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Best Proximity Results on Dualistic Partial Metric Spaces

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Abstract: We introduce the generalized almost (φ, θ)-contractions by means of comparison type functions and another kind of mappings endowed with specific properties in the setting of dualistic partial metric spaces. Also, generalized almost θ-Geraghty contractions in the setting of dualistic partial metric spaces are defined by the use of a function of Geraghty type and another adequate auxiliary function. For these classes of generalized contractions, we have stated and proved the existence and uniqueness of a best proximity point.

Keywords: best proximity point; generalized contraction; dualistic partial metric spaces

MSC: 47H10; 54H25

1. Introduction

The problem of whether the equation $Tx = x$ has solutions or not, where T is a self mapping on X, has been studied thoroughly, since the famous principle of Banach was stated. The issue has been developed even further: find the point x which realizes the smallest distance from a subset A of a set X, with a metric d, to another subset B, by means of a nonself mapping T, that is $d(A, B) = d(x, Tx)$. Such a point is called point of best proximity. The importance of this problem increases if we think that, if the sets A and B are mutually disjoint, the equation $x = Tx$ does not have any solution. Many researchers have been attracted by the study of best proximity points. In [1], some best proximity results are stated, also an algorithm to determine such a point is given. Best proximity properties for nonself nonexpansive mappings are proved in [2]. In [3], some existence and uniqueness results with regard to appropriate nonself mappings, by means of some control functions, are stated. In [4], a convergence theorem was used in order to prove an existence property with regard to an adequate map, without using the famous lemma of Zorn. In [5], conditions which ensure the existence and uniqueness of a best proximity point are stated. In [6], a suitable property is used to state best proximity theorems for generalized types of contractions.

The use of generalized metric spaces moved further the theory of fixed point and that of best proximity point. In this respect, in [7–9] results are stated in the framework of different types of b-metric spaces. The setting of Menger probabilistic metric spaces has been used in [10] in order to obtain some best proximity results. Generalized weak contractions in partial metric spaces are developed in [11]. In [12], the framework of dualistic metric spaces is used in order to prove some properties on a pair of adequate mappings. In [13], dual contractions of rational type are studied in the context of dualistic partial metric spaces. Important aspects of fixed point results can be found, for example, in [14] or [15].

The aim of this paper is to introduce generalized contraction mappings in the framework of dualistic metric spaces and to state and prove some best proximity results for such kind of contractions. We have in view generalized contractivity conditions defined by means of a comparison type function to which we add a term which involves a continuous function endowed with suitable properties. Also, we refer to generalized contractions formed by a Geraghty type term and a second one involving adequate continuous mappings.

2. Preliminaries

The framework we use, dualistic partial metrics, is due to O'Neill. One of the main conditions fulfilled by such spaces is the symmetry, a property which is of crucial importance in proving our statements.

Definition 1 ([16])**.** *Let $\mathcal{M}$ be a nonempty set. A function $\mathcal{D}\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}$ is called a dualistic partial metric if*

(D_1) $\jmath=\lambda$ *if and only if* $\mathcal{D}(\jmath,\jmath)=\mathcal{D}(\lambda,\lambda)=\mathcal{D}(\jmath,\lambda)$;

(D_2) $\mathcal{D}(\lambda,\lambda)\le\mathcal{D}(\lambda,\jmath)$;

(D_3) $\mathcal{D}(\jmath,\lambda)=\mathcal{D}(\lambda,\jmath)$;

(D_4) $\mathcal{D}(\jmath,\kappa)\le\mathcal{D}(\jmath,\lambda)+\mathcal{D}(\lambda,\kappa)-\mathcal{D}(\lambda,\lambda)$,
for each $\jmath,\lambda,\kappa\in\mathcal{M}$.

Along our work, $\mathcal{M}\neq\emptyset$, $(\mathcal{M},\mathcal{D})$ denotes a dualistic partial metric space, and $\mathcal{A},\mathcal{B}\subseteq\mathcal{M}$, $\mathcal{A},\mathcal{B}\neq\emptyset$, unless specifically stated otherwise.

In this case,

$$d_{\mathcal{D}}\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}^+,\quad d_{\mathcal{D}}(\lambda,\kappa)=\mathcal{D}(\lambda,\kappa)-\mathcal{D}(\lambda,\lambda), \tag{1}$$

is a quasi metric on M. With respect to the induced topologies, there is the relation $\tau(\mathcal{D})=\tau(d_{\mathcal{D}})$. Moreover, $d^s_{\mathcal{D}}(\jmath,\lambda)=\max\{d_{\mathcal{D}}(\jmath,\lambda),d_{\mathcal{D}}(\lambda,\jmath)\}$ is a metric on $\mathcal{M}$.

To any partial metric (please, see [17]), one can associate a dualistic partial metric space, as the next example shows.

Example 1. *Let $\mathcal{P}$ be a partial metric defined on $\mathcal{M}$.*

$$\mathcal{D}\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R},\quad \mathcal{D}(\jmath,\kappa)=\mathcal{P}(\jmath,\kappa)-\mathcal{P}(\jmath,\jmath)-\mathcal{P}(\kappa,\kappa),\ \text{for each}\ \jmath,\kappa\in M,$$

is a dualistic partial metric.

Remark 1. *$D(\jmath,\kappa)$ might not be positive. Furthermore, the equality $\mathcal{D}(\jmath,\kappa)=0$ does not necessarily compel $\jmath=\kappa$. In this frame, $\mathcal{D}(\jmath,\jmath)$ is considered as the weight of $\jmath$, and indicates the quantity of information $\jmath$ has. However, $\mathcal{D}\Big|_{\mathbb{R}^+_0\times\mathbb{R}^+_0}$ is a partial metric.*

Remark 2. *Each partial metric is obviously a dualistic partial one, but the reverse does not hold true. Define*

$$\mathcal{D}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R},\quad \mathcal{D}(\lambda,\ell)=\max\{\lambda,\ell\},\ \text{for all}\ \lambda,\ \ell\in\mathbb{R}.$$

Obviously $\mathcal{D}$ satisfies the dualistic partial metric hypotheses. But $\mathcal{D}$ is not a partial metric; it is enough to see that $D(-7,-2)=-2\notin\mathbb{R}^+_0$.

In accordance with [16], $\mathcal{D}$ generates a T_0 topology on $\mathcal{M}$, denoted by $\tau(\mathcal{D})$, in which the open balls are $\{B_{\mathcal{D}}(\lambda,\epsilon) : \lambda \in \mathcal{M}, \epsilon > 0\}$, where $B_{\mathcal{D}}(\lambda,\epsilon) = \{\kappa \in \mathcal{M} : \mathcal{D}(\lambda,\kappa) < \epsilon + \mathcal{D}(\lambda,\lambda)\}$.

Now we can introduce the notions of convergence, Cauchy sequences, and completness in this setting of dualistic partial metric spaces.

Definition 2 ([18])**.** *(1) A sequence $\{\jmath_n\}$ in $(\mathcal{M},\mathcal{D})$ is a Cauchy sequence if $\lim_{n,m\to\infty}\mathcal{D}(\jmath_n,\jmath_m)$ exists and it is finite.*

(2) A sequence $\{\jmath_n\} \subset \mathcal{M}$ converges to a point $\jmath^*$ if and only if $\mathcal{D}(\jmath^*,\jmath^*) = \lim_{n\to\infty}\mathcal{D}(\jmath^*,\jmath_n)$.

(3) $(\mathcal{M},\mathcal{D})$ is complete if each Cauchy sequence $\{\jmath_n\}$ in M converges, in $\tau(\mathcal{D})$, to $\jmath^* \in \mathcal{M}$ such that $\mathcal{D}(\jmath^*,\jmath^*) = \lim_{n,m\to\infty}\mathcal{D}(\jmath_n,\jmath_m)$.

The next lemma establishes some connections between the original dualistic partial metric and the induced one.

Lemma 1 ([18])**.** *The following statements hold:*

(1) Each Cauchy sequence in the induced metric space $(\mathcal{M},d^s_{\mathcal{D}})$ is also a Cauchy sequence in the dualistic partial metric space $(\mathcal{M},\mathcal{D})$.

(2) If the completness of $(\mathcal{M},\mathcal{D})$ is fulfilled then the completness of $(\mathcal{M},d^s_{\mathcal{D}})$ is satisfied also. Moreover, the converse holds true.

(3) A sequence $\{\jmath_n\} \subseteq \mathcal{M}$ converges to $\jmath^* \in \mathcal{M}$ in $\tau(d^s_{\mathcal{D}})$ if and only if

$$\lim_{n,m\to\infty}\mathcal{D}(\jmath_n,\jmath_m) = \lim_{n\to\infty}\mathcal{D}(\jmath^*,\jmath_n) = D(\jmath^*,\jmath^*).$$

To introduce our new results, we need several specific concepts related to a nonself mapping T in the setting of dualistic partial metric spaces.

Two subsets of $\mathcal{A}$ and $\mathcal{B}$ respectively will be of paramount importance, namely

$$\mathcal{A}_0 = \{a \in \mathcal{A} : \text{ there exists } b \in \mathcal{B}, \mathcal{D}(a,b) = \mathcal{D}(\mathcal{A},\mathcal{B})\},$$

$$\mathcal{B}_0 = \{b \in \mathcal{B} : \text{ there exists } a \in \mathcal{A}, \mathcal{D}(a,b) = \mathcal{D}(\mathcal{A},\mathcal{B})\},$$

where

$$\mathcal{D}(\mathcal{A},\mathcal{B}) := \inf\{\mathcal{D}(a,b) : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Definition 3. *Let T be a nonself mapping from A to B. An element $\ell \in \mathcal{A}$ is called a best proximity point of T if the following equality is fulfilled*

$$\mathcal{D}(\ell, T\ell) = \mathcal{D}(\mathcal{A},\mathcal{B}).$$

A tool of significant importance in the development of the best proximity point results is the notion of a pair endowed with the (weak) (P)-property.

Following the definition of [19], the next notion can be introduced in this framework.

Definition 4. *Presume that $\mathcal{A}_0 \neq \emptyset$. If for each $\jmath_1$, $\jmath_2 \in \mathcal{A}$, and κ_1, $\kappa_2 \in \mathcal{B}$, the following implication holds*

$$\begin{pmatrix} \mathcal{D}(\jmath_1,\kappa_1) = \mathcal{D}(\mathcal{A},\mathcal{B}) \\ \mathcal{D}(\jmath_2,\kappa_2) = \mathcal{D}(\mathcal{A},\mathcal{B}) \end{pmatrix} \text{implies} \quad |\mathcal{D}(\jmath_1,\jmath_2)| \le |\mathcal{D}(\kappa_1,\kappa_2)|,$$

the pair $(\mathcal{A}, \mathcal{B})$ *is endowed with the weak* (P)*-property.*

By replacing the inequality in the right-hand side of the implication by an equality, a less general notion is obtained, namely that of a pair endowed with the (P)-property. In the next lines, we present an example of a pair endowed with the weak (P)-property.

Example 2. *Consider* $\mathcal{A} = [-1, 0]$, *and* $\mathcal{B} = [-3, -2]$ *subsets of* $\mathbb{R}$, *and* $\mathcal{D}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mathcal{D}(\jmath, \kappa) = \max\{\jmath, \kappa\}$. *It can be noticed that* $\mathcal{D}(\mathcal{A}, \mathcal{B}) = -1$. *Suppose* $\mathcal{D}(\jmath_1, \kappa_1) = -1$, *and* $\mathcal{D}(\jmath_2, \kappa_2) = -1$. *Then* $\jmath_1 = -1$, *and* $\jmath_2 = -1$. *Hence,* $\mathcal{D}(\jmath_1, \jmath_2) = -1$, *while* $\mathcal{D}(\kappa_1, \kappa_2) = \max\{\kappa_1, \kappa_2\}$. *Since* κ_1, *and* κ_2 *belong to* $[-3, -2]$, *the inequality* $|\mathcal{D}(\jmath_1, \jmath_2)| \leq |\mathcal{D}(\kappa_1, \kappa_2)|$ *is satisfied. Therefore, the pair* $(\mathcal{A}, \mathcal{B})$ *is endowed with the weak* (P)*-property.*

3. Almost (φ, θ)-Contractions

In [6], there has been stated a result on best proximity with respect to an almost contraction if some additional hypotheses are satisfied. In order to introduce their theorem, we point out the next definition.

Definition 5 ([20])**.** *If a map* $\varphi\colon [0, +\infty) \to [0, +\infty)$ *fulfills the following two conditions*

1. φ *is increasingly monotone,*
2. *For any* $t \in [0, +\infty)$, $\sum_{n=0}^{+\infty} \varphi^n(t) < \infty$,

then φ *is said to be a c-comparison.*

By replacing the second condition by imposing that $\lim_{n \to +\infty} \varphi^n(t) = 0$ for all $t \geq 0$, the concept of comparison functions may be retrieved. Interesting properties of different types of comparison functions can be found, for example, in [21].

The set of all c-comparison functions is denoted by Φ.

Let $\mathcal{T}$ be the set of all continuous functions $\theta\colon [0, +\infty)^4 \to [0, +\infty)$ endowed with the properties

$$\begin{aligned} &\theta(0, \jmath, \lambda, \kappa) = 0, \text{ for each } \jmath, \lambda, \kappa \in [0, +\infty), \\ &\theta(\jmath, \lambda, 0, \kappa) = 0, \text{ for each } \jmath, \lambda, \kappa \in [0, +\infty), \\ &\theta(\jmath, \jmath, \jmath, \jmath) = 0, \text{ for each } \jmath \in [0, +\infty). \end{aligned}$$

Example 3. *Define the function*

$$\theta\colon [0, +\infty)^4 \to [0, +\infty), \quad \theta(\jmath, \lambda, \kappa, \ell) = \tau\big(\jmath\lambda\kappa\ell - \inf{}^4\{\jmath, \lambda, \kappa, \ell\}\big), \quad \tau > 0.$$

Then $\theta \in \mathcal{T}$.

By means of c-comparison functions and mappings which fulfill some of the properties of the set $\mathcal{T}$ in [6] there has been defined a generalized contraction. There has been proved that such a kind of a generalized contraction possesses a best proximity point, provided that it satisfies also an inclusion condition type, and the (P) property is satisfied by a pair of nonempty closed sets adequately chosen.

We are ready now to move on to the first outcome of our work, namely develop best proximity theorems in a wider setting, that of dualistic partial metric spaces. In the following, $(\mathcal{M}, \mathcal{D})$ is a dualistic partial metric space, $\mathcal{A}$, and $\mathcal{B}$ are nonempty subsets of $\mathcal{M}$, and $\mathcal{A}_0 \neq \emptyset$. Also denote by $\tilde{\mathcal{D}}(\jmath, \lambda) = \mathcal{D}(\jmath, \lambda) - \mathcal{D}(\mathcal{A}, \mathcal{B})$.

The generalized almost (φ, θ)-contraction in this framework is introduced as follows

Definition 6. *Let $\varphi \in \Phi$, and $\theta \in \mathcal{T}$. Consider $T\colon \mathcal{A} \to \mathcal{B}$ a nonselfmapping so that if for each $\jmath, \kappa \in \mathcal{A}$,*

$$|\mathcal{D}(T\jmath, T\kappa)| \quad \leq \quad \varphi(|\mathcal{D}(\jmath,\kappa)|) + \theta\big(\tilde{\mathcal{D}}(\kappa, T\jmath), \tilde{\mathcal{D}}(\jmath, T\kappa), \tilde{\mathcal{D}}(\jmath, T\jmath), \tilde{\mathcal{D}}(\kappa, T\kappa)\big).$$

Such a nonself mapping is a generalized almost (φ,θ)-contraction.

We provide now an example of such a generalized contraction.

Example 4. *Let $\mathbb{R}$ endowed with the dualistic partial metric $\mathcal{D}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $D(\jmath,\kappa) = \max\{\jmath,\kappa\}$. Consider the mapping $T\colon [-\frac{1}{2}, \frac{1}{2}] \to [-\frac{1}{4}, \frac{1}{4}]$, $Tx = 2x^3$. Furthermore, let us take*

$$\varphi\colon [0,\infty) \to [0,\infty), \qquad \varphi(t) = \begin{cases} t^2, & \text{if } t \in \left[0, \dfrac{2}{3}\right]; \\ t - \dfrac{2}{9}, & \text{if } t > \dfrac{2}{3}, \end{cases}$$

and $\theta \in \mathcal{T}$. It can be easily checked that T is an almost (φ,θ)-contraction.

With regard to such kind of generalized contractions, the next result holds true.

Theorem 1. *Consider a generalized almost (φ,θ)-contraction $T\colon \mathcal{A} \to \mathcal{B}$. Additionally, presume that:*

(1) $T\mathcal{A}_0 \subseteq \mathcal{B}_0$;
(2) *$(\mathcal{A},\mathcal{B})$ is a pair endowed with the weak (P)-property.*

Then, T possesses a unique best proximity point.

Proof. Consider $\jmath_0 \in \mathcal{A}_0$. Since $T\mathcal{A}_0 \subseteq \mathcal{B}_0$, then $T\jmath_0 \in \mathcal{B}_0$, and there is $\jmath_1 \in \mathcal{A}_0$ such that $\mathcal{D}(\jmath_1, T\jmath_0) = \mathcal{D}(\mathcal{A},\mathcal{B})$. By continuing this procedure, we obtain a sequence $\{\jmath_n\} \subseteq \mathcal{A}_0$, which satisfies the equality

$$\mathcal{D}(\jmath_{n+1}, T\jmath_n) = \mathcal{D}(\mathcal{A},\mathcal{B}), \quad n \in \mathbb{N} \cup \{0\}.$$

$(\mathcal{A},\mathcal{B})$ is a pair which possesses the weak (P)-property, therefore, for $n \in \mathbb{N}$, the inequality $|\mathcal{D}(\jmath_n, \jmath_{n+1})| \leq |\mathcal{D}(T\jmath_{n-1}, T\jmath_n)|$ holds true.

Using the generalized contractivity of T, we obtain

$$\begin{aligned} & |\mathcal{D}(\jmath_n, \jmath_{n+1})| \leq |\mathcal{D}(T\jmath_{n-1}, T\jmath_n)| \\ \leq \;& \varphi(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|) + \theta(\tilde{\mathcal{D}}(\jmath_n, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_n), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_n, T\jmath_n)) \\ = \;& \varphi(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|) + \theta(0, \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_n), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_n, T\jmath_n)) \\ = \;& \varphi(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|), \quad n \in \mathbb{N}. \end{aligned}$$

Hence, it has been proved that

$$|\mathcal{D}(\jmath_n, \jmath_{n+1})| \quad \leq \quad \varphi(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|), \; n \in \mathbb{N}.$$

Using the monotone of φ, we get

$$|\mathcal{D}(\jmath_n, \jmath_{n+1})| \quad \leq \quad \varphi^n(|\mathcal{D}(\jmath_0, \jmath_1)|), \quad n \in \mathbb{N}. \tag{2}$$

Considering $n \to \infty$ and having in mind the properties of the c-comparison function, it follows that $|\mathcal{D}(\jmath_n, \jmath_{n+1})| \to 0$, so $\mathcal{D}(\jmath_n, \jmath_{n+1}) \to 0$.

Let us focus now on the self distance. By using the generalized contraction property of T, it follows

$$\begin{aligned} & |\mathcal{D}(\jmath_n, \jmath_n)| \le |\mathcal{D}(T\jmath_{n-1}, T\jmath_{n-1})| \\ \le\; & \varphi(|\mathcal{D}(\jmath_{n-1}, \jmath_{n-1})|) + \theta(\tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1})) \\ =\; & \varphi(|\mathcal{D}(\jmath_{n-1}, \jmath_{n-1})|), \quad n \in \mathbb{N}. \end{aligned}$$

Obviously, it can be easily proved that

$$|\mathcal{D}(\jmath_n, \jmath_n)| \le \varphi^n(|\mathcal{D}(\jmath_0, \jmath_0)|),\ n \in \mathbb{N}, \tag{3}$$

which compels also that $\mathcal{D}(\jmath_n, \jmath_n) \to 0$.

Taking into account the inequalities

$$\begin{aligned} \mathcal{D}(A, B) &\le \mathcal{D}(\jmath_n, T\jmath_n) \le \mathcal{D}(\jmath_n, \jmath_{n+1}) + \mathcal{D}(\jmath_{n+1}, T\jmath_n) - \mathcal{D}(\jmath_{n+1}, \jmath_{n+1}) \\ &= \mathcal{D}(\jmath_n, \jmath_{n+1}) + \mathcal{D}(\mathcal{A}, \mathcal{B}) - \mathcal{D}(\jmath_{n+1}, \jmath_{n+1}), \end{aligned}$$

and letting $n \to +\infty$, we obtain

$$\lim_{n \to +\infty} \mathcal{D}(\jmath_n, T\jmath_n) = \mathcal{D}(\mathcal{A}, \mathcal{B}).$$

We continue by proving that $\{\jmath_n\}$ is a Cauchy sequence with respect to D. In order to do that, it is enough to show that $\{\jmath_n\}$ is a Cauchy sequence with respect to $d^s_{\mathcal{D}}$, as Lemma 1 states.

Relation (1) from the induced quasi metric d_D implies the following relation

$$d_{\mathcal{D}}(\jmath_n, \jmath_{n+1}) = \mathcal{D}(\jmath_n, \jmath_{n+1}) - \mathcal{D}(\jmath_n, \jmath_n),\ n \in \mathbb{N} \cup \{0\}.$$

Furthermore, taking advantage of inequalities (2), and (3), we obtain

$$\begin{aligned} d_{\mathcal{D}}(\jmath_n, \jmath_{n+1}) &\le |\mathcal{D}(\jmath_n, \jmath_{n+1})| + |\mathcal{D}(\jmath_n, \jmath_n)| \\ &\le \varphi^n(|\mathcal{D}(\jmath_0, \jmath_1)|) + \varphi^n(|\mathcal{D}(\jmath_0, \jmath_0)|). \end{aligned}$$

Re-writing this relation, it follows, for n, $\ell \in \mathbb{N}$,

$$d_{\mathcal{D}}(\jmath_{n+\ell}, \jmath_{n+\ell+1}) \le \varphi^{n+\ell}(|\mathcal{D}(\jmath_0, \jmath_1)|) + \varphi^{n+\ell}(|\mathcal{D}(\jmath_0, \jmath_0)|).$$

Since $d_{\mathcal{D}}$ is a quasi metric, for n, $p \in \mathbb{N}$, by the use of the previous relation, it may be noticed that

$$\begin{aligned} d_{\mathcal{D}}(\jmath_n, \jmath_{n+p}) &\le \sum_{\ell=0}^{p-1} d_{\mathcal{D}}(\jmath_{n+\ell}, \jmath_{n+\ell+1}) \\ &\le \sum_{\ell=0}^{p-1} \Big(\varphi^{n+\ell}(|\mathcal{D}(\jmath_0, \jmath_1)|) + \varphi^{n+\ell}(|\mathcal{D}(\jmath_0, \jmath_0)|) \Big). \end{aligned}$$

But φ is a c-comparison function, so $d_{\mathcal{D}}(\jmath_n, \jmath_{n+p}) \to 0$. Similarly, $d_{\mathcal{D}}(\jmath_{n+p}, \jmath_n) \to 0$. Therefore, we are able to conclude that $\{\jmath_n\}$ is a Cauchy sequence with respect to $d^s_{\mathcal{D}}$, and hence also with regard to $\mathcal{D}$. Since A is closed, we obtain that $\{\jmath_n\}$ is convergent to an element $\jmath \in A$. Therefore,

$$\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath) = \mathcal{D}(\jmath, \jmath) = \lim_{n,m\to\infty} \mathcal{D}(\jmath_n, \jmath_m).$$

Since $\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath_n) = 0$, we get also that $\mathcal{D}(\jmath, \jmath) = 0$.

Taking advantage of the fact that $\mathcal{D}(\jmath_n, T\jmath_n) \to \mathcal{D}(\mathcal{A}, B)$, by letting $n \to +\infty$ in the inequality

$$|\mathcal{D}(T\jmath, T\jmath_n)| \leq \varphi(|\mathcal{D}(\jmath, \jmath_n)|) + \theta(\tilde{\mathcal{D}}(\jmath_n, T\jmath), \tilde{\mathcal{D}}(\jmath, T\jmath_n), \tilde{\mathcal{D}}(\jmath_n, T\jmath_n), \tilde{\mathcal{D}}(\jmath, T\jmath)),$$

it gets $\lim_{n\to+\infty} |\mathcal{D}(T\jmath, T\jmath_n)| = 0$, so $\lim_{n\to+\infty} \mathcal{D}(T\jmath, T\jmath_n) = 0$.

Also, keeping in mind that $\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath_n) = 0$, similarly it can be proved that $\lim_{n\to\infty} \mathcal{D}(T\jmath_n, T\jmath_n) = 0$.

Using the triangle inequality, it follows

$$\mathcal{D}(\jmath, T\jmath) \leq \mathcal{D}(\jmath, \jmath_n) + \mathcal{D}(\jmath_n, T\jmath_n) + \mathcal{D}(T\jmath, T\jmath_n) - \mathcal{D}(\jmath_n, \jmath_n) - \mathcal{D}(T\jmath_n, T\jmath_n). \tag{4}$$

Taking $n \to +\infty$ in relation (4), we obtain that $\mathcal{D}(\jmath, T\jmath) \leq \mathcal{D}(\mathcal{A}, \mathcal{B})$. The converse inequality being obvious, $\mathcal{D}(\jmath, T\jmath) = \mathcal{D}(\mathcal{A}, \mathcal{B})$.

We can move on to the uniqueness part of the theorem. Suppose T has two different best proximity points, $\jmath$, and κ. The following relations hold

$$\begin{aligned} |\mathcal{D}(\jmath, \kappa)| &\leq |\mathcal{D}(T\jmath, T\kappa)| \\ &\leq \varphi(|\mathcal{D}(\jmath, \kappa)|) + \theta(\tilde{\mathcal{D}}(\kappa, T\jmath), \tilde{\mathcal{D}}(\jmath, T\kappa), \tilde{\mathcal{D}}(\jmath, T\jmath), \tilde{\mathcal{D}}(\kappa, T\kappa)) \\ &= \varphi(|\mathcal{D}(\jmath, \kappa)|) + \theta(\tilde{\mathcal{D}}(\kappa, T\jmath), \tilde{\mathcal{D}}(\jmath, T\kappa), 0, 0) \\ &= \varphi(|\mathcal{D}(\jmath, \kappa)|), \end{aligned}$$

which implies $\mathcal{D}(\jmath, \kappa) = 0$.

On the other hand,

$$\begin{aligned} |\mathcal{D}(\jmath, \jmath)| &\leq |\mathcal{D}(T\jmath, T\jmath)| \\ &\leq \varphi(|\mathcal{D}(\jmath, \jmath)|) + \theta(\tilde{\mathcal{D}}(\jmath, T\jmath), \tilde{\mathcal{D}}(\jmath, T\jmath), \tilde{\mathcal{D}}(\jmath, T\jmath), \tilde{\mathcal{D}}(\jmath, T\jmath)) \\ &= \varphi(|\mathcal{D}(\jmath, \jmath)|), \end{aligned}$$

hence $\mathcal{D}(\jmath, \jmath) = 0$. Similarly, we obtain $\mathcal{D}(\kappa, \kappa) = 0$. As $D(\jmath, \jmath) = \mathcal{D}(\kappa, \kappa) = \mathcal{D}(\jmath, \kappa)$, we get that $\jmath = \kappa$, and the conclusion follows. □

By taking $\theta \equiv 0$, and $\varphi \in \Phi$ as particular choices, the next result is reached.

Corollary 1. *$T\colon \mathcal{A} \to \mathcal{B}$ is a nonself mapping, and $\varphi \in \Phi$ so that*

$$|\mathcal{D}(T\lambda, T\kappa)| \leq \varphi(|\mathcal{D}(\lambda, \kappa)|), \ \ \lambda, \kappa \in \mathcal{A}.$$

In addition, presume that

(1) *$T\mathcal{A}_0 \subseteq \mathcal{B}_0$;*
(2) *the pair $(\mathcal{A}, \mathcal{B})$ has the weak (P)-property.*

Then, T possesses a unique best proximity point.

Moreover, by having in view the case $\mathcal{A} = \mathcal{B} = \mathcal{M}$, we may obtain some fixed point results in the framework of dualistic partial metric spaces.

Corollary 2. *Consider $T\colon \mathcal{M} \to \mathcal{M}$ a generalized almost (φ, θ)-contraction. Then, there exists a unique fixed point of T, $\jmath \in \mathcal{M}$.*

By taking now the case of $\theta \equiv 0$, we get the following corollary which refers to fixed points.

Corollary 3. *Suppose $T\colon \mathcal{M} \to \mathcal{M}$, endowed with the property that there exists a c-comparison function φ, so that*

$$|\mathcal{D}(T\lambda, T\kappa)| \leq \varphi(|\mathcal{D}(\lambda, \kappa)|), \;\; \lambda, \kappa \in \mathcal{M}.$$

Then T possesses a unique fixed point.

Example 5. *Let $\mathcal{M} = \left[-\frac{1}{2}, \frac{1}{2}\right]$ with the dualistic partial metric $D(\jmath, \kappa) = \max\{\jmath, \kappa\}$. Consider the mapping $T\colon \mathcal{M} \to \mathcal{M}$, $T\jmath = \jmath^3$. By taking $\varphi\colon [0, \infty) \to [0, \infty)$,*

$$\varphi(t) = \begin{cases} t^2, & \text{if } t < \dfrac{1}{2}; \\ t - \dfrac{1}{4}, & \text{if } t \geq \dfrac{1}{2}, \end{cases}$$

the conditions from Corollary 3 are fulfilled. Therefore, T has a unique fixed point.

4. Generalized Geraghty Type Contractions

This section is dedicated to some generalized contractions in the sense of Geraghty, defined by means of the set $\mathcal{T}$. The result of Geraghty [22] has attracted many researchers. In [23], generalized Geraghty type contractions are studied in various generalized metric spaces. In [24], common fixed point results are provided, in the setting of partial metric spaces, for improved α-Geraghty contractions; these generalized contractions refer to mappings which form an α-admissible triangular pair, and also have as argument the maximum of four terms involving partial distances between the considered points and their images through the two contractive maps. In [19] a best proximity result is in view, for Geraghty-contractions.

Here we are going to introduce a generalized type of Geraghty contractions, by using functions from the set $\mathcal{T}$. In the following, we focus on existence and uniqueness of a best proximity point for such kind of nonself mapps.

Let $\mathcal{G}$ be the set containing all functions $\beta\colon [0, \infty) \to [0, 1)$, satisfying an implication type condition

$$\Big(\beta(t_n) \to 1\Big) \quad \text{implies} \quad \Big(t_n \to 0\Big).$$

We provide now some examples of such functions from the set $\mathcal{G}$.

Example 6. *The function $\beta\colon [0, \infty) \to [0, 1)$,*

$$\beta\colon [0, \infty) \to [0, 1),\; \beta(t) = \begin{cases} \frac{\sin t}{t}, & \text{if } t \neq 0; \\ 0, & \text{if } t = 0, \end{cases}$$

is an element of $\mathcal{G}$.

Throughout the section, $(\mathcal{M}, \mathcal{D})$ is a dualistic metric space, $\beta \in \mathcal{G}$, $\theta \in \mathcal{T}$, and $\mathcal{A}, \mathcal{B}$ nonempty subsets of $\mathcal{M}$, $\mathcal{A}_0 \neq \emptyset$.

To generalize further the Geraghty type contractions, we introduce the following definition.

Definition 7. *A nonself mapping mapping $T\colon \mathcal{A} \to \mathcal{B}$ having the property that for each $\jmath, \kappa \in \mathcal{A}$,*

$$|\mathcal{D}(T\jmath, T\kappa)| \leq \beta(|\mathcal{D}(\jmath,\kappa)|)|\mathcal{D}(\jmath,\kappa)| + \theta\big(\tilde{\mathcal{D}}(\kappa, T\jmath), \tilde{\mathcal{D}}(\jmath, T\kappa), \tilde{\mathcal{D}}(\jmath, T\jmath), \tilde{\mathcal{D}}(\kappa, T\kappa)\big),$$

is said to be a generalized almost θ-Geraghty contraction.

Example 7. *Let $\mathcal{M} = \mathbb{R}$ with the dualistic partial metric $\mathcal{D}(\jmath,\kappa) = \max\{\jmath,\kappa\}$, and the mapping $T\colon [2,3] \to [1,2]$, $T\jmath = \ln(\jmath+1)$. T is a generalized almost θ-Geraghty contraction, for*

$$\beta\colon [0,\infty) \to [0,1),\ \beta(\jmath) = \begin{cases} \dfrac{1}{\jmath}\ln(1+\jmath) & \text{if } \jmath \neq 0; \\ 0, & \text{if } \jmath = 0, \end{cases}$$

and $\theta \in \mathcal{T}$.

The next lemma is a feature of sequences which are not Cauchy sequences; it will be used in the proof of our result.

Lemma 2 ([23]). *Let $\{x_n\}$ be a sequence in X, and d a metric on X, so that $\{x_{2n}\}$ is not a Cauchy sequence, and*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Then there exist $\delta > 0$, and the subsequences $\{m_k\}$, $\{n_k\}$ so that the subsequences

$$d(x_{2m_k}, x_{2n_k}), \quad d(x_{2m_k}, x_{2n_k+1}), \quad d(x_{2m_k-1}, x_{2n_k})$$

have the limit δ as $k \to \infty$.

Based on this lemma, we obtain its variant in dualistic partial metric spaces, as follows.

Lemma 3. *Let $\{\jmath_n\}$ in $\mathcal{M}$, so that $\{\jmath_{2n}\}$ is not a Cauchy sequence, and*

$$\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath_{n+1}) = 0, \quad \text{and} \quad \lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath_n) = 0.$$

Then there exist $\delta > 0$, and the subsequences $\{m_k\}$, $\{n_k\}$ so that the subsequences

$$\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k}), \quad \mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k+1}), \quad \mathcal{D}(\jmath_{2m_k-1}, \jmath_{2n_k})$$

have the limit δ as $k \to \infty$.

Proof. Let $d^s_{\mathcal{D}}$ be the metric induced by $\mathcal{D}$. By using Lemma 1, $\{\jmath_{2n}\}$ is not a Cauchy sequence with respect to $d^s_{\mathcal{D}}$ either. According to Lemma 2, there exists $\delta > 0$, and some subsequences $\{m_k\}$, $\{n_k\}$ so that the subsequences

$$d^s_{\mathcal{D}}(\jmath_{2m_k}, \jmath_{2n_k}), \quad d^s_{\mathcal{D}}(\jmath_{2m_k}, \jmath_{2n_k+1}), \quad d^s_{\mathcal{D}}(\jmath_{2m_k-1}, \jmath_{2n_k})$$

have the limit δ as $k \to \infty$. Having in mind that $\lim_{n\to\infty} D(\jmath_n, \jmath_n) = 0$, $\mathcal{D}$ is symmetric and the definition of the induced metric $d^s_{\mathcal{D}}$, it follows the conclusion of the lemma. □

Our result in this respect is

Theorem 2. *Consider a generalized almost θ-Geraghty contraction T from $\mathcal{A}$ to $\mathcal{B}$. In addition, assume that*

(1) $T\mathcal{A}_0 \subseteq \mathcal{B}_0$;
(2) $(\mathcal{A}, \mathcal{B})$ *is a pair which satisfies the weak (P)-property.*

Then, T possesses a best proximity point which is unique.

Proof. Let $\jmath_0 \in \mathcal{A}_0$. Having in view that $T\mathcal{A}_0 \subseteq \mathcal{B}_0$, then $T\jmath_0 \in \mathcal{B}_0$, so there exists $\jmath_1 \in \mathcal{A}_0$ such that $\mathcal{D}(\jmath_1, T\jmath_0) = D(\mathcal{A}, \mathcal{B})$. By repeating this procedure, we obtain a sequence $\{\jmath_n\} \subseteq \mathcal{A}_0$, which satisfies the equality

$$\mathcal{D}(\jmath_{n+1}, T\jmath_n) = \mathcal{D}(\mathcal{A}, \mathcal{B}), \quad n \in \mathbb{N} \cup \{0\}.$$

$(\mathcal{A}, \mathcal{B})$ is a pair which fulfills the weak (P)-property, therefore we obtain, for $n \in \mathbb{N}$, the inequality $|\mathcal{D}(\jmath_n, \jmath_{n+1})| \leq |\mathcal{D}(T\jmath_{n-1}, T\jmath_n)|$.

Using the generalized contractivity of T, we get

$$\begin{aligned} & |\mathcal{D}(\jmath_n, \jmath_{n+1})| \leq |\mathcal{D}(T\jmath_{n-1}, T\jmath_n)| \\ \leq\ & \beta(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|)|\mathcal{D}(\jmath_{n-1}, \jmath_n)| + \theta(\tilde{\mathcal{D}}(\jmath_n, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_n), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_n, T\jmath_n)) \\ =\ & \beta(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|)|\mathcal{D}(\jmath_{n-1}, \jmath_n)| + \theta(0, \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_n), \tilde{\mathcal{D}}(\jmath_{n-1}, T\jmath_{n-1}), \tilde{\mathcal{D}}(\jmath_n, T\jmath_n)) \\ =\ & \beta(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|)|\mathcal{D}(\jmath_{n-1}, \jmath_n)| \\ \leq\ & |\mathcal{D}(\jmath_{n-1}, \jmath_n)|, \quad n \in \mathbb{N}. \end{aligned} \tag{5}$$

We get that the sequence $\{|\mathcal{D}(\jmath_n, \jmath_{n+1})|\}$ is decreasing. Let $r \geq 0$ be its limit. Suppose that $r \neq 0$. From relation (5), we obtain, for n large enough,

$$\frac{|\mathcal{D}(\jmath_n, \jmath_{n+1})|}{|\mathcal{D}(\jmath_{n-1}, \jmath_n)|} \leq \beta|\mathcal{D}(\jmath_{n-1}, \jmath_n)| < 1, \tag{6}$$

and taking $n \to \infty$ it follows that $\lim_{n\to\infty} \beta(|\mathcal{D}(\jmath_{n-1}, \jmath_n)|) = 1$. Having in mind the properties of the function β, we get that $\lim_{n\to\infty} |\mathcal{D}(\jmath_{n+1}, \jmath_n)| = 0$, a contradiction to our assumption. It follows that $\mathcal{D}(\jmath_n, \jmath_{n+1}) \to 0$.

Similarly, it can be proved that $\mathcal{D}(\jmath_n, \jmath_n) \to 0$, as $n \to \infty$.

Having in view that

$$\mathcal{D}(A, B) \leq \mathcal{D}(\jmath_n, T\jmath_n) \leq \mathcal{D}(\jmath_n, \jmath_{n+1}) + \mathcal{D}(\jmath_{n+1}, T\jmath_n) - \mathcal{D}(\jmath_{n+1}, \jmath_{n+1}),$$

by considering $n \to \infty$ we get that

$$\lim_{n\to\infty} \mathcal{D}(\jmath_n, T\jmath_n) = \mathcal{D}(\mathcal{A}, \mathcal{B}). \tag{7}$$

We continue by proving that $\{\jmath_n\}$ is a Cauchy sequence in $\tau(\mathcal{D})$.

Presume $\{\jmath_{2n}\}$ is not a Cauchy sequence in $\tau(D)$. By Lemma 3, there exists $\delta > 0$, m_k, $n_k \in \mathbb{N}$, so that

$$\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k}) \to \delta, \qquad \mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k+1}) \to \delta, \qquad \mathcal{D}(\jmath_{2m_k-1}, \jmath_{2n_k}) \to \delta,$$

as $k \to \infty$.

Having in mind the generalized triangle inequality, it follows

$$\mathcal{D}(\jmath_{2m_k+1}, \jmath_{2n_k}) \leq \mathcal{D}(\jmath_{2m_k+1}, \jmath_{2m_k}) + \mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k}) - \mathcal{D}(\jmath_{2m_k}, \jmath_{2m_k}),$$

and also

$$\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k}) \leq \mathcal{D}(\jmath_{2m_k}, \jmath_{2m_k+1}) + \mathcal{D}(\jmath_{2m_k+1}, \jmath_{2n_k}) - \mathcal{D}(\jmath_{2m_k+1}, \jmath_{2m_k+1}).$$

By taking $k \to \infty$, we get that $\lim_{k\to\infty} \mathcal{D}(\jmath_{2m_k+1}, \jmath_{2n_k}) = \delta$.

Similarly, it can be proved that $\lim_{k\to\infty} \mathcal{D}(\jmath_{2m_k+1}, \jmath_{2n_k+1}) = \delta$.

It can be noticed that

$$\begin{aligned} &|\mathcal{D}(\jmath_{2m_k+1}, \jmath_{2n_k+1})| \leq |\mathcal{D}(T\jmath_{2m_k}, T\jmath_{2n_k})| \\ \leq\ &\beta(|\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k})|)|\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k})| + \theta(\tilde{\mathcal{D}}(\jmath_{2n_k}, T\jmath_{2m_k}), \tilde{\mathcal{D}}(\jmath_{2m_k}, T\jmath_{2n_k}), \tilde{\mathcal{D}}(\jmath_{2n_k}, T\jmath_{2n_k}), \tilde{\mathcal{D}}(\jmath_{2m_k}, T\jmath_{2m_k}). \end{aligned} \tag{8}$$

By dividing the inequality by $|\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k})|$ and then taking $k \to \infty$, we get that

$$\lim_{k\to\infty} \frac{|\mathcal{D}(\jmath_{2m_k+1}, \jmath_{2n_k+1})|}{|\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k})|} \leq \lim_{k\to\infty} \beta(|\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k})|) \leq 1,$$

hence $\lim_{k\to\infty} \beta(|\mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k})|) = 1$. It follows that $\lim_{k\to\infty} \mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k}) = 0$, which contradicts the fact that $\lim_{k\to\infty} \mathcal{D}(\jmath_{2m_k}, \jmath_{2n_k}) = \delta > 0$. Therefore, $\{\jmath_{2n}\}$ is a Cauchy sequence. Moreover, having also in mind that $\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath_{n+1}) = 0$, and $\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath_n) = 0$, we obtain that $\{\jmath_n\}$ is a Cauchy sequence with respect to $\mathcal{D}$. From the completness of $(\mathcal{M}, \mathcal{D})$ and the fact that $\mathcal{A}$ is closed, it follows that there is $\jmath \in \mathcal{A}$ so that

$$\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath) = \mathcal{D}(\jmath, \jmath) = \lim_{n,m\to\infty} \mathcal{D}(\jmath_n, \jmath_m).$$

As $\lim_{n\to\infty} \mathcal{D}(\jmath_n, \jmath_n) = 0$, we get that $\mathcal{D}(\jmath, \jmath) = 0$.

The next relation holds true

$$|\mathcal{D}(T\jmath_n, T\jmath)| \leq \beta(|\mathcal{D}(\jmath_n, \jmath)|)|\mathcal{D}(\jmath_n, \jmath)| + \theta(\tilde{\mathcal{D}}(\jmath, T\jmath_n), \tilde{\mathcal{D}}(\jmath_n, T\jmath), \tilde{\mathcal{D}}(\jmath_n, T\jmath_n), \tilde{\mathcal{D}}(\jmath, T\jmath)),\ n \in \mathbb{N},$$

having in mind that $\beta(t) \in [0, 1)$, for any t, and taking $n \to \infty$, we get $\lim_{n\to\infty} \mathcal{D}(T\jmath_n, T\jmath) = 0$.

Analogously, $\lim_{n\to\infty} \mathcal{D}(T\jmath_n, T\jmath_n) = 0$.

Furthermore, since

$$\mathcal{D}(A, B) \leq \mathcal{D}(T\jmath, \jmath_n) \leq \mathcal{D}(T\jmath, T\jmath_n) + \mathcal{D}(T\jmath_n, \jmath_n) - \mathcal{D}(T\jmath_n, T\jmath_n),\ n \in \mathbb{N},$$

by considering $n \to \infty$, we obtain that $\lim_{n\to\infty} \mathcal{D}(T\jmath, \jmath_n) = \mathcal{D}(\mathcal{A}, \mathcal{B})$.

Moreover,

$$\mathcal{D}(A, B) \leq \mathcal{D}(\jmath, T\jmath) \leq \mathcal{D}(\jmath, \jmath_n) + \mathcal{D}(\jmath_n, T\jmath) - \mathcal{D}(\jmath_n, \jmath_n),\ n \in \mathbb{N},$$

hence $\mathcal{D}(\jmath, T\jmath) = \mathcal{D}(\mathcal{A}, \mathcal{B})$.

With regard to the uniqueness, assume that there are $\jmath \neq \kappa$ two best proximity points of T. We obtain

$$\begin{aligned} |\mathcal{D}(\jmath,\kappa)| &\leq |\mathcal{D}(T\jmath,T\kappa)| \\ &\leq \beta(|\mathcal{D}(\jmath,\kappa)|)|\mathcal{D}(\jmath,\kappa)| + \theta(\tilde{\mathcal{D}}(\kappa,T\jmath),\tilde{\mathcal{D}}(\jmath,T\kappa),\tilde{\mathcal{D}}(\jmath,T\jmath),\tilde{\mathcal{D}}(\kappa,T\kappa)) \\ &\leq |\mathcal{D}(\jmath,\kappa)| + \theta(\tilde{\mathcal{D}}(\kappa,T\jmath),\tilde{\mathcal{D}}(\jmath,T\kappa),0,0)) \\ &\leq |\mathcal{D}(\jmath,\kappa)|, \end{aligned}$$

which implies $\mathcal{D}(\jmath,\kappa) = 0$.

On the other hand,

$$\begin{aligned} |\mathcal{D}(\jmath,\jmath)| &\leq |\mathcal{D}(T\jmath,T\jmath)| \\ &\leq \beta(|\mathcal{D}(\jmath,\jmath)|)|\mathcal{D}(\jmath,\jmath)| + \theta(\tilde{\mathcal{D}}(\jmath,T\jmath),\tilde{\mathcal{D}}(\jmath,T\jmath),\tilde{\mathcal{D}}(\jmath,T\jmath),\tilde{\mathcal{D}}(\jmath,T\jmath)) \\ &\leq |\mathcal{D}(\jmath,\jmath)|, \end{aligned}$$

hence $\mathcal{D}(\jmath,\jmath) = \beta(|\mathcal{D}(\jmath,\jmath)|)|\mathcal{D}(\jmath,\jmath)|$. Since $\beta(|\mathcal{D}(\jmath,\jmath)|) \in [0,1)$, it follows that $D(\jmath,\jmath) = 0$. Similarly, we obtain $\mathcal{D}(\kappa,\kappa) = 0$. As $\mathcal{D}(\jmath,\jmath) = \mathcal{D}(\kappa,\kappa) = \mathcal{D}(\jmath,\kappa)$, we get that $\jmath = \kappa$. □

Having in view the particular situation of $\theta \equiv 0$, and $\beta \in \mathcal{G}$, the next corollary follows.

Corollary 4. *Let T be a nonself mapping from $\mathcal{A}$ to $\mathcal{B}$, for which there is $\beta \in \mathcal{G}$ so that*

(1) $|\mathcal{D}(T\jmath,T\lambda)| \leq \beta(|\mathcal{D}(\jmath,\lambda)|)|\mathcal{D}(\jmath,\lambda)|$, *for all $\jmath$, $\lambda \in A$;*
(2) $T\mathcal{A}_0 \subseteq \mathcal{B}_0$;
(3) $(\mathcal{A},\mathcal{B})$ *fulfills the weak (P)-property.*

Then, T has a unique best proximity point.

Some fixed point results may be obtained with regard to similar altering distances functions.

Corollary 5. *Consider $\beta \in \mathcal{G}$, and $\theta \in \mathcal{T}$. Let $T\colon \mathcal{A} \to \mathcal{B}$ be a mapping which is a generalized almost θ-Geraghty contraction. Then, there exists a unique fixed point of T, $\jmath \in \mathcal{M}$.*

Corollary 6. *Suppose there is $\beta \in \mathcal{G}$ so that, for any $\jmath$, λ, the following inequality holds*

$$|D(T\jmath,T\lambda)| \leq \beta(|D(\jmath,\lambda)|)|D(\jmath,\lambda)|.$$

Then T has a unique fixed point.

We end up this section with an example.

Example 8. *Let $\mathcal{M} = \mathbb{R}$ with the dualistic partial metric $\mathcal{D}(\jmath,\kappa) = \max\{\jmath,\kappa\}$, and the mapping $T\colon [0,1] \to [0,1]$, $T\jmath = \ln(\jmath+1)$. T is a generalized almost θ-Geraghty contraction, for*

$$\beta\colon [0,\infty) \to [0,1),\ \beta(\lambda) = \begin{cases} \dfrac{1}{\lambda}\ln(1+\lambda), & \text{if } \lambda \neq 0; \\ 0, & \text{if } \lambda = 0, \end{cases}$$

and $\theta \in \mathcal{T}$. By applying Corollary 6, it follows that the mapping T has a unique fixed point.

5. Conclusions

We introduced a generalized almost (φ, θ)-contraction by means of a comparison type function and one of four variables, with adequate properties. Also, a generalized almost θ-Geraghty contraction was defined. Using the concept of (P)-property, we stated existence and uniqueness results of some best proximity points with respect to these two kinds of generalized contractions.

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