}))\times\vec{p})\vec{\Theta}\right]\right]\right] + 0(\frac{1}{m})^4 \end{array}. \tag{70}$$

By substituting Equations (67), (68) and (70) into Equation (65), with $\vec{x}\times\vec{\eta} = \vec{\overline{X}}$, we arrive at

$$\begin{array}{c} \tilde{H}^{nc} = \left(\vec{\alpha}\times\vec{x}\right)\vec{\eta} - eA_0 + m\beta_0 + i\frac{eF_{\rho\mu}}{2m}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right) + e((grad A_0)\times\vec{p})\vec{\Theta} - \beta_0\frac{\vec{\pi}^2}{2m} \\ +e\left(div\left(\vec{\alpha}\vec{A} - \frac{iF_{\rho\mu}}{2m}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right)\right)\times\vec{p}\right)\vec{\Theta} - \frac{i}{m}\vec{\beta}\frac{\partial\vec{\pi}}{\partial t} - \frac{e}{m}\vec{\beta}\, grad A_0 + \frac{1}{m}\left[\vec{\beta}\vec{\pi}, \vec{\overline{X}}\vec{\alpha}\right] \\ +\frac{e}{2m^2}\left[\vec{\beta}\vec{\pi}, \beta^0\vec{\Sigma}\vec{H} + (\vec{\beta}\times\vec{\alpha})\vec{H}\right] + \frac{e\vec{\Sigma}}{m^2}\left(\vec{\beta}\times\vec{E}\right)(\beta^0)^2 - \frac{ie}{m^2}(\vec{\beta}\vec{E})\{2(\vec{\beta}\vec{\pi})(\beta^0)^2 - \vec{\beta}\vec{\pi}\} \\ +\frac{e}{m}\left[\vec{\beta}\vec{\pi}, (div(\vec{\alpha}\vec{A} - \frac{iF_{\rho\mu}}{2m}(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}))\times\vec{p})\vec{\Theta}\right] - \frac{e\vec{\Sigma}}{2m^2}(\vec{\pi}\times\frac{\partial\vec{A}}{\partial t}) - \frac{e\vec{\Sigma}}{2m^2}(\vec{\pi}\times grad A_0) \\ +\frac{\vec{\pi}^2(\vec{\alpha}\vec{\pi})}{2m^2} + \frac{1}{2m^2}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \vec{\overline{X}}\alpha\right]\right] + \frac{ie}{4m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, -2(\vec{E}\vec{\beta})(\beta^0)^2 - i\beta^0\vec{S}\vec{H} - i(\vec{\beta}\times\vec{\alpha})\vec{H}\right]\right] \\ +\frac{e}{2m^2}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, (div(\vec{\alpha}\vec{A} - \frac{i}{2m}F_{\rho\mu}(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}))\times\vec{p})\vec{\Theta}\right]\right] - \frac{e}{6m^3}\left[\vec{\beta}\vec{\pi}, \vec{\Sigma}(\vec{\pi}\times\frac{\partial\vec{A}}{\partial t})\right] \\ -\frac{e}{12m^3}\left[\vec{\beta}\vec{\pi}, \vec{\Sigma}(\vec{\pi}\times grad A_0)\right] + \frac{1}{12m^3}\left[\vec{\beta}\vec{\pi}, \vec{\pi}^2(\vec{\alpha}\vec{\pi})\right] + \frac{1}{12m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \vec{\overline{X}}\vec{\alpha}\right]\right]\right] \\ +\frac{1}{12m^2}\left[\vec{\beta}\vec{\pi}, \beta_0\vec{\pi}^2\right] + \frac{e}{12m^3}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, (div(\vec{\alpha}\vec{A} - \frac{iF_{\rho\mu}}{2m}(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}))\times\vec{p})\vec{\Theta}\right]\right]\right]. \end{array} \tag{71}$$

Precisely, we consider only terms until the order of $(\frac{1}{m})^2$. Therefore, terms up to or equal to $(\frac{1}{m})^3$ can be neglected, and, with Equation (A6), we obtain the following Hamiltonian:

$$\begin{aligned}\tilde{H}^{nc} = m\beta_0 - e(A_0 + \left(\left(gradA_0 + div(\vec{\alpha}\vec{A})\right) \times \vec{p}\right)\vec{\Theta}) + \frac{\vec{\pi}^2}{2m}(\vec{\alpha}\frac{\vec{\pi}}{m} - \beta_0) - \frac{\vec{\beta}}{m}\left(i\frac{\partial\vec{\pi}}{\partial t} + e.gradA_0\right) \\ -\frac{ie}{m}\left\{(\vec{E}\vec{\beta})(\beta^0)^2 - \left(div\left((\vec{E}\vec{\beta})(\beta^0)^2\right) \times \vec{p}\right)\vec{\Theta}\right\} + \frac{e}{2m}\left\{\beta^0(\vec{\Sigma}\vec{H}) - \left(div\left(\beta^0\vec{\Sigma}\vec{H}\right) \times \vec{p}\right)\vec{\Theta}\right\} \\ +\frac{e}{2m}\left\{(\vec{\beta} \times \vec{\alpha})\vec{H} - \left(div\left((\vec{\beta} \times \vec{\alpha})\vec{H}\right) \times \vec{p}\right)\vec{\Theta}\right\} + \frac{e}{2m^2}\left[\vec{\beta}\vec{\pi}, (\beta^0\vec{\Sigma} + \vec{\beta} \times \vec{\alpha})\vec{H}\right] + \vec{\overline{X}}\vec{\alpha} \\ -\frac{ie}{m^2}(\vec{\beta}.\vec{E})\left(2(\vec{\beta}\vec{\pi})(\beta^0)^2 - \vec{\beta}\vec{\pi}\right) + \frac{e\vec{\Sigma}}{m^2}\left(\vec{\beta} \times \vec{E}\right)(\beta^0)^2 - \frac{e\vec{\Sigma}}{2m^2}\vec{\pi} \times (\frac{\partial\vec{A}}{\partial t} + gradA_0) + \frac{1}{2m}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\frac{\vec{\pi}}{m}, \vec{\overline{X}}\vec{\alpha}\right]\right] \\ +\frac{e}{m}\left[\vec{\beta}\vec{\pi}, \left(div\left(\vec{\alpha}\vec{A} + (\frac{i}{m}(\vec{E}\vec{\beta})(\beta^0)^2 - \beta^0\frac{\vec{\Sigma}}{2m}\vec{H} - \frac{1}{2m}(\vec{\beta} \times \vec{\alpha})\vec{H})\right) \times \vec{p}\right)\vec{\Theta}\right] + \frac{1}{m}\left[\vec{\beta}\vec{\pi}, \beta_0\frac{\pi^2}{12m} + \vec{\overline{X}}\vec{\alpha}\right] \\ +\frac{e}{2m^2}\left[\vec{\beta}\vec{\pi}, \left[\vec{\beta}\vec{\pi}, \left(div\left(\vec{\alpha}\vec{A} + (\frac{i}{m}(\vec{E}\vec{\beta})(\beta^0)^2 - \frac{1}{2m}\beta^0\vec{\Sigma}\vec{H} - \frac{1}{2m}(\vec{\beta} \times \vec{\alpha})\vec{H})\right) \times \vec{p}\right)\vec{\Theta}\right]\right] + 0(\frac{1}{m})^3.\end{aligned} \quad (72)$$

The above equation will be admitted basically as the non-commutative Schrödinger–Pauli Hamiltonian for a classical particle of zero or unity spin interacting with an EMF. The appearance of terms proportional to the explicit phase (even space) non-commutative terms involved in the Schrödinger–Pauli Hamiltonian because of the fact of the effect of the phase-space non-commutativity on the DKP equation, which means they appeared as terms containing the non-commutativity parameters (η, Θ). Then, after using the classical limit via the unitary Foldy–Wouthuysen transformation, those terms that appeared being responsible for generating new terms and correction terms containing the non-commutativity parameters.

In the above Equations (71) and (72), Σ stands for the spin operator of the bosons (with the eigenvalues of 0 or 1), and H, E are the magnetic and the electric fields, respectively.

We denote and interpret the separate terms in our non-commutative Schrödinger–Pauli Hamiltonian as follows. We can identify each term separately, starting with non-diagonal term $m\beta_0$ as the rest energy (which can be eliminated simply from another FW transformation).

Then, $e(A_0 + ((gradA_0 + div(\vec{\alpha}\vec{A})) \times \vec{p})\vec{\Theta}) = e\overset{nc}{\Phi}$ as the non-commutative electrostatic energy term, followed by $\frac{\vec{\pi}^2}{2m}(\vec{\alpha}\frac{\vec{\pi}}{m} - \beta_0) + \frac{1}{2m}[\vec{\beta}\vec{\pi}, [\vec{\beta}\frac{\vec{\pi}}{m}, \vec{\overline{X}}\vec{\alpha}]]$, which are the non-commutative modified kinetic energy, with its NC correction term $\frac{1}{2m}[\vec{\beta}\vec{\pi}, \beta_0\frac{\vec{\pi}^2}{6m} + 2\vec{\overline{X}}\vec{\alpha}]$. Using the same steps which gave us Equation (16) through Equation (15), and, with $(\eta = 0,\ \vec{\overline{X}}\vec{\alpha} \sim 0)$, the term of kinetic energy is totally diagonalized and can be written as:

$$\frac{\vec{\pi}^2}{2m}(\frac{\pi_0}{m}(2\beta_0^2 - 1). \quad (73)$$

The most important result we care about is the existence of the orbital angular momentum and the spin couplings with the external magnetic field, but they are modified and affected by the non-commutativity influence as it is obvious in the terms

$$\frac{e}{2m}\{\beta^0(\vec{\Sigma}\vec{H}) - (div(\beta^0\vec{\Sigma}\vec{H}) \times \vec{p})\vec{\Theta}\},\ and\ \frac{e}{2m}\{(\vec{\beta} \times \vec{\alpha})\vec{H} - (div((\vec{\beta} \times \vec{\alpha})\vec{H}) \times \vec{p})\vec{\Theta}\}. \quad (74)$$

The following terms represent the diagonal spin-orbit coupling by the electric field, but they are affected and modified also by the NC influence

$$-\frac{e\vec{\Sigma}}{2m^2}\vec{\pi} \times (\frac{\partial\vec{A}}{\partial t} + gradA_0) - \frac{ie}{m}\{(\vec{E}\vec{\beta})(\beta^0)^2 - (div((\vec{E}\vec{\beta})(\beta^0)^2) \times \vec{p})\vec{\Theta}\}. \quad (75)$$

The following terms can be explained by being analogous to the terms of Darwin for particles with spin-1/2 in interaction with an EMF

$$-\frac{ie}{m^2}(\vec{\beta}\vec{E})\{2(\vec{\beta}\vec{\pi})(\beta^0)^2 - \vec{\beta}\vec{\pi}\} + \frac{e\vec{\Sigma}}{m^2}(\vec{\beta} \times \vec{E})(\beta^0)^2. \quad (76)$$

The rest of the terms represent higher-order corrections: one of the FW transformation and of one of the PSNC influence.

Under the condition $\eta = \Theta = 0$, Equation (72) becomes

$$\begin{aligned} \tilde{H}^{nc} &= m\beta_0 - eA_0 + \frac{\vec{\pi}^2}{2m}\left(\frac{1}{m}\vec{\alpha}\vec{\pi} - \beta_0\right) + \frac{e}{2m}\beta^0(\vec{\Sigma}\vec{H}) + \frac{e}{2m}(\vec{\beta}\times\vec{\alpha})\vec{H} \\ &- \frac{\vec{\beta}}{m}\left(i\frac{\partial\vec{\pi}}{\partial t} + e.gradA_0\right) - \frac{ie}{m^2}(\vec{\beta}\vec{E})\left(2(\vec{\beta}\vec{\pi})(\beta^0)^2 - \vec{\beta}\vec{\pi}\right) + \frac{e}{m^2}\vec{\Sigma}\left(\vec{\beta}\times\vec{E}\right)(\beta^0)^2 \\ &- \frac{ie}{m}(\vec{E}\vec{\beta})(\beta^0)^2 - \frac{e}{2m^2}\vec{\Sigma}\vec{\pi}\times\left(\frac{\partial\vec{A}}{\partial t} + gradA_0\right) + \frac{1}{2m^2}\left[\vec{\beta}\vec{\pi}, e(\beta^0\vec{\Sigma} + \vec{\beta}\times\vec{\alpha})\vec{H} + \frac{\beta_0}{6}\vec{\pi}^2\right]. \end{aligned} \tag{77}$$

Equation (77) is similar to the Schrödinger–Pauli Hamiltonian extracted from the Dirac equation in interaction with an external electromagnetic field.

5. Conclusions

In previous sections, we have studied the non-relativistic limit of the DKP equation which provides description of the zero or unity spin particles in the DKP representation using the FW unitary transformation in NCPS, where we introduced the phase-space non-commutativity influence. Then, subsequently applying the FW transformation to to take the system (in interaction with an EMF) to a non-relativistic regime, where we found the Schrödinger–Pauli equation (at least to the order of approximation we have considered), knowing that we investigated the non-relativistic limit of the DKP equation in two cases. In the first case, we considered only the non-commutativity in phase with the absence of the interaction with fields, but, for the second case, we considered the full NCPS in the presence of the external electromagnetic field.

In the first case, the concerned equation was the non-relativistic Schrödinger equation, knowing that the effect of the phase non-commutativity vanished in the DKP Hamiltonian but appeared in the corresponding wave-function. At the second case, the concerned equation was the phase-space non-commutative Schrödinger–Pauli equation, where the effect of the NCPS appeared widely in the obtained equation, and it modified most of the equation terms, and affected especially the spin and the orbital angular momentum terms that characterize the Pauli equation. Taking into account the fact that the non-commutativity influence was injected using both the Bopp-shift transformation through the Heisenberg-like commutation relations and the Gronewold–Moyal product. The use of the FW transformation always enables bringing the system of relativistic quantum mechanics to a non-relativistic regime, and it is confirmed by our present work that the FW transformation is applicable even when the non-commutativity is considered.

In the topic of the DKP theory, historically, the first authors who have studied the non-relativistic limit of the DKP equation were Nikitin and Fushchych in their paper, in which they used a different technique for diagonalizing the Hamiltonian as they pointed out in their paper [38], and others also have investigated the non-relativistic Kemmer equation through a Galilean covariance approach [14], in which they used the Galilean covariance to diagonalize the Kemmer Hamiltonian, without forgetting the authors Moshin and Tomazelli who have investigated the non-relativistic of the DKP equation in a commutative space [39].

We may compare our results with that of the other authors as follows:

Firstly, we compared our results with that of the authors Moshin and Tomazelli [39]. Under the condition ($\eta = \Theta = 0$), and by taking into account only terms until the order of $(\frac{1}{m})^2$ (terms up to or equal to $(\frac{1}{m})^3$ can be neglected), we found almost the same results.

Secondly, we made a comparison with the work of the author Silenko [40]. We found that the author has based research on the equation of the particle spin motion described by the Bargmann–Michel–Telegdi equation. Then, in order to check the wave equations for the spin-1 particles, the author took the Lagrangian that describes the spin effects for the particles of an arbitrary spin which interacted with an EMF. Note that, in the general form of his Hamiltonian, he considered an additional term with the odd and even terms in the Hamiltonian to make the application of the

FW transformation easier, so that Equation (19) is similar to that of ours with some exceptions, as in our transformed Hamiltonian in the case of the commutativity ($\eta = \Theta = 0$) (but with a second FW transformation to eliminate the first term of our transformed Hamiltonian). Our Hamiltonian is more detailed than that of Silenko, and it contains corrections that are related to the order of $(\frac{1}{m})^3$. The author has done two of the FW transformations. On the other hand, we made only one single FW transformation (it was enough for us to use a single transformation to find what was interesting).

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Appendix A. The Simplification of Vector Squared of Non-Commutative Momentum

Starting with

$$(\overrightarrow{P}^{nc})^2 = (p_i + \frac{1}{2}\eta_{ij}x_j)^2 = p_i^2 + \frac{1}{2}\eta_{ij}p_i x_j + \frac{1}{2}\eta_{ij}x_j p_i + \frac{1}{4}\eta_{ij}\eta_{ik}x_j x_k, \tag{A1}$$

considering only the 1st order of the phase non-commutativity $0(\eta^2)$.

With $\eta_{ij} = -\eta_{ji} = \eta\epsilon_{ij}$, $\eta_k = \frac{1}{2}\epsilon_{kij}\eta_{ij} \longrightarrow \eta_{ij} = 2\eta_k\epsilon_{kij}$, knowing that $(\epsilon_{kij})^2 = 1$, and $(U \times V)_k = \epsilon_{kij}U_i V_j$, $\overrightarrow{L} = \overrightarrow{x} \times \overrightarrow{P}$, we simplify as follows:

$$\frac{1}{2}\eta_{ij}p_i x_j = \eta_k\epsilon_{kij}p_i x_j = (\overrightarrow{P} \times \overrightarrow{x})\overrightarrow{\eta} = -\overrightarrow{L}\overrightarrow{\eta}, \tag{A2}$$

$$\frac{1}{2}\eta_{ij}x_j p_i = -\eta_k\epsilon_{kji}x_j p_i = -(\overrightarrow{x} \times \overrightarrow{P})\overrightarrow{\eta} = -\overrightarrow{L}\overrightarrow{\eta}. \tag{A3}$$

Combining these results, we have

$$(\overrightarrow{P}^{nc})^2 = \overrightarrow{P}^2 - 2\overrightarrow{L}\overrightarrow{\eta} + 0(\eta^2). \tag{A4}$$

Appendix B. The Useful Commutation and Tensor Relations, Matrix Product

Using the DKP algebra Equations (8) and (9), we have

$$F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right) = F_{00}\beta^0\beta^0\beta^0 + F_{ij}\beta^i\beta^0\beta^j + F_{\rho\mu}\beta^\mu g^{0\rho}, \tag{A5}$$

$$F_{\rho\mu}\left(\beta^\mu\beta^0\beta^\rho + \beta^\mu g^{0\rho}\right) = -2(\overrightarrow{E}\overrightarrow{\beta})(\beta^0)^2 - i\beta^0\overrightarrow{\Sigma}\overrightarrow{H} - i(\overrightarrow{\beta} \times \overrightarrow{\alpha})\overrightarrow{H}, \tag{A6}$$

with $F_{\rho\mu}\beta^\mu g^{0\rho} = -\overrightarrow{E}.\overrightarrow{\beta}$, $\quad \Sigma_{ij} = \beta^i\beta^0\beta^j - \beta^j\beta^0\beta^i$, $i,j = 1,2,3$.

The useful commutation and vector relations

$$\left[\overrightarrow{\beta}\overrightarrow{\pi}, \overrightarrow{\alpha}\overrightarrow{\pi}\right] = -\beta_0\overrightarrow{\pi}^2, \tag{A7}$$

$$\left[\overrightarrow{\beta}\overrightarrow{\pi}, \beta_0\right] = -\overrightarrow{\alpha}\overrightarrow{\pi}, \tag{A8}$$

$$\left[\overrightarrow{\beta}\overrightarrow{\pi}, A_0(\overrightarrow{x})\right] = -i\overrightarrow{\beta}\overrightarrow{\nabla}A_0 = -i\overrightarrow{\beta}divA_0, \tag{A9}$$

$$\left[\overrightarrow{\beta}\overrightarrow{\pi}, \frac{\partial(\overrightarrow{\beta}\overrightarrow{\pi})}{\partial t}\right] = ie\overrightarrow{\Sigma}(\overrightarrow{\pi} \times \frac{\partial\overrightarrow{A}}{\partial t}), \tag{A10}$$

$$\left[\overrightarrow{\beta}\overrightarrow{\pi}, ((\overrightarrow{\nabla}A_0) \times \overrightarrow{p})\overrightarrow{\Theta}\right] = \left[\overrightarrow{\beta}\overrightarrow{\pi}, \overline{\kappa}\right] = 0, \ with \ ((\overrightarrow{\nabla}A_0) \times \overrightarrow{p})\overrightarrow{\Theta} = \overline{\kappa}\epsilon\mathbb{R}, \tag{A11}$$

$$\left[\vec{\beta}\vec{\pi},(\vec{\beta}\vec{E})(\beta^0)^2\right]=i\vec{\Sigma}\left(\vec{\beta}\times\vec{E}\right)(\beta^0)^2+(\vec{\beta}\vec{E})\{2(\vec{\beta}\vec{\pi})(\beta^0)^2-\vec{\beta}\vec{\pi}\}, \quad \text{(A12)}$$

$$\left[\vec{\beta}\vec{\pi},\beta_0\vec{\pi}^2\right]=-\vec{\pi}^2(\vec{\alpha}\vec{\pi}), \quad \text{(A13)}$$

$$\left[\vec{\beta}\vec{\pi},\vec{\beta}\vec{\pi}\vec{\alpha}\vec{\pi}\right]=-\beta_0\vec{\pi}^2, \quad \text{(A14)}$$

$$\left[\vec{\beta}\vec{\pi},\vec{\beta}.gradA_0\right]=\vec{\Sigma}(\vec{\pi}\times gradA_0). \quad \text{(A15)}$$

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