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The Gravity of the Classical Klein-Gordon Field

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Abstract: The work shows that the evolution of the field of the free Klein–Gordon equation (KGE), in the hydrodynamic representation, can be represented by the motion of a mass density $\propto |\psi|^2$ subject to the Bohm-type quantum potential, whose equation can be derived by a minimum action principle. Once the quantum hydrodynamic motion equations have been covariantly extended to the curved space-time, the gravity equation (GE), determining the geometry of the space-time, is obtained by minimizing the overall action comprehending the gravitational field. The derived Einstein-like gravity for the KGE field shows an energy-impulse tensor density (EITD) that is a function of the field with the spontaneous emergence of the "cosmological" pressure tensor density (CPTD) that in the classical limit leads to the cosmological constant (CC). The energy-impulse tensor of the theory shows analogies with the modified Brans–Dick gravity with an effective gravity constant G divided by the field squared. Even if the classical cosmological constant is set to zero, the model shows the emergence of a theory-derived quantum CPTD that, in principle, allows to have a stable quantum vacuum (out of the collapsed branched polymer phase) without postulating a non-zero classical CC. In the classical macroscopic limit, the gravity equation of the KGE field leads to the Einstein equation. Moreover, if the boson field of the photon is considered, the EITD correctly leads to its electromagnetic energy-impulse tensor density. The work shows that the cosmological constant can be considered as a second order correction to the Newtonian gravity. The outputs of the theory show that the expectation value of the CPTD is independent by the zero-point vacuum energy density and that it takes contribution only from the space where the mass is localized (and the space-time is curvilinear) while tending to zero as the space-time approaches to the flat vacuum, leading to an overall cosmological effect on the motion of the galaxies that may possibly be compatible with the astronomical observations.

Keywords: non-Minkowskian hydrodynamic representation of quantum equations; Einstein gravity of classical fields; energy-impulse tensor of classical Klein–Gordon field; cosmological constant

1. Introduction

One of the serious problems of gravity physics nowadays [1] refers to the connection between the quantum fields theory (QFT) and the gravity equation (GE). The problem has come to a partial solution in the semi-classical approximation where the energy-impulse tensor density is substituted by its expectation value [2–6].

Even if unable to give answers in a fully quantum regime, the semiclassical approximation has brought successful results such as the explanation of the Hawking radiation and black hole (BH) evaporation [7].

The difficulties about the integration of QFT and the GE become really evident in the so called cosmological constant problem, a term that Einstein added to its equation to give stability to the solution of universe evolution that in general relativity would lead to its final collapse. The introduction, by hand, of the cosmological constant was then refused by Einstein himself who defined it as "the biggest mistake of my life" [8]. Actually, the CC has been introduced [1] to explain the astronomical observations about the motion of galaxies [9] and to give stability to the quantum vacuum bringing it out from the unphysical collapsed branched polymer phase [10], being the physical vacuum of the strong gravity phase related to a positive, not null, CC [11]. Moreover, the energy-impulse tensor density (EITD) for classical bodies in the GE owns a point-dependence by the mass density (i.e., $\propto |\psi|^2$), without any analytically complete connection with the field of matter $\psi = |\psi| \exp \frac{i}{\hbar} S$. As discussed by Thiemann [12], this connection cannot be build up by simply replacing the EITD by its Minkowskian vacuum expectation value. If we do so, we end up with a non-Minkowskian metric tensor solution that has to feed back into the vacuum expectation value and so on with the iteration that does not converge in general.

As a consequence of this fault, modifications to the Einstein equation have been proposed both from the theoretical point of view, such as the Brans-Dicke modified gravity [13], and by using a semi-empirical approach such as the covariant running G or the slip function QFT [14]; However, the cosmological constant itself in the quantum pure gravity can be considered a modification of the general relativity for defining a GE compatible with the needs of a quantum theory.

Due to the undefined connection between the GE and the particle fields, the integration between the QFT and the GE is still an open question that is the object of intense theoretical investigation. Generally speaking, the link between the matter fields and the GE can be obtained by:

1. Defining an adequate GE for matter fields (as, for instance, happens for the photon field);
2. Defining the link between the GE and the QFT by quantizing the action of the new GE.

At glance with the first point, the paper shows that it is possible to obtain the GE with analytical connection with the KGE field:

$$\psi = |\psi| \exp \frac{i}{\hbar} S \tag{1}$$

To this end, the hydrodynamic representation of the field equation as a function of the variables:

$$|\psi| \text{ and } \partial_\mu S = -p_\mu \tag{2}$$

that leads to the classical-like description of a mass density $|\psi|^2$ owing the hydrodynamic impulse p_μ subject to the non-local quantum potential interaction) is utilized. Then, by using the minimum action principle applied to the hydrodynamic model, the gravity generated by the KGE field ψ is derived.

The paper is organized as follows: in Section 2 the Lagrangean version of the hydrodynamic KGE is developed; in Section 3 the gravity equation is derived by the minimum action principle; in Section 4, the perturbative approach to the GE–KGE system of evolutionary equations is derived; in Section 5, the expectation value of the cosmological constant of the quantum KGE massive field is calculated; in Section 6, some features of the GE as well as the check of the theory are discussed.

2. The Hydrodynamic Representation of the Klein–Gordon Equation (KGE)

In this section we derive the hydrodynamic representation of the Klein–Gordon equation (KGE) in the form of Lagrangean equations that allow to define the minimum action principle for the hydrodynamic formalism.

Following the method firstly proposed by Madelung [15] and then generalized by other authors [16–18], the hydrodynamic form of the KGE:

$$\partial_\mu \partial^\mu \psi = -\frac{m^2 c^2}{\hbar^2} \psi \tag{3}$$

in the Minkowskian space, is given by the system of two differential Equations [19]: The Hamilton–Jacobi type one:

$$\partial_\mu S\, \partial^\mu S - \hbar^2 \frac{\partial_\mu \partial^\mu |\psi|}{|\psi|} - m^2 c^2 = 0 \tag{4}$$

coupled to the current conservation equation:

$$\frac{\partial}{\partial q_\mu}\left(|\psi|^2 \frac{\partial S}{\partial q^\mu}\right) = m \frac{\partial J_\mu}{\partial q_\mu} = 0 \tag{5}$$

where,

$$S = \frac{\hbar}{2i} \ln[\frac{\psi}{\psi^*}] \tag{6}$$

and where the 4-current reads,

$$J_\mu = (c\rho, -J_i) = \frac{i\hbar}{2m}(\psi^* \frac{\partial \psi}{\partial q^\mu} - \psi \frac{\partial \psi^*}{\partial q^\mu}) \tag{7}$$

Moreover, being the 4-impulse in the hydrodynamic analogy:

$$p_\mu = (\frac{E}{c}, -p_i) = -\frac{\partial S}{\partial q^\mu} \tag{8}$$

it follows that,

$$J_\mu = (c\rho, -J_i) = -|\psi|^2 \frac{p_\mu}{m} \tag{9}$$

where,

$$\rho = \frac{J_0}{c} = \frac{|\psi|^2}{mc^2} \frac{\partial S}{\partial t} \tag{10}$$

Moreover, by using (8), Equation (4) reads:

$$\frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q_\mu} = p_\mu p^\mu = \left(\frac{E^2}{c^2} - p^2\right) = m^2 c^2 \left(1 - \frac{V_{qu}}{mc^2}\right) \tag{11}$$

(where $p^2 = p_i p_i$ is the modulus squared of the hydrodynamic spatial momentum (italic indexes run from 1 to 3)) that, for k-th eigenstate $\psi_k = |\psi_k| \exp \frac{i}{\hbar} S_{(k)}$, leads to:

$$\begin{aligned} \frac{E_{(k)}{}^2}{c^2} - p_{(k)}{}^2 &= m^2 \gamma^2 \dot{q}_\mu \dot{q}^\mu \left(1 - \frac{V_{qu(k)}}{mc^2}\right) \\ &= m^2 \gamma^2\, c^2 \left(1 - \frac{V_{qu(k)}}{mc^2}\right) - m^2 \gamma^2\, \dot{q}^2 \left(1 - \frac{V_{qu(k)}}{mc^2}\right) \end{aligned} \tag{12}$$

where,

$$\gamma = \frac{1}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}} = \frac{1}{\sqrt{\frac{g_{\mu\nu} \dot{q}^\nu \dot{q}^\mu}{c^2}}} \tag{13}$$

Moreover, by denoting the negative-energy state by the minus subscript, so that $E_{(k)\pm} = \pm E_{(k)}$, from (12) it follows that:

$$E_{(k)} = m\gamma c^2 \sqrt{1 - \frac{V_{qu(k)}}{mc^2}} \tag{14}$$

and more generally, by defining $p_{(k)_{\mu\pm}} = \pm p_{(k)_\mu}$, the hydrodynamic impulse [19] reads:

$$p_{(k)_\mu} = m\gamma_{(k)}\, \dot{q}_{(k)_\mu} \sqrt{1 - \frac{V_{qu(k)}}{mc^2}} = \frac{E_{(k)}}{c^2} \dot{q}_{(k)_\mu} \tag{15}$$

where $V_{qu(k)} = V_{qu(\psi_k)}$ is the quantum potential that reads:

$$V_{qu(\psi)} = -\frac{\hbar^2}{m}\frac{\partial_\mu \partial^\mu |\psi|}{|\psi|} \tag{16}$$

2.1. The Lagrangean Form of the KGE

Equation (4) in the low velocity limit leads to the Madelung quantum hydrodynamic analogy of Schrodinger equation [19] that in the classical limit (i.e., $\hbar = 0$) leads to the classical Lagrangean equation of motion [20]. For the purpose of this work, we generalize the Lagrangean formulation to the hydrodynamic KGEs (4–5).

Since in curved space-time under the gravitational field we may have discrete energy values, in the following we use the discrete formalism (i.e., $\iint \frac{d^3k}{(2\pi)^3} \to \frac{1}{\sqrt{V}} \sum_{k=-\infty}^{\infty}$) that is also useful in the numerical approach.

For the generic superposition of eigenstates

$$\psi = \sum_{k=k_{\min}}^{k=k_{\max}} b_k |\psi_k| \exp[\frac{iS_{(k)}}{\hbar}] = |\psi| \exp[\frac{iS}{\hbar}] \tag{17}$$

where

$$S = \frac{\hbar}{2i} \ln[\frac{\psi}{\psi^*}] = \frac{\hbar}{2i} \left(\begin{array}{c} \ln[\sum\limits_{k=k_{\min}}^{k=k_{\max}} b_k |\psi_k| \exp(\frac{iS_{(k)}}{\hbar})] \\ -\ln[\sum\limits_{k=k_{\min}}^{k=k_{\max}} b^*{}_k |\psi_k| \exp(-\frac{iS_{(k)}}{\hbar})] \end{array} \right) \tag{18}$$

by using (6)–(8), it follows that the hydrodynamic momentum p_μ, and the Lagrangean function, respectively, reads:

$$\begin{aligned} p_\mu = -\partial_\mu S \quad &= -\frac{1}{2} \frac{\sum\limits_{k=k_{\min}}^{k=k_{\max}} b_k |\psi_k| \exp[\frac{iS_{(k)}}{\hbar}] \left(\frac{\hbar}{i} \partial_\mu \ln|\psi_k| - p_{(k)_\mu} \right)}{\sum\limits_k b_k |\psi_k| \exp[\frac{iS_{(k)}}{\hbar}]} \\ &+ \frac{1}{2} \frac{\sum\limits_{k=k_{\min}}^{k=k_{\max}} b^*{}_k |\psi_k| \exp[\frac{-iS_{(k)}}{\hbar}] \left(\frac{\hbar}{i} \partial_\mu \ln|\psi_k| + p_{(k)_\mu} \right)}{\sum\limits_k b^*{}_k |\psi_k| \exp[\frac{-iS_{(k)}}{\hbar}]} = Tr\left(\boldsymbol{p}_\mu \right) \end{aligned} \tag{19}$$

where the matrix $\boldsymbol{p}_\mu$ reads:

$$\boldsymbol{p}_\mu \equiv \left(\boldsymbol{p}_\mu \right)_{jk} = \left(\begin{array}{c} -\frac{1}{2} \frac{b_k |\psi_k| \exp[\frac{iS_{(k)}}{\hbar}] \left(\frac{\hbar}{i} \partial_\mu \ln|\psi_k| - p_{(k)_\mu} \right)}{\sum\limits_j b_j |\psi_j| \exp[\frac{iS_{(j)}}{\hbar}]} \\ +\frac{1}{2} \frac{b^*{}_k |\psi_k| \exp[\frac{-iS_{(k)}}{\hbar}] \left(\frac{\hbar}{i} \partial_\mu \ln|\psi_k| + p_{(k)_\mu} \right)}{\sum\limits_j b^*{}_j |\psi_j| \exp[\frac{-iS_{(j)}}{\hbar}]} \end{array} \right) \delta_{jk} = \widetilde{p}_{(k)\mu} \delta_{jk} \tag{20}$$

and where j, k run from $k_{\min}$ and $k_{\max}$;

$$\begin{aligned} L = \quad & \tfrac{dS}{dt} = \tfrac{\partial S}{\partial t} + \tfrac{\partial S}{\partial q_i}\dot{q}_i = -Tr\left(\boldsymbol{p}_\mu \dot{\boldsymbol{q}}^\mu\right) = Tr(\boldsymbol{L}) \\ & = \tfrac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\dot{q}^\mu_{(k)}\partial_\mu \ln|\psi_k| + L_{(k)}\right)}{\sum\limits_k b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} \\ & -\tfrac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\dot{q}^\mu_{(k)}\partial_\mu \ln|\psi_k| - L_{(k)}\right)}{\sum\limits_k b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \end{aligned} \tag{21}$$

where,

$$\boldsymbol{L} = -\boldsymbol{p}_\mu \dot{\boldsymbol{q}}^\mu = -\widetilde{p}_{(h)\mu}\dot{q}^\mu_{(k)}\delta_{hk} = \widetilde{L}_{(k)}\delta_{hk} \tag{22}$$

where,

$$\widetilde{L}_{(k)} = \frac{1}{2}\left(\begin{array}{c} \dfrac{b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\dot{q}^\mu_{(k)}\partial_\mu \ln|\psi_k| + L_{(k)}\right)}{\sum\limits_k b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} \\ -\dfrac{b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\dot{q}^\mu_{(k)}\partial_\mu \ln|\psi_k| - L_{(k)}\right)}{\sum\limits_k b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \end{array}\right) \tag{23}$$

$$\dot{\boldsymbol{q}}^\mu = \dot{q}^\mu_{(k)}\delta_{jk} \tag{24}$$

Thence, by using the identities $L_{(k)} = -p_{(k)_\mu}\dot{q}^\mu_{(k)}$ and $\left(\frac{\partial}{\partial \dot{\boldsymbol{q}}^\mu}\right)_{jh} = \left(\frac{\partial}{\partial \dot{q}^\mu_{(j)}}\right)\delta_{jh}$ it follows that,

$$\boldsymbol{p}_\mu = \left(\boldsymbol{p}_\mu\right)_{jk} = -\left(\frac{\partial}{\partial \dot{\boldsymbol{q}}^\mu}\right)_{jh}(\boldsymbol{L})_{hk} = -\left(\frac{\partial}{\partial \dot{q}^\mu_{(j)}}\right)\widetilde{L}_{(k)}\delta_{jk} = -\frac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{q}}^\mu} \tag{25}$$

$$\begin{aligned} \widetilde{p}_{(k)\mu} \quad & = -\tfrac{1}{2}\frac{b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\partial_\mu \ln|\psi_k| + \frac{\partial L_{(k)}}{\partial \dot{q}^\mu}\right)}{\sum\limits_k b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} \\ & +\tfrac{1}{2}\frac{b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\partial_\mu \ln|\psi_k| - \frac{\partial L_{(k)}}{\partial \dot{q}^\mu}\right)}{\sum\limits_k b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \\ & = -\frac{\partial \widetilde{L}_{(k)}}{\partial \dot{q}^\mu_{(k)}} \end{aligned} \tag{26}$$

Moreover, given that, for stationary states of time-independent systems (i.e., eigenstates) [19] the hydrodynamic Lagrangean function (21) does not explicitly depend on time, it holds that,

$$\widetilde{L}_{(k)} = L_{(k)} \tag{27}$$

$$\widetilde{p}_{(k)\mu} = p_{(k)\mu} \tag{28}$$

$$-\frac{\partial L_{(k)}}{\partial q^\mu} = -\frac{\partial}{\partial q^\mu}\frac{dS_{(k)}}{dt} = -\left(0, -\frac{\partial}{\partial q_i}\frac{dS_{(k)}}{dt}\right) = -\left(0, -\frac{d}{dt}\frac{\partial S_{(k)}}{\partial q_i}\right) = \frac{d}{dt}\left(-\frac{\partial S_{(k)}}{\partial q^\mu}\right) = \dot{p}_{(k)_\mu} \tag{29}$$

so that, for the eigenstates, the system of Lagrangean hydrodynamic motion equations read

$$p_{(k)_\mu} = -\frac{\partial L_{(k)}}{\partial \dot{q}^\mu_{(k)}} \tag{30}$$

$$\dot{p}_{(k)_\mu} = -\frac{\partial L_{(k)}}{\partial q^\mu} \tag{31}$$

where, by using (29), it follows that,

$$L_{(k)} = -\frac{mc^2}{\gamma_{(k)}}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} = L_{(k)Class}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} = L_{(k)_{Class}} + L_{(k)_Q} \tag{32}$$

where,

$$L_{(k)_{Class}} = -\frac{mc^2}{\gamma_{(k)}} \tag{33}$$

where,

$$L_{(k)_Q} = -\alpha_{(V_{qu(k)})}L_{(k)_{Class}} = -\left(1-\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\right)L_{(k)_{Class}} \tag{34}$$

where,

$$\alpha_{(V_{qu(k)})} = \left(1-\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\right) \tag{35}$$

and that,

$$\frac{d}{dt}\left(-\frac{\partial L_{(k)}}{\partial \dot{q}^{\mu}_{(k)}}\right) = -\frac{\partial L_{(k)}}{\partial q^{\mu}} \tag{36}$$

For sake of accuracy, it must be observed that the solutions of (36) have to be submitted to quantization (given by the irrotational property [17,19]) and to the current conservation condition (5). As shown in [19] the stationary states of (36) obey to the current conservation (5) and are irrotational solutions (i.e., the eigenstates) for the field of the KGE.

Equation (36) is the same both for positive and negative energy states since the Lagrangean function $L_{(k)_-}$ for negative energy states reads $L_{(k)_-} = -p_{(k)_{\mu_-}}\dot{q}^{\mu}_{(k)} = p_{(k)_\mu}\dot{q}^{\mu}_{(k)} = -L_{(k)}$

It is useful to observe that the hydrodynamic equation of motion (36) depends by q, $\dot{q}$ and by the mass distribution $|\psi|^2$ (and its derivatives) contained in V_{qu}.

For $\hbar \to 0$ (i.e., $V_{qu} \to 0$) the classical motion of the mass distribution for the so-called dust matter [17] are obtained just as a function of q, $\dot{q}$.

Moreover, generally speaking, given the integrability of $\widetilde{L}_{(k)}$, the motion equation of the generic superposition of state (17) reads:

$$-\frac{\partial \widetilde{L}_{(k)}}{\partial q^{\mu}} = \frac{d}{dt}\left(-\frac{\partial \widetilde{L}_{(k)}}{\partial \dot{q}^{\mu}_{(k)}}\right) + \widetilde{p}_{(k)_\nu}\frac{\partial \dot{q}_{(k)_\nu}}{\partial q^{\mu}} = \dot{\widetilde{p}}_{(k)_\mu} + \widetilde{p}_{(k)_\nu}\frac{\partial \dot{q}_{(k)_\nu}}{\partial q^{\mu}} \tag{37}$$

that making the summation over k leads to:

$$Tr(\dot{\boldsymbol{p}}_{\mu}) = -\frac{\partial Tr(\boldsymbol{L})}{\partial q^{\mu}} - Tr\left(\boldsymbol{p}_{\nu}\partial_{\mu}\dot{\boldsymbol{q}}_{\nu}\right) \tag{38}$$

and to,

$$\dot{p}_{\mu} = -\frac{\partial L}{\partial q^{\mu}} - \sum_k \widetilde{p}_{(k)_\nu}\frac{\partial q_{(k)_\nu}}{\partial q^{\mu}} \tag{39}$$

where,

$$Tr(\boldsymbol{L}) = L = L_{Class} + L_Q + L_{mix} \tag{40}$$

where,

$$L_{Class} = \frac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]L_{(k)Class}}{\sum\limits_{k} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} + \frac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]L_{(k)Class}}{\sum\limits_{k} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \tag{41}$$

where,

$$L_Q = \frac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\alpha_{(V_{qu(k)})}L_{(k)Class}}{\sum\limits_{k} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} + \frac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\alpha_{(V_{qu(k)})}L_{(k)Class}}{\sum\limits_{k} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \tag{42}$$

and where,

$$L_{mix} = \frac{\hbar}{2i}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]L_{(k)_{mix}}}{\sum\limits_{k} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} - \frac{\hbar}{2i}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]L_{(k)_{mix}}}{\sum\limits_{k} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \tag{43}$$

where,

$$L_{(k)_{mix}} = \dot{q}^{\mu}_{(k)}\,\partial_\mu \ln|\psi_k| \tag{44}$$

2.2. The Hydrodynamic Energy-Impulse Tensor

By using the hydrodynamic energy-impulse tensor (EIT) $\mathrm{T}_{\mu}{}^{\nu}$

$$\begin{aligned}\mathrm{T}_{(k)\mu}{}^{\nu} = & -\left(\dot{q}^{\nu}\frac{\partial L_{(k)}}{\partial \dot{q}^{\mu}} - L_{(k)}\delta_{\mu}{}^{\nu}\right) = \left(\dot{q}^{\nu}_{(k)}p_{(k)\mu} - \dot{q}^{\alpha}_{(k)}p_{(k)\alpha}\delta_{\mu}{}^{\nu}\right) \\ & = \frac{mc^2}{\gamma_{(k)}}\sqrt{1-\frac{V_{qu}}{mc^2}}\left(u_{(k)\mu}u_{(k)}{}^{\nu} - u_{(k)}{}^{\alpha}u_{(k)\alpha}\delta_{\mu}{}^{\nu}\right)\end{aligned} \tag{45}$$

where $u_\mu = \frac{\gamma}{c}\dot{q}_\mu$, with the help of (32), the quantum hydrodynamic motion equation (36) reads:

$$mc\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\frac{du_\mu}{dt} = -mcu_{(k)\mu}\frac{d}{dt}\left(\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\right) + \frac{mc^2}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\left(\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\right) \tag{46}$$

leading to the equation of motion of the eigenstates:

$$mc\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\frac{du_{(k)\mu}}{dt} = \frac{\partial \mathrm{T}_{(k)\mu}{}^{\nu}}{\partial q^\nu} = \mathrm{T}_{(k)\mu}{}^{\nu}{}_{;\nu} \tag{47}$$

where we pose,

$$\mathrm{T}_{(k)\mu}{}^{\nu} = \mathrm{T}_{(k)Class\mu}{}^{\nu} + L_{(k)Class}\delta_{\mu}{}^{\nu} + \mathrm{T}_{(k)Q_{\mu}}{}^{\nu} \tag{48}$$

where,

$$\mathrm{T}_{(k)Class\mu}{}^{\nu} = \frac{mc^2}{\gamma_{(k)}}u_\mu u^\nu \tag{49}$$

and where,

$$\mathrm{T}_{(k)Q\mu}{}^{\nu} = -\alpha_{(V_{qu(k)})}\frac{mc^2}{\gamma_{(k)}}\left(u_{(k)\mu}u_{(k)}{}^{\nu} - u_{(k)}{}^{\alpha}u_{(k)\alpha}\delta_{\mu}{}^{\nu}\right) \tag{50}$$

For the generic quantum state (17), the energy-impulse tensor reads:

$$\begin{aligned}\mathrm{T}_{\mu}{}^{\nu} &= -\left(\dot{q}^{\nu}\frac{\partial L}{\partial \dot{q}^{\mu}} - L\delta_{\mu}{}^{\nu}\right) = -\dot{q}^{\nu}p_{\mu} + L\delta_{\mu}{}^{\nu} \\ &= -\frac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\left(\dot{q}^{\nu}_{(k)}\partial_{\mu}|\psi_k|+\dot{q}^{\alpha}_{(k)}\partial_{\alpha}|\psi_k|\delta_{\mu}{}^{\nu}\right)-\mathrm{T}_{(k)\mu}{}^{\nu}\right)}{\sum\limits_{k} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} \\ &+\frac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\left(\frac{\hbar}{i}\left(\dot{q}^{\nu}_{(k)}\partial_{\mu}|\psi_k|+\dot{q}^{\alpha}_{(k)}\partial_{\alpha}|\psi_k|\delta_{\mu}{}^{\nu}\right)+\mathrm{T}_{(k)\mu}{}^{\nu}\right)}{\sum\limits_{k} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]}\end{aligned} \tag{51}$$

and it can be recast as:

$$\mathrm{T}_{\mu}{}^{\nu} = \mathrm{T}_{Class\mu}{}^{\nu} + L_{Class}\delta_{\mu}{}^{\nu} + \mathrm{T}_{Q\mu}{}^{\nu} + \mathrm{T}_{mix\mu}{}^{\nu} \tag{52}$$

where,

$$\begin{aligned}\mathrm{T}_{Class\mu}{}^{\nu} &= \frac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\mathrm{T}_{(k)Class\mu}{}^{\nu}}{\sum\limits_{k} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} \\ &+\frac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\mathrm{T}_{(k)Class\mu}{}^{\nu}}{\sum\limits_{k} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]}\end{aligned} \tag{53}$$

where,

$$\begin{aligned}\mathrm{T}_{Q\mu}{}^{\nu} &= \frac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\mathrm{T}_{Q(k)\mu}{}^{\nu}}{\sum\limits_{k} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} \\ &+\frac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\mathrm{T}_{Q(k)\mu}{}^{\nu}}{\sum\limits_{k} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]}\end{aligned} \tag{54}$$

and

$$\begin{aligned}\mathrm{T}_{mix\mu}{}^{\nu} &= -\frac{\hbar}{2i}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\left(\dot{q}^{\nu}_{(k)}\partial_{\mu}|\psi_k|+\dot{q}^{\alpha}_{(k)}\partial_{\alpha}|\psi_k|\delta_{\mu}{}^{\nu}\right)}{\sum\limits_{k} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} \\ &+\frac{\hbar}{2i}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\left(\dot{q}^{\nu}_{(k)}\partial_{\mu}|\psi_k|+\dot{q}^{\alpha}_{(k)}\partial_{\alpha}|\psi_k|\delta_{\mu}{}^{\nu}\right)}{\sum\limits_{k} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]}\end{aligned} \tag{55}$$

So far, since we want to covariantly generalize the theory in curvilinear space-time, we have not introduced in the formulas above the explicit form of the Minkowskian KGE field (i.e., $\psi_{k(q_i,t)} = A\exp\left(ik^{\mu}q_{\mu}\right) = A\exp[-i(k_iq_i - \omega t)]$) with the consequential conditions $\partial^{\nu}|\psi_k| = 0$, $k_{\min} = -\infty$ and $k_{\max} = \infty$. Nevertheless, it is useful to derive such Minkowskian expressions that, by posing:

$$\eta_{\nu\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{56}$$

reads,

$$\lim_{g_{\mu\nu}\to\eta_{\mu\nu}} L = L_0 = \frac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]L_{(k)}}{\sum\limits_k b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} + \frac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]L_{(k)}}{\sum\limits_k b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \tag{57}$$

$$\lim_{g_{\mu\nu}\to\eta_{\mu\nu}} L_{mix} = 0 \tag{58}$$

$$\lim_{g_{\mu\nu}\to\eta_{\mu\nu}} \mathrm{T}_\mu{}^\nu = \mathrm{T}_{0\mu}{}^\nu = \frac{1}{2}\frac{\sum\limits_{k=-\infty} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\mathrm{T}_{0(k)\mu}{}^\nu}{\sum\limits_k b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]} + \frac{1}{2}\frac{\sum\limits_{k=-\infty} b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\mathrm{T}_{0(k)\mu}{}^\nu}{\sum\limits_k b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]} \tag{59}$$

where $\lim_{g_{\mu\nu}\to\eta_{\mu\nu}} \mathrm{T}_{(k)\mu}{}^\nu = \mathrm{T}_{0(k)\mu}{}^\nu$, and

$$\lim_{g_{\mu\nu}\to\eta_{\mu\nu}} \mathrm{T}_{mix\mu}{}^\nu = 0 \tag{60}$$

In the quantum case, due to the force generated by the quantum potential (e.g., responsible of the ballistic expansion of an isolated Gaussian packet) for the gradient of the EITD, it follows that:

$$T_\mu{}^\nu{}_{;\nu} = \left(|\psi_k|^2\mathrm{T}_\mu{}^\nu\right)_{;\nu} \neq 0 \tag{61}$$

Finally, it is worth noting that, since in the classical limit (whose resolution length is much bigger than the De Broglie length) due to the quantum decoherence [21] (produced by fluctuations) the superposition of states (17) undergo collapse to an eigenstate, the classical macroscopic limit is obtained by the limiting procedure:

$$\lim_{macro} \equiv \lim_{dec}\lim_{\hbar\to 0} = \lim_{\hbar\to 0}\lim_{dec} \tag{62}$$

where the subscript "*dec*" stands for *decoherence* and where

$$\begin{aligned}\lim_{dec}\psi = \lim_{dec} & \sum_{k=k_{\min}}^{k=k_{\max}} b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}] \\ & = b_{\tilde{k}}|\psi_{\tilde{k}}|\exp[\frac{iS_{(\tilde{k})}}{\hbar}]\end{aligned} \tag{63}$$

where $k_{min} \leq \tilde{k} \leq k_{max}$. The exchange of the order of the two limits it is possible since, by (36), the $\lim_{\hbar\to 0}$ applied to the eigenstates motion equation leads to the classical limit.

Since the detailed stochastic hydrodynamic derivation of (62) shows that the quantum non-local interactions can extend themselves beyond the De Broglie length in the case of strong coupling [21], here we assume that (62) generally holds in a curved space-time taking into account the possibility that the macroscopic scale may go very much beyond the De Broglie length.

Finally, it is useful to note that in the Minkowskian case, by using (35), (50) and (60), the identity (52) reads:

$$\begin{aligned}\lim_{macro}\mathrm{T}_{0\mu}{}^\nu{}_{;\nu} &= \lim_{\hbar\to 0}\lim_{dec}\left(\mathrm{T}_{0Class\mu}{}^\nu{}_{;\nu} + \left(L_{0Class}\delta_\mu{}^\nu\right)_{;\nu} + \mathrm{T}_{0Q\mu}{}^\nu{}_{;\nu} + \mathrm{T}_{0mix\mu}{}^\nu{}_{;\nu}\right) \\ &= \lim_{\hbar\to 0}\left(\mathrm{T}_{(k)0Class\mu}{}^\nu{}_{;\nu} + \left(L_{(k)0Class}\delta_\mu{}^\nu\right)_{;\nu} + \mathrm{T}_{0(k)Q\mu}{}^\nu{}_{;\nu}\right) \\ &= \mathrm{T}_{(k)0Class\mu}{}^\nu{}_{;\nu} + \left(L_{(k)0Class}\delta_\mu{}^\nu\right)_{;\nu} = 0\end{aligned} \tag{64}$$

2.3. The Minimum Action in the Hydrodynamic Formalism

Since the hydrodynamic Lagrangean depends also by the quantum potential and hence by $|\psi| = f_1(|\psi_k|)$ and $\partial_\mu|\psi| = f_2(|\psi_k|, \partial_\mu|\psi_k|)$, the problem of defining the equation of motion can be generally carried out by using the set of variables: $x_{(k)^\mu} = (q_\mu, \dot{q}_{\mu(k)}, |\psi_k|, \partial_\mu|\psi_k|)$. Thence, the variation of the hydrodynamic action $\mathcal{S} = \int \iiint \mathcal{L} dVdt$ (between the fixed starting and end points, $q_{\mu\ a}\ q_{\mu\ b}$, respectively) where $\mathcal{L} = |\psi|^2 L$, reads [19]:

$$\begin{aligned}
\delta\mathcal{S} = &\int \iiint |\psi|^2 \left(\sum_k \frac{\partial \widetilde{L}_{(k)}}{\partial x(k)^\mu} \right) \delta x(k)^\mu dVdt \\
&= \int \iiint |\psi|^2 \sum_k \left(\frac{\partial \widetilde{L}_{(k)}}{\partial q^\mu} \delta q^\mu + \frac{\partial \widetilde{L}_{(k)}}{\partial q^\mu_{(k)}} \delta q^\mu_{(k)} + \frac{\partial \widetilde{L}_{(k)}}{\partial |\psi_k|} \delta |\psi_k| + \frac{\partial \widetilde{L}_{(k)}}{\partial \partial^\mu |\psi_k|} \delta \partial^\mu |\psi_k| \right) dVdt \\
&= \frac{1}{c} \int \iiint |\psi|^2 \sum_k \left(\left(\frac{\partial \widetilde{L}_{(k)}}{\partial q^\mu} - \frac{d}{dt} \frac{\partial \widetilde{L}_{(k)}}{\partial q^\mu_{(k)}} \right) \delta q^\mu + \left(\frac{\partial \widetilde{L}_{(k)}}{\partial |\psi_k|} - \partial^\mu \frac{\partial \widetilde{L}_{(k)}}{\partial \partial^\mu |\psi_k|} \right) \delta |\psi_k| \right) d\Omega
\end{aligned} \tag{65}$$

Given that the quantum motion equations for eigenstates (30–31) satisfy the condition:

$$\left(\frac{\partial \widetilde{L}_{(k)}}{\partial q^\mu} - \frac{d}{dt} \frac{\partial \widetilde{L}_{(k)}}{\partial \dot{q}^\mu_{(k)}} \right) = \left(\frac{\partial L_{(k)}}{\partial q^\mu} - \frac{d}{dt} \frac{\partial L_{(k)}}{\partial \dot{q}^\mu_{(k)}} \right) = 0 \tag{66}$$

(that explicitly defines $\psi_{(q,t)}$) the variation of the action $\delta\mathcal{S}$ for the k-th eigenstates reads:

$$\delta\mathcal{S} = \delta\left(\Delta\mathcal{S}_{Q(k)}\right) = \frac{1}{c} \int \iiint |\psi_k|^2 \left(\frac{\partial L_{(k)}}{\partial |\psi_k|} - \partial^\mu \frac{\partial L_{(k)}}{\partial \partial_\mu |\psi_k|} \right) \delta |\psi_k| d\Omega \tag{67}$$

that is not null since it takes contribution from the quantum potential contained into the hydrodynamic Lagrangean $L_{(k)}$.

Thence, for the quantum hydrodynamic evolution it follows that:

$$\delta\mathcal{S} - \delta\left(\Delta\mathcal{S}_{Q(k)}\right) = 0 \tag{68}$$

that, since in the classical limit, for $\hbar \to 0$, $V_{qu} \to 0$, it holds that:

$$\frac{\partial\left(\lim_{\hbar\to 0} L_{(k)}\right)}{\partial |\psi_k|} = 0 \tag{69}$$

$$\frac{\partial\left(\lim_{\hbar\to 0} L_{(k)}\right)}{\partial^\mu |\psi_k|} = 0 \tag{70}$$

the classical extremal principle,

$$\lim_{\hbar\to 0} \delta\mathcal{S} = \lim_{\hbar\to 0} \delta\left(\Delta\mathcal{S}_{Q(k)}\right) = 0 \tag{71}$$

is recovered.

Moreover, generally speaking, by using (37), for the general superposition of state (17) the variation of the action reads:

$$\begin{aligned}
\delta\mathcal{S} &= -\frac{1}{c} \int \iiint |\psi|^2 \sum_k \left(\left(\widetilde{p}_{(k)} \frac{\partial q_{(k)_\nu}}{\partial q^\mu} \right) \delta q^\mu - \left(\frac{\partial \widetilde{L}_{(k)}}{\partial |\psi_k|} - \partial^\mu \frac{\partial \widetilde{L}_{(k)}}{\partial \partial^\mu |\psi_k|} \right) \delta |\psi_k| \right) d\Omega \\
&= \delta\left(\Delta\mathcal{S}_{Q_{mix}} + \Delta\mathcal{S}_Q\right) = \delta(\Delta\mathcal{S})
\end{aligned} \tag{72}$$

where

$$\Delta \mathcal{S}_Q = \frac{1}{c} \int \iiint |\psi|^2 \sum_k \left(\frac{\partial \widetilde{L}_{(k)}}{\partial |\psi_k|} - \partial^\mu \frac{\partial \widetilde{L}_{(k)}}{\partial \partial^\mu |\psi_k|} \right) \delta |\psi_k| d\Omega \tag{73}$$

and where

$$\delta\left(\Delta \mathcal{S}_{Q_{mix}}\right) = -\frac{1}{c} \int \iiint |\psi|^2 \sum_k \widetilde{p}_{(k)} \frac{\partial q_{(k)\nu}}{\partial q^\mu} \delta |\psi_k| d\Omega \tag{74}$$

is due to the quantum mixing of superposition of states.

Thence, the condition (72) can be generalized to:

$$\delta \mathcal{S} - \delta(\Delta \mathcal{S}) = 0 \tag{75}$$

that, being:

$$\lim_{macro} \delta\left(\Delta \mathcal{S}_{Qmix}\right) = \lim_{\hbar \to 0} \lim_{dec} \delta\left(\Delta \mathcal{S}_{Qmix}\right) = 0 \tag{76}$$

in the classical limit leads to:

$$\begin{aligned} \lim_{macro} \delta \mathcal{S} = \ & \lim_{macro} \delta(\Delta \mathcal{S}) = \lim_{\hbar \to 0} \lim_{dec} \delta\left(\Delta \mathcal{S}_Q + \Delta \mathcal{S}_{Q_{mix}}\right) \\ & = \lim_{\hbar \to 0} \delta\left(\Delta \mathcal{S}_{Q_{(k)}}\right) = 0 \end{aligned} \tag{77}$$

2.4. The Hydrodynamic KGE in Curvilinear Space-Time

As far as the motion Equations (30), (31) and (37) are concerned, there is no way to univocally define them in the non-Minkowskian space-time without a postulate that fixes the criterion of generalization.

In the classical general relativity this criterion is given by the equivalence of inertial and gravitational mass that, in fact, is equivalent to postulate that the classical equation of motion is covariant in general relativity [19].

Analogously, assuming the covariance of the KGE [19] that reads:

$$\psi_{;\mu}^{;\mu} = (g^{\mu\nu} \partial_\nu \psi)_{;\mu} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} (g^{\mu\nu} \partial_\nu \psi) = -\frac{m^2 c^2}{\hbar^2} \psi \tag{78}$$

also the hydrodynamic motion Equations (4) and (5) own the covariant form [19]:

$$g_{\mu\nu} \partial^\nu S\, \partial^\mu S - \hbar^2 \frac{1}{|\psi| \sqrt{-g}} \partial_\mu \sqrt{-g} (g^{\mu\nu} \partial_\nu |\psi|) - m^2 c^2 = 0 \tag{79}$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial q^\mu} \sqrt{-g} \left(g^{\mu\nu} |\psi|^2 \frac{\partial S}{\partial q^\nu} \right) = 0 \tag{80}$$

where the quantum potential reads:

$$V_{qu} = -\frac{\hbar^2}{m} \frac{1}{|\psi| \sqrt{-g}} \partial_\mu \sqrt{-g} (g^{\mu\nu} \partial_\nu |\psi|) \tag{81}$$

where $g_{\nu\mu}$ is the metric tensor and where $g = |g_{\nu\mu}|^{-1}$.

Moreover, given the covariance of (79) and (80) also the motion equation (36) or (37) as well as $T_{(k)\mu\nu}$ and $L_{(k)}$, that read respectively:

$$\begin{aligned} T_{(k)\mu\nu} \ & = \left(\dot{q}_\nu \frac{\partial L_{(k)}}{\partial \dot{q}^\mu} - L_{(k)} \delta_\mu{}^\nu \right) \\ & = mc^2 \sqrt{\frac{g_{\kappa\lambda} \dot{q}^\lambda_{(k)} \dot{q}^\kappa_{(k)}}{c^2}} \sqrt{1 - \frac{V_{qu(k)}}{mc^2}} \left(u_\mu u_\nu - g_{\beta\alpha} u^\alpha u^\beta g_{\mu\nu} \right) \end{aligned} \tag{82}$$

where,

$$\begin{aligned} L_{(k)} &= -\frac{mc^2}{\gamma_{(k)}}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} = -mc^2\sqrt{\frac{g_{\mu\nu}\dot{q}^{\nu}_{(k)}\dot{q}^{\mu}_{(k)}}{c^2}}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} \\ &= -g_{\mu\alpha}\dot{q}^{\alpha}_{(k)}p^{\mu}_{(k)} = -g_{\mu\alpha}c^2\left(\partial_t S_{(k)}\right)^{-1}p^{\alpha}_{(k)}p^{\mu}_{(k)} \end{aligned} \tag{83}$$

are covariant.

Once the quantum equations are defined in non-Minkowskian space-time, their meaning is fully determined when the metric of the space-time is determined by the GE based on additional condition (e.g., on the hydrodynamic action covariantly generalized).

Moreover, it is useful to note that, due to the biunique relation between the quantum hydrodynamic Equations (79) and (80), subject to the irrotational conditions, and the KGE (78) [17], the system of Equations (79) and (80) coupled to the GE, are equivalent to the KGE–GE system. Furthermore, since Equation (79) can be expressed in the Lagrangean form (36) or (37), by using the relations (14), (15) and (45), both the energy-impulse tensor (EIT) (51) and the Lagrangean (21) can be expressed as a function of the field ψ with the help of the following relations

$$\begin{aligned} L_{(k)} &= -c^2\left(\partial_t S_{(k)}\right)^{-1}g_{\mu\alpha}p^{\alpha}_{(k)}p^{\mu}_{(k)} = -c^2\left(\partial_t S_{(k)}\right)^{-1}g_{\mu\alpha}\partial^{\alpha}S_{(k)}\partial^{\mu}S_{(k)} \\ &= -c^2\frac{i\hbar}{2}\left(\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial t}\right)^{-1}g_{\mu\alpha}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\nu} \end{aligned} \tag{84}$$

$$\begin{aligned} \mathrm{T}_{(k)\mu\nu} &= c^2\left(\frac{\partial S_{(k)}}{\partial t}\right)^{-1}\left(\frac{\partial S_{(k)}}{\partial q^\mu}\frac{\partial S_{(k)}}{\partial q^\nu} - g_{\alpha\beta}\frac{\partial S_{(k)}}{\partial q_\beta}\frac{\partial S_{(k)}}{\partial q_\alpha}g_{\mu\nu}\right) \\ &= -m^2c^4\left(\frac{\frac{\hbar}{2i}\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial t}\right)^{-1}\left(\left(\frac{\hbar}{2mc}\right)^2\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q^\mu}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q^\nu} + \left(1-\frac{V_{qu(k)}}{mc^2}\right)g_{\mu\nu}\right) \end{aligned} \tag{85}$$

and

$$\dot{q}_{(k)_\mu} = c^2\frac{p_{(k)_\mu}}{E} = -c^2\frac{\partial_\mu S_{(k)}}{\partial_t S_{(k)}} = -c^2\frac{\partial_\mu \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial_t \ln[\frac{\psi_k}{\psi_k{}^*}]} \tag{86}$$

where the last identity has been obtained by inverting (15).

It is worth mentioning that the KGE-GE system of evolutionary equations has the advantage of containing only the irrotational states that satisfy the current conservation condition.

3. The Minimum Action in Curved Space-Time and the Gravity Equation for the Hydrodynamic KGE

In order to follow an analytical procedure we derive the gravity equation, by applying the minimum action principle to the quantum hydrodynamic matter evolution associated to the field ψ.

Given that the quantum hydrodynamic equations in Minkowskian space-time [19] satisfies the minimum action principle (75), when we consider the covariant formulation in the curved space-time, such variation takes a contribution from the variability of the metric tensor. When we consider the gravity and we assume that the geometry of space-time is that one which makes null the overall variation of the action (75), we define the condition that leads to the definition of the GE.

By considering the variation of the action due to the curvilinear coordinates [22] and the functional dependence by ψ, it follows that:

$$\delta\mathcal{S} = \frac{1}{c}\int\iiint|\psi|^2\sum_k\left(\begin{array}{c}\left(\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}\widetilde{L}_{(k)}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^\lambda}\frac{\partial\sqrt{-g}\widetilde{L}_{(k)}}{\partial\frac{\partial g^{\mu\nu}}{\partial q^\lambda}} - \frac{\partial}{\partial|\psi|}\frac{\partial\sqrt{-g}\widetilde{L}_{(k)}}{\partial\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}}\right)\right)\delta g^{\mu\nu} \\ +\widetilde{p}_{(k)_\nu}q^{\nu}_{(k)\ ;\mu}\delta q^\mu + \left(\frac{\partial\widetilde{L}_{(k)}}{\partial|\psi_k|} - \partial^\mu\frac{\partial\widetilde{L}_{(k)}}{\partial\partial^\mu|\psi_k|}\right)\delta|\psi_k|\end{array}\right)\sqrt{-g}d\Omega \tag{87}$$

and,

$$\delta\mathcal{S}-\delta(\Delta\mathcal{S})=\frac{1}{c}\int\iiint|\psi|^2\sum_k\left(\frac{\partial\sqrt{-g}\widetilde{L}_{(k)}}{\partial g^{\mu\nu}}-\frac{\partial}{\partial q^\lambda}\frac{\partial\sqrt{-g}\widetilde{L}_{(k)}}{\partial\frac{\partial g^{\mu\nu}}{\partial q^\lambda}}-\frac{\partial}{\partial|\psi_k|}\frac{\partial\sqrt{-g}\widetilde{L}_{(k)}}{\partial\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}}\right)\delta g^{\mu\nu}d\Omega \tag{88}$$

If we postulate that the variation of action of the gravitational field,

$$\delta\mathcal{S}_g=\frac{c^3}{16\pi G}\int\iiint\left(R_{uv}-\frac{1}{2}Rg_{uv}\right)\delta g^{uv}\sqrt{-g}d\Omega \tag{89}$$

offsets that one produced by the KGE field so that:

$$\delta\mathcal{S}-\delta(\Delta\mathcal{S})+\delta\mathcal{S}_g=\int\iiint\left(R_{uv}-\frac{1}{2}Rg_{uv}-\frac{8\pi G}{c^4}\left|\psi\right|^2\boldsymbol{\tau}_{uv}\right)\delta g^{uv}\sqrt{-g}d\Omega=0 \tag{90}$$

we obtain the gravitational equation,

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi G}{c^4}|\psi|^2\tau_{\mu\nu} \tag{91}$$

where,

$$\begin{aligned}\tau_{\mu\nu}\ &=\frac{1}{2}\frac{\sum\limits_{k=-\infty}b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]\left(\tau_{(k)\mu\nu_{class}}+\tau_{(k)\mu\nu_Q}+\tau_{(k)\mu\nu_{mix}}\right)}{\sum\limits_k b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]}\\&+\frac{1}{2}\frac{\sum\limits_{k=-\infty}b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]\left(\tau_{(k)\mu\nu_{class}}+\tau_{(k)\mu\nu_Q}-\tau_{(k)\mu\nu_{mix}}\right)}{\sum\limits_k b^*{}_k|\psi_k|\exp[\frac{-iS_{(k)}}{\hbar}]}\\&+\tau_{\mu\nu_{curv}}\\&=\tau_{\mu\nu_{class}}+\tau_{\mu\nu_Q}+\tau_{\mu\nu_{mix}}+\tau_{\mu\nu_{curv}}\end{aligned} \tag{92}$$

where,

$$\tau_{(k)\mu\nu_{class}}=2\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial g^{\mu\nu}}-\frac{\partial}{\partial q^\lambda}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial q^\lambda}}-\frac{\partial}{\partial|\psi_k|}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}}\right)\sqrt{-g}L_{(k)class} \tag{93}$$

$$\tau_{(k)\mu\nu_Q}=2\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial g^{\mu\nu}}-\frac{\partial}{\partial q^\lambda}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial q^\lambda}}-\frac{\partial}{\partial|\psi_k|}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}}\right)\sqrt{-g}L_{(k)Q} \tag{94}$$

$$\tau_{(k)\mu\nu_{mix}}=2\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial g^{\mu\nu}}-\frac{\partial}{\partial q^\lambda}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial q^\lambda}}-\frac{\partial}{\partial|\psi_k|}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}}\right)\sqrt{-g}L_{(k)mix} \tag{95}$$

$$\begin{aligned}
\tau_{\pm i\mu\nu_{curv}} &= \frac{1}{2}\sum_{k=-\infty}\begin{pmatrix} L_{(k)\mu\nu_{class}} \\ +L_{(k)\mu\nu_{Q}} \\ +L_{(k)\mu\nu_{mix}} \end{pmatrix}\begin{pmatrix} \begin{pmatrix} \frac{\partial}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^{\lambda}}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial q^{\lambda}}} \\ -\frac{\partial}{\partial|\psi_{\pm i(k)}|}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial|\psi_{\pm i(k)}|}} \end{pmatrix}\Xi_{(k)} \\ -\begin{pmatrix} \frac{\partial\Xi_{(k)}}{\partial q^{\lambda}}\frac{\partial\ln\left(L_{(k)class}\right)}{\frac{\partial g^{\mu\nu}}{\partial q^{\lambda}}} + \frac{\partial\ln\left(L_{(k)class}\right)}{\partial q^{\lambda}}\frac{\partial\Xi_{(k)}}{\frac{\partial g^{\mu\nu}}{\partial q^{\lambda}}} \\ \frac{\partial\Xi_{(k)}}{\partial|\psi_k|}\frac{\partial\ln\left(L_{(k)class}\right)}{\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}} + \frac{\partial\ln\left(L_{(k)class}\right)}{\partial|\psi_k|}\frac{\partial\Xi_{(k)}}{\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}} \end{pmatrix} \end{pmatrix} \\
&+\frac{1}{2}\sum_{k=-\infty}\begin{pmatrix} L_{(k)\mu\nu_{class}} \\ +L_{(k)\mu\nu_{Q}} \\ -L_{(k)\mu\nu_{mix}} \end{pmatrix}\begin{pmatrix} \begin{pmatrix} \frac{\partial}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^{\lambda}}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial q^{\lambda}}} \\ -\frac{\partial}{\partial|\psi_{\pm i(k)}|}\frac{\partial}{\partial\frac{\partial g^{\mu\nu}}{\partial|\psi_{\pm i(k)}|}} \end{pmatrix}\Xi^{*}{}_{(k)} \\ -\begin{pmatrix} \frac{\partial\Xi^{*}{}_{(k)}}{\partial q^{\lambda}}\frac{\partial\ln\left(L_{(k)class}\right)}{\frac{\partial g^{\mu\nu}}{\partial q^{\lambda}}} + \frac{\partial\ln\left(L_{(k)class}\right)}{\partial q^{\lambda}}\frac{\partial\Xi^{*}{}_{(k)}}{\frac{\partial g^{\mu\nu}}{\partial q^{\lambda}}} \\ \frac{\partial\Xi^{*}{}_{(k)}}{\partial|\psi_k|}\frac{\partial\ln\left(L_{(k)class}\right)}{\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}} + \frac{\partial\ln\left(L_{(k)class}\right)}{\partial|\psi_k|}\frac{\partial\Xi^{*}{}_{(k)}}{\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}} \end{pmatrix} \end{pmatrix}
\end{aligned} \tag{96}$$

where,

$$\Xi_{(k)} = \frac{b_k|\psi_k|\exp[\frac{iS_{(k)}}{\hbar}]}{\sum_j b_j|\psi_j|\exp[\frac{iS_{(k)}}{\hbar}]}$$

.

Moreover, given that [22],

$$\frac{1}{2c}\int\iiint\tau_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}d\Omega = -\frac{1}{c}\int\iiint\tau^{\nu}_{\mu;\nu}\xi^{\mu}\sqrt{-g}d\Omega \tag{97}$$

where,

$$\delta g^{\mu\nu} = \xi^{\mu;\nu} + \xi^{\nu;\mu} \tag{98}$$

for the infinitesimal transformation of coordinates,

$$q\prime^{\mu} = q^{\mu} + \xi^{\mu} \tag{99}$$

around the Minkowskian case (97) leads to,

$$\delta\mathcal{S} - \delta(\Delta\mathcal{S}) = -\frac{1}{c}\int\iiint\left(|\psi|^2\tau^{\nu}_{\mu}\right)_{;\nu}\xi^{\mu}\sqrt{-g}d\Omega = 0 \tag{100}$$

and, in the classical limit, to,.

$$\begin{aligned}
&\lim_{decoherence}\lim_{\hbar\to 0} - \tfrac{1}{c}\int\iiint\left(|\psi|^2\tau^{\nu}_{\mu}\right)_{;\nu}\xi^{\mu}\sqrt{-g}d\Omega = \\
&-\tfrac{1}{c}\int\iiint\left(\lim_{decoherence}\lim_{\hbar\to 0}\left(|\psi|^2\tau^{\nu}_{\mu}\right)_{;\nu}\right)\xi^{\mu}\sqrt{-g}d\Omega = 0
\end{aligned} \tag{101}$$

and thence, for the arbitrariness of ξ^{μ}, to,

$$\lim_{dec}\lim_{\hbar\to 0}\left(|\psi|^2\tau^{\nu}_{\mu}\right)_{;\nu} = 0 \tag{102}$$

Furthermore, given that from (35), (50), (52)–(55) in the classical (Minkowskian) limit it holds that:

$$\begin{aligned}
&\lim_{dec}\lim_{\hbar\to 0}\left(|\psi|^2T^{\nu}_{\mu}\right)_{;\nu} = \left(\lim_{dec}\lim_{\hbar\to 0}|\psi|^2T^{\nu}_{\mu}\right)_{;\nu} \\
&= \left(\lim_{dec}|\psi|^2\left(T^{\nu}_{\mu_{class}} + L_{class}\delta^{\nu}_{\mu} + T_{mix}{}^{\nu}_{\mu}\right)\right)_{;\nu} = |\psi_k|^2T_{(k)}{}^{\nu}_{\mu;\nu_{class}} + |\psi_k|^2\left(L_{(k)class}\delta^{\nu}_{\mu}\right)_{;\nu} = 0
\end{aligned} \tag{103}$$

from (92) it follows that:

$$\begin{aligned} \lim_{dec}\lim_{\hbar\to 0}|\psi|^2\tau_{\mu}{}^{\nu} &= |\psi_k|^2\tau_{(k)\mu}{}^{\nu}{}_{class} \\ &= |\psi_k|^2 T_{(k)\mu}{}^{\nu}{}_{class} + |\psi_k|^2 L_{(k)class}\delta_{\mu}^{\nu} + |\psi_k|^2 C_{(\dot{q}_\mu)}\delta_{\mu}{}^{\nu} \end{aligned} \tag{104}$$

and, that:

$$\lim_{dec}\lim_{\hbar\to 0}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = -\frac{8\pi G}{c^4}\left(T_{(k)\mu\nu_{class}} + |\psi|^2 L_{(k)_{class}} g_{\mu\nu} + |\psi|^2 C g_{\mu\nu}\right) \tag{105}$$

where,

$$\mathrm{T}_{(k)\mu\nu_{class}} = |\psi_k|^2 \mathrm{T}_{(k)\mu\nu_{class}} \tag{106}$$

Moreover, by posing:

$$\Lambda_{class} = -L_{(k)class_{(\dot{q}_\mu)}} - C_{(\dot{q}_\mu)} \tag{107}$$

the Einstein equation (where the k variable is needless),

$$\lim_{decoherence}\lim_{\hbar\to 0}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = R^{macro}_{\mu\nu} - \frac{1}{2}R^{macro}g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{(k)\mu\nu_{class}} - \Lambda g_{\mu\nu}\right) \tag{108}$$

where,

$$\Lambda = |\psi_k|^2 \Lambda_{class} \tag{109}$$

is recovered in the classical limit.

3.1. The Gravity Equation (GE) for the KGE Eigenstates

Moreover, by using the identity,

$$\tau_{(k)\mu}{}^{\nu}{}_{class} = T_{(k)\mu}{}^{\nu}{}_{class} - \Lambda_{class}\delta_{\mu}{}^{\nu} \tag{110}$$

obtained from (104) and (107), and by using (35) and (94) it follows that:

$$\begin{aligned} \tau_{(k)\mu\nu_Q} &= -\alpha_{(V_{qu})}\left(\tau_{(k)\mu\nu_{class}} + L_{(k)class}\Delta_{\mu\nu(k)}\right) \\ &= -\alpha_{(V_{qu})}\left(T_{(k)\mu\nu_{class}} - \Lambda_{class}g_{\mu\nu} + L_{(k)class}\Delta_{\mu\nu(k)}\right) \end{aligned} \tag{111}$$

where,

$$\Delta_{\mu\nu(k)} = \frac{2}{\alpha_{(V_{qu(k)})}}\left(\begin{array}{c} \left(\frac{\partial}{\partial g^{\mu\nu}} - \frac{\partial}{\partial q^\lambda}\frac{\partial}{\frac{\partial g^{\mu\nu}}{\partial q^\lambda}} - \frac{\partial}{\partial|\psi_k|}\frac{\partial}{\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}}\right)\alpha_{(V_{qu(k)})} \\ -\left(\begin{array}{c} \frac{\partial\alpha_{(V_{qu(k)})}}{\partial q^\lambda}\frac{\partial\ln\left(L_{(k)class}\right)}{\frac{\partial g^{\mu\nu}}{\partial q^\lambda}} + \frac{\partial\ln\left(L_{(k)class}\right)}{\partial q^\lambda}\frac{\partial\alpha_{(V_{qu(k)})}}{\frac{\partial g^{\mu\nu}}{\partial q^\lambda}} \\ \frac{\partial\alpha_{(V_{qu(k)})}}{\partial|\psi_k|}\frac{\partial\ln\left(L_{(k)class}\right)}{\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}} + \frac{\partial\ln\left(L_{(k)class}\right)}{\partial|\psi_k|}\frac{\partial\alpha_{(V_{qu(k)})}}{\frac{\partial g^{\mu\nu}}{\partial|\psi_k|}} \end{array}\right) \end{array}\right) \tag{112}$$

and, being both $\tau_{\mu\nu_{curv}} = 0$ and $\tau_{\mu\nu_{mix}} = 0$, that:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{8\pi G}{c^4}\Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\left(\begin{array}{c} T_{(k)\mu\nu_{class}}\left(1 - \alpha_{(V_{qu(k)})}\right) \\ +\alpha_{(V_{qu(k)})}\left(\Lambda g_{\mu\nu} - |\psi_k|^2 L_{(k)class}\Delta_{\mu\nu(k)}\right) \end{array}\right) \tag{113}$$

Finally, by separating $\Delta_{\mu\nu}$ in the isotropic and stress part $\Delta_{s\mu\nu}$ as follows:

$\Delta_{\mu\nu} = \frac{\Delta_{\lambda\lambda}}{4} g_{\mu\nu} + \Delta_{s\mu\nu}$, the GE, for eigenstates, reads:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{8\pi G}{c^4} \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} \begin{pmatrix} T_{(k)\mu\nu_{class}} \left(1 - \alpha_{(V_{qu(k)})}\right) \\ +\alpha_{(V_{qu(k)})} \begin{pmatrix} \left(\Lambda - |\psi_k|^2 L_{(k)class} \frac{\Delta_{\lambda\lambda}}{4}\right) g_{\mu\nu} \\ -|\psi_k|^2 L_{(k)class} \Delta_{s\mu\nu(k)} \end{pmatrix} \end{pmatrix} \tag{114}$$

3.2. The GE of the General KGE Field

Finally, by using (92)–(96) the GE as a function of the general KGE field (17) reads:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{8\pi G}{c^4} \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} |\psi|^2 Tr\left(\boldsymbol{\tau}_{\mu\nu_{class}} - \boldsymbol{\Lambda}_Q g_{\mu\nu} + \boldsymbol{\Delta\tau}_{\mu\nu_{stress}}\right) \tag{115}$$

where,

$$\boldsymbol{\tau}_{\mu\nu_{class}} = \varsigma_{(k)} \left(1 - \alpha_{(V_{qu(k)})}\right) T_{(k)\mu\nu_{class}} \delta_{hk} \tag{116}$$

where,

$$\varsigma_{(k)} = \begin{pmatrix} \frac{1}{2} \frac{b_k |\psi_k| \exp[\frac{iS_{(k)}}{\hbar}]}{\sum_j b_j |\psi_j| \exp[\frac{iS_{(j)}}{\hbar}]} \\ +\frac{1}{2} \frac{b^*{}_k |\psi_k| \exp[\frac{-iS_{(k)}}{\hbar}]}{\sum_j b^*{}_j |\psi_j| \exp[\frac{-iS_{(j)}}{\hbar}]} \end{pmatrix} \tag{117}$$

$$\boldsymbol{\Lambda}_Q = -\left(\begin{pmatrix} \frac{1}{2} \frac{b_k |\psi_k| \exp[\frac{iS_{(k)}}{\hbar}] \begin{pmatrix} \alpha_{(V_{qu(k)})} \left(\Lambda_{class} - L_{(k)class} \frac{\Delta_{\kappa\kappa(k)}}{4}\right) \\ + \frac{\tau_{(k)\kappa\kappa_{mix}}}{4} \end{pmatrix}}{\sum_j b_j |\psi_j| \exp[\frac{iS_{(j)}}{\hbar}]} \\ +\frac{1}{2} \frac{b^*{}_k |\psi_k| \exp[\frac{-iS_{(k)}}{\hbar}] \begin{pmatrix} \alpha_{(V_{qu(k)})} \left(\Lambda_{class} - L_{(k)class} \frac{\Delta_{\kappa\kappa(k)}}{4}\right) \\ - \frac{\tau_{(k)\kappa\kappa_{mix}}}{4} \end{pmatrix}}{\sum_j b^*{}_j |\psi_j| \exp[\frac{-iS_{(j)}}{\hbar}]} \end{pmatrix} + \frac{\tau_{\lambda\lambda_{curv}}}{4} \right) \delta_{hk} \tag{118}$$

and,

$$\boldsymbol{\Delta\tau}_{\mu\nu_{stress}} = \left(\begin{pmatrix} \frac{1}{2} \frac{b_k |\psi_k| \exp[\frac{iS_{(k)}}{\hbar}] \begin{pmatrix} \left(\alpha_{(V_{qu(k)})} L_{(k)class} \Delta_{s\mu\nu(k)}\right) \\ -\tau_{s(k)}{}_{\mu\nu_{mix}} \end{pmatrix}}{\sum_j b_j |\psi_j| \exp[\frac{iS_{(j)}}{\hbar}]} \\ +\frac{1}{2} \frac{b^*{}_k |\psi_k| \exp[\frac{-iS_{(k)}}{\hbar}] \begin{pmatrix} \left(\alpha_{(V_{qu(k)})} L_{(k)class} \Delta_{s\mu\nu(k)}\right) \\ +\tau_{s(k)}{}_{\mu\nu_{mix}} \end{pmatrix}}{\sum_j b^*{}_j |\psi_j| \exp[\frac{-iS_{(j)}}{\hbar}]} \end{pmatrix} + \tau_{s\mu\nu_{curv}} \right) \delta_{hk} \tag{119}$$

where both $\tau_{(k)\mu\nu_{mix}}$ and $\tau_{\mu\nu_{curv}}$ have been split into the isotropic and stress parts as in the following:

$$\tau_{(k)\mu\nu_{mix}} = \frac{\tau_{(k)\lambda\lambda_{mix}}}{4} g_{\mu\nu} + \tau_{s(k)}{}_{\mu\nu_{mix}} \tag{120}$$

$$\tau_{\mu\nu_{curv}} = \frac{\tau_{\lambda\lambda\,curv}}{4} g_{\mu\nu} + \tau_{s\mu\nu_{curv}} \tag{121}$$

Equations (113)–(115) can be expressed as a function of the KGE field by using the relations (48), (84)–(86). Actually, the hydrodynamic approach has been used as a "Trojan horse" to find the GE where the non-physical states are implicitly excluded by writing it as a function of the KGE field (i.e., we do not need to impose the irrotational condition).

The main difference with the Einstein equation is given by the terms $\alpha_{(V_{qu})}$, $\boldsymbol{\Delta\tau}_{\mu\nu_{stress}}$,$\tau_{\mu\nu_{mix}}$ and $\tau_{\mu\nu_{curv}}$ whose quantum-mechanical origin can be noticed by passing to the macroscopic classical scale being $lim_{\hbar\to 0}\alpha_{(V_{qu})} = 0$, $lim_{macro}\boldsymbol{\Delta\tau}_{\mu\nu_{stress}} = 0$, $lim_{macro}\tau_{\mu\nu_{curv}} = 0$ and $lim_{macro}\tau_{\mu\nu_{mix}} = 0$.

Finally, it must be noted that Equation (114) represents the decoherent limit of (115).

4. Perturbative Approach to the GE–KGE System

For particles very far from the Planckian mass density $\frac{m_p}{l_p{}^3} = \frac{c^5}{\hbar G^2}$, it is possible to solve the system of the equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{8\pi G}{c^4}\Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}|\psi|^2 Tr\left(\boldsymbol{\tau}_{\mu\nu_{class}(\psi)} - \boldsymbol{\Lambda}_Q g_{\mu\nu(\psi)} + \boldsymbol{\Delta\tau}_{\mu\nu_{stress}(\psi)}\right) \tag{122}$$

$$(g^{\mu\nu}\partial_\nu\psi)_{;\mu} = -\frac{m^2c^2}{\hbar^2}\psi \tag{123}$$

by a perturbative iteration,

$$R_{\mu\nu(\varepsilon_{\mu\nu})} \cong R^{(0)}{}_{\mu\nu(\varepsilon_{\mu\nu})} + R^{(1)}{}_{\mu\nu(\varepsilon_{\mu\nu})} + R^{(2)}{}_{\mu\nu(\varepsilon_{\mu\nu})} + \ldots. \tag{124}$$

$$\psi = \psi_0 + \psi' + \psi'' + \ldots.. \tag{125}$$

$$g_{\nu\mu} = \eta_{\nu\mu} + \varepsilon_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \ldots \tag{126}$$

($\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = |\varepsilon|^2 << 1$) where $\eta_{\nu\mu}$ satisfies the static solution $R^{(0)}_{\mu\nu}{}_{(\eta_{\nu\mu})} = 0$,$R^{(0)}{}_{(\eta_{\nu\mu})} = 0$, of the zero order GE

$$R^{(0)}_{\mu\nu} - \frac{1}{2}R^{(0)} = 0 \tag{127}$$

ψ_0 is the solution of the zero order KGE,

$$\partial_\mu\partial^\mu\psi_0 = -\frac{m^2c^2}{\hbar^2}\psi_0 \tag{128}$$

ψ' the solution of the first order KGE,

$$\partial_\mu\partial^\mu\psi' + \frac{m^2c^2}{\hbar^2}\psi' = -\partial_\mu h^{(1)\mu\nu}\partial_\nu\psi \cong -\partial_\mu h^{(1)\mu\nu}\partial_\nu\psi_0 \tag{129}$$

(at first order $\varepsilon_{\mu\nu} = h^{(1)}_{\mu\nu}$) and $h^{(1)}_{\mu\nu}$ is the solution of the first order GE,

$$R^{(1)}_{\mu\nu}{}_{(h^{(1)}{}_{\mu\nu})} - \frac{1}{2}R^{(1)}_{\mu\nu}{}_{(h^{(1)}{}_{\mu\nu})}g_{\mu\nu} + \frac{8\pi G}{c^4}\Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}|\psi|^2 Tr\left(\boldsymbol{\tau}^{(0)}_{\mu\nu_{class}} - \boldsymbol{\Lambda}^{(0)}_Q g_{\mu\nu} + \boldsymbol{\Delta\tau}^{(0)}_{\mu\nu_{stress}}\right) \tag{130}$$

where the Christoffel symbol reads [22]:

$$\Gamma^\alpha_{\nu\mu} = \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\varepsilon_{\beta\nu} + \partial_\nu\varepsilon_{\beta\mu} - \partial_\beta\varepsilon_{\nu\mu}) \tag{131}$$

leading to [22]:

$$R_{\mu\nu(h_{\mu\nu})} = \left(\partial_l\Gamma^l_{\mu\nu} - \partial_\nu\Gamma^l_{\mu l} + \Gamma^l_{\mu\nu}\Gamma^m_{lm} - \Gamma^m_{\mu l}\Gamma^l_{\nu m}\right) \tag{132}$$

Moreover, by using the zero-order relations $V_{qu_0(k)} = 0$, $\alpha_{(V_{qu_0(k)})} = 0$, $\tau^{(0)}{}_{(k)\,\mu\nu_{mix}} = 0$, the components $\boldsymbol{\tau}^{(0)}_{\mu\nu_{class}}$, $\boldsymbol{\Delta\tau}^{(0)}_{\mu\nu_{stress}}$ and $\boldsymbol{\Lambda}^{(0)}_{Q} g_{\mu\nu}$ respectively read:

$$\begin{aligned}\boldsymbol{\tau}^{(0)}_{\mu\nu_{class}} &= \varsigma_{0(k)} \frac{mc^2}{\gamma} g_{\mu\alpha} u^{\alpha} u^{\nu} \delta_{hk} \\ &= \varsigma_{0(k)} c^2 \frac{i\hbar}{2} g_{\mu\alpha} \frac{\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\alpha} \frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\nu}}{\sqrt{1-\frac{V_{qu_0(k)}}{mc^2}} \frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial t}} \delta_{hk} \\ &= \varsigma_{0(k)} c^2 g_{\mu\alpha} \frac{k^{\alpha} k^{\nu}}{\omega} \delta_{hk}\end{aligned} \tag{133}$$

where,

$$\varsigma_{0(k)} = \varsigma_{(k)_{(\psi 0k)}} \tag{134}$$

and, being $\tau_{\mu\nu_{curv}(\psi_{0(k)})} = 0$,

$$\boldsymbol{\Delta\tau}^{(0)}_{\mu\nu_{stress}} = -\left(\left(\begin{array}{c} \frac{1}{2}\frac{b_k|\psi_{0k}|\exp[\frac{iS_{(k)}}{\hbar}]\left(\begin{array}{c}\left(\alpha_{(V_{qu_0(k)})} L_{(k)class} \Delta_{s\mu\nu(k)}\right) \\ +\tau^{(0)}{}_{s(k)\mu\nu_{mix}}\end{array}\right)}{\sum_j b_j|\psi_j|\exp[\frac{iS_{(j)}}{\hbar}]} \\ +\frac{1}{2}\frac{b^*{}_k|\psi_{0k}|\exp[\frac{-iS_{(k)}}{\hbar}]\left(\begin{array}{c}\left(\alpha_{(V_{qu_0(k)})} L_{(k)class} \Delta_{s\mu\nu(k)}\right) \\ -\tau^{(0)}{}_{s(k)\mu\nu_{mix}}\end{array}\right)}{\sum_j b^*{}_j|\psi_j|\exp[\frac{-iS_{(j)}}{\hbar}]}\end{array}\right) + \tau_{s\mu\nu_{curv}}\right)\delta_{hk} = 0 \tag{135}$$

$$\boldsymbol{\Lambda}^{(0)}_{Q} = -\left(\left(\begin{array}{c} \frac{b_k|\psi_{0k}|e^{[\frac{iS_{(k)}}{\hbar}]}}{2\sum_j b_j|\psi_j|e^{[\frac{iS_{(k)}}{\hbar}]}}\left(\begin{array}{c}\alpha_{(V_{qu_0(k)})}\left(\begin{array}{c}\Lambda_{class} - \frac{\Delta_{\kappa\kappa(k)}}{4} c^2 \frac{i\hbar}{2} g_{\alpha\beta} \\ \frac{\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu_0(k)}}{mc^2}}\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial t}}\end{array}\right) \\ +\frac{\tau^{(0)}{}_{(k)\kappa\kappa mix}}{4}\end{array}\right) \\ +\frac{b^*{}_k|\psi_{0k}|e^{[-\frac{iS_{(k)}}{\hbar}]}}{2\sum_j b^*{}_j|\psi_j|e^{[-\frac{iS_{(k)}}{\hbar}]}}\left(\begin{array}{c}\alpha_{(V_{qu_0(k)})}\left(\begin{array}{c}\Lambda_{class} - \frac{\Delta_{\kappa\kappa(k)}}{4} c^2 \frac{i\hbar}{2} g_{\alpha\beta} \\ \frac{\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu_0(k)}}{mc^2}}\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial t}}\end{array}\right) \\ -\frac{\tau^{(0)}{}_{(k)\kappa\kappa mix}}{4}\end{array}\right)\end{array}\right) + \frac{\tau_{\lambda\lambda_{curv}}}{4}\right)\delta_{hk} = 0 \tag{136}$$

leading to the first-order GE:

$$R_{\mu\nu\,(h^{(1)}_{\mu\nu})} - \frac{1}{2} R_{(h^{(1)}_{\mu\nu})} g_{\mu\nu} + \frac{8\pi G}{c^4} \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} Tr\left(\boldsymbol{\tau}^{(0)}_{\mu\nu_{class}}\right) \tag{137}$$

By making the macroscopic limit of (137) (with $\Lambda = 0$) we obtain:

$$\begin{aligned}\lim_{macro}\left(R_{\mu\nu\,(h^{(1)}_{\mu\nu})} - \tfrac{1}{2} R_{(h^{(1)}_{\mu\nu})} g_{\mu\nu}\right) &= \lim_{\hbar\to 0} \tfrac{8\pi G}{c^4} |\psi_{\tilde{k}}|^2 \tfrac{mc^2}{\gamma} g_{\mu\alpha} u^{\alpha} u^{\nu} \\ R^{macro}_{\mu\nu}{}_{(h^{(1)}_{\mu\nu})} - \tfrac{1}{2} R^{macro}{}_{(h^{(1)}_{\mu\nu})} g_{\mu\nu} &= \tfrac{8\pi G}{c^4} |\psi|^2 \tfrac{mc^2}{\gamma} g_{\mu\alpha} u^{\alpha}_{class} u^{\nu}_{class}\end{aligned} \tag{138}$$

from which we can readily see that the weak gravity limit, on macroscopic scale at the first-order, leads to the Newtonian potential of gravity. The first contribution to the cosmological constant comes from the second order of approximation:

$$R^{(2)}_{\mu\nu}{}_{(\varepsilon_{\mu\nu})} - \frac{1}{2} R^{(2)}{}_{(\varepsilon_{\mu\nu})} g_{\mu\nu} = \frac{8\pi G}{c^4} |\psi|^2 Tr\left(\boldsymbol{\tau}^{(1)}_{\mu\nu_{class}} - \boldsymbol{\Lambda}^{(1)}_{Q} g_{\mu\nu} + \boldsymbol{\Delta\tau}^{(1)}_{\mu\nu_{stress}}\right) \tag{139}$$

where $\varepsilon_{\mu\nu} = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$ and the EITD is calculated by using $\psi\prime$ obtained by (129), where,

$$\Lambda^{(1)}_{Q} = -\left(\left(\begin{array}{c} \frac{b_k|\psi_k|e^{[\frac{iS_{(k)}}{\hbar}]}}{2\sum_j b_j|\psi_j|e^{[\frac{iS_{(k)}}{\hbar}]}}\left(\alpha_{(V_{qu(k)})}\left(\begin{array}{c}\Lambda - \frac{\Delta_{\lambda\lambda(k)}}{4}c^2\frac{i\hbar}{2}g_{\alpha\beta} \\ \frac{\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu(k)}}{mc^2}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial t}}}\end{array}\right) + \frac{\tau_{(k)\lambda\lambda_{mix}}}{4}\right) \\ +\frac{b^*{}_k|\psi_k|e^{[-\frac{iS_{(k)}}{\hbar}]}}{2\sum_j b^*{}_j|\psi_j|e^{[-\frac{iS_{(k)}}{\hbar}]}}\left(\alpha_{(V_{qu(k)})}\left(\begin{array}{c}\Lambda - \frac{\Delta_{\lambda\lambda(k)}}{4}c^2\frac{i\hbar}{2}g_{\alpha\beta} \\ \frac{\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu(k)}}{mc^2}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial t}}}\end{array}\right) - \frac{\tau_{(k)\lambda\lambda_{mix}}}{4}\right)\end{array}\right) + \frac{\tau_{\lambda\lambda_{curv}}}{4}\right)\delta_{hk} \tag{140}$$

where $\psi_k = \psi_{0k} + \psi\prime_k$.

Moreover, being $\lim_{decoherence}\tau_{\mu\nu_{curv}} = 0$, the decoherent (macroscopic) limit (with $\Lambda = 0$) of the GE at the second order reads

$$R^{(2)dec}_{\mu\nu(\varepsilon_{\mu\nu})} - \frac{1}{2}R^{(2)dec}_{(\varepsilon_{\mu\nu})}g_{\mu\nu} = \frac{8\pi G}{c^4}|\psi_{\tilde{k}}|^2\left(\left(1-\alpha^{(1)}_{(\tilde{k})}\right)\frac{mc^2}{\gamma}g_{\mu\alpha}u^\alpha u^\nu + \Lambda^{(1)dec}{}_{(\tilde{k})}{}_{Q}g_{\mu\nu} + \Delta\tau^{(1)dec}_{\mu\nu_{stress}}\right) \tag{141}$$

where,

$$\frac{mc^2}{\gamma}u^\alpha u^\nu = c^2\frac{i\hbar}{2}\frac{\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\nu}}{\sqrt{1-\frac{V_{qu(k)}}{mc^2}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial t}}} = \frac{mc^2}{\gamma}(u^\alpha_{class}u^\nu_{class} + \Delta u^\alpha \Delta u^\nu) \tag{142}$$

where,

$$\frac{mc^2}{\gamma}u^\alpha_{class}u^\nu_{class} = c^2\frac{i\hbar}{2}\frac{\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial q_\nu}}{\sqrt{1-\frac{V_{qu_0(k)}}{mc^2}\frac{\partial \ln[\frac{\psi_{0k}}{\psi_{0k}{}^*}]}{\partial t}}} = c^2\frac{k^\alpha k^\nu}{\omega_k} \tag{143}$$

where, at first order,

$$\begin{aligned}\frac{mc^2}{\gamma}\Delta u^\alpha \Delta u^\nu \quad &\cong c^2\frac{k^\alpha k^\nu}{\omega_k{}^2}\left(-\partial_t\left(\frac{\psi\prime_k}{\psi_{0k}} - \frac{\psi\prime_k{}^*}{\psi_{0k}{}^*}\right) + \frac{\omega_k V^{(1)}_{qu(\psi\prime_k)}}{2mc^2}\right) \\ &+c^2\frac{i\hbar}{2\omega_k}\left(k^\alpha\partial^\nu\left(\frac{\psi\prime_k}{\psi_{0k}} - \frac{\psi\prime_k{}^*}{\psi_{0k}{}^*}\right) + \partial^\alpha\left(\frac{\psi\prime_k}{\psi_{0k}} - \frac{\psi\prime_k{}^*}{\psi_{0k}{}^*}\right)k^\nu\right)\end{aligned} \tag{144}$$

and where,

$$\begin{aligned}\Lambda^{(1)dec}_{(k)}{}_{Q} = \quad & -\alpha^{(1)}_{(V_{qu_1(k)})}\Delta^{(1)}_{\kappa\kappa(k)}g_{\alpha\beta}c^2\frac{i\hbar}{8}\frac{\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\alpha}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu(k)}}{mc^2}\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial t}}} \\ &= -\alpha^{(1)}_{(V_{qu_1(k)})}\frac{\Delta^{(1)}_{\kappa\kappa(k)}}{4}g_{\alpha\beta}\frac{mc^2}{\gamma}u^\alpha u^\beta \\ &= -\alpha^{(1)}_{(V_{qu_1(k)})}\frac{\Delta^{(1)}_{\kappa\kappa(k)}}{4}g_{\alpha\beta}\frac{mc^2}{\gamma}\left(u^\alpha_{class}u^\beta_{class} + \Delta u^\alpha \Delta u^\beta\right) \\ &\cong -\alpha^{(1)}_{(V_{qu_1(k)})}\frac{\Delta^{(1)}_{\kappa\kappa(k)}}{4}g_{\alpha\beta}\frac{mc^2}{\gamma}u^\alpha_{class}u^\beta_{class} \\ &\cong -\alpha^{(1)}_{(V_{qu_1(k)})}\Delta^{(1)}_{\kappa\kappa(k)}g_{\alpha\beta}c^2\frac{k^\alpha k^\beta}{\omega_k}\end{aligned} \tag{145}$$

and

$$\begin{aligned}
\Delta\tau_{\mu\nu_{stress}}^{(1)dec} &= -\alpha_{(V_{qu_1(k)})}^{(1)} \Delta_{s_{\mu\nu(k)}}^{(1)} g_{\alpha\beta} c^2 \frac{i\hbar}{2} \frac{\frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\alpha} \frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu(k)}}{mc^2}} \frac{\partial \ln[\frac{\psi_k}{\psi_k{}^*}]}{\partial t}} \\
&= -\alpha_{(V_{qu_1(k)})}^{(1)} \Delta_{s_{\mu\nu(k)}}^{(1)} g_{\alpha\beta} \frac{mc^2}{\gamma} u^\alpha u^\beta \\
&= -\alpha_{(V_{qu_1(k)})}^{(1)} \Delta_{s_{\mu\nu(k)}}^{(1)} g_{\alpha\beta} \frac{mc^2}{\gamma} \left(u_{class}^\alpha u_{class}^\beta + \Delta u^\alpha \Delta u^\beta \right) \\
&\cong -\alpha_{(V_{qu_1(k)})}^{(1)} \Delta_{s_{\mu\nu(k)}}^{(1)} g_{\alpha\beta} \frac{mc^2}{\gamma} u_{class}^\alpha u_{class}^\beta \\
&\cong -\alpha_{(V_{qu_1(k)})}^{(1)} \Delta_{s_{\mu\nu(k)}}^{(1)} g_{\alpha\beta} c^2 \frac{k^\alpha k^\beta}{\omega_k}
\end{aligned} \tag{146}$$

If the macroscopic GE (138) and the Einstein equation of the general relativity coincide themselves at first order and the Newtonian gravity is purely classic, the second order GE (141) contains contributions (among those the cosmological isotropic pressure $\Lambda_{(k)\ Q}^{(1)}$) that go to zero if $\hbar$ is set to zero so that $\Lambda_{(k)\ Q}^{(1)}$ actually is the macroscopic quantum-mechanical contribution to the Newtonian gravity.

It is worth mentioning that the macroscopic GE shows the additional contribution $\Delta\tau_{\mu\nu_{stress}}^{(1)dec}$ to the cosmological isotropic pressure. The dependence of such term by both $\alpha_{(V_{qu_1(k)})}^{(1)}$ and $\Delta_{s_{\mu\nu(k)}}^{(1)}$, that become relevant in very high-curvature space-time, suggests that this term gives detectable effects near the big black holes at the center of the galaxies.

It is noteworthy that the EIT stress component $\Delta\tau_{\mu\nu_{stress}}$, that leads to a non-zero slip function [23,24], is specific of the (microscopic) quantum-coherent curved space-time but it decays to $\Delta\tau_{\mu\nu_{stress}}^{(1)dec}$ for the decoherence at the macroscopic scale.

5. The 'Cosmological' Pressure Tensor Density (CPTD) Expectation-Value of the Quantum KGE Field

Generally speaking, when the KGE field is quantized, the EITD on the right side of the GE becomes a quantum operator, and thence also the Ricci's tensor (as well as the metric tensor) of the GE, on the left side, become quantum operators. As can be easily shown for pure gravity [25], the commutating rules for the KGE field quantization fixes the commuting relations for the metric tensor.

At zero order, the GE equation leads to a Minkowskian KGE field and, hence, when the KGE field is quantized, the standard QFT outputs are obtained.

If at zero order the GE is decoupled by the field of the massive KGE, at higher order it is not.

The quantization of the GE–KGE system of equations is not the goal of this work, nevertheless, it is interesting to evaluate the "cosmological" pressure tensor density (CPTD) expectation value of the vacuum in order to evaluate if it can lead to the lowering of the theoretical value of the CC on cosmological scale and can help to solve the problem of the disagreement of the QFT with the experimental observations.

In order to evaluate the macroscopic cosmological constant of the quantum KGE field (i.e., at the zero order Minkowskian limit of the GE–KGE system of equations for the ordinary QFT) we need to calculate the expectation value $< 0_k|\lim_{decoherence} \Lambda_Q g_{\mu\nu}|0_k >=< 0_k|\Lambda_{(k)Q}^{(1)dec} g_{\mu\nu}|0_k >$. To this end, we need to express the quantum potential as a function of the annihilation and creation operators $a_{(k)}$ and $a^\dagger{}_{(k)}$ of the Fourier decomposition of the free KGE quantum field:

$$\psi = \iint \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left(a_{(k)} \exp[ik_\alpha q^\alpha] + a^\dagger{}_{(k)} \exp[-ik_\alpha q^\alpha] \right) \tag{147}$$

that, by using the discrete form of field Fourier decomposition (i.e., $\iint \frac{d^3k}{(2\pi)^3} \rightarrow \frac{1}{\sqrt{V}} \sum_{k=0}$) reads:

$$\psi = \frac{1}{\sqrt{V}} \sum_{k=0}^{\infty} \frac{1}{2\omega_k} \left(a_{(k)} \exp[\frac{ip_\alpha q^\alpha}{\hbar}] + a^\dagger{}_{(k)} \exp[-\frac{ip_\alpha q^\alpha}{\hbar}] \right) \tag{148}$$

where $\frac{p_\mu}{\hbar} = k_\mu$, where the identities,

$$a(k) \equiv 2\omega_k b(k) |\psi_k| \tag{149}$$

and,

$$a^\dagger(k) \equiv 2\omega_k b(-k) |\psi_{-k}| \tag{150}$$

can be established with notation in (17) (where for $k < 0 \Rightarrow$ both $a(k) \rightarrow a^\dagger(k)$ and $a^\dagger(k) \rightarrow a(k)$) and leads to:

$$\begin{aligned} V_{qu_0(k>0)} &= -\frac{\hbar^2}{m} \frac{1}{|\psi_k|\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu |a(k) \exp[ik_\alpha q^\alpha]| \\ &= -\frac{\hbar^2}{m} \frac{1}{|\psi_k|} \partial_\mu \partial^\mu |a(k) \exp[ik_\alpha q^\alpha]| \\ &= -\frac{\hbar^2}{m} \frac{1}{|\psi_k|} \partial_\mu \partial^\mu \sqrt{a(k) a^\dagger(k)} = V_{qu_0(k<0)} \end{aligned} \tag{151}$$

where, now, $a(k), a^\dagger(k)$ are quantum operators obeying to the commutation relations:

$$[a(k), a^\dagger(k\prime)] = \delta_{kk\prime} \tag{152}$$

However, even if the squared root of operators in the quantum potential (151) can be defined by making use of the Taylor expansion series, the higher order terms of such expansion possess the ordering problem of the quantum operators. In order to remark this freedom in the definition of the quantum potential operator, we name it as $\hat{V}_{qu}^{Q-ord}$ and reads:

$$\hat{V}_{qu}^{Q-ord} = -\frac{\hbar^2}{m} \frac{1}{(|\psi_k|)^{Q-ord} \sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu (|\psi_k|)^{Q-ord} \tag{153}$$

leading to the Minkowskian limit:

$$\hat{V}_{qu0}^{Q-ord} = -\frac{\hbar^2}{m} \frac{1}{\left(\sqrt{a(k) a^\dagger(k)}\right)^{Q-ord} \sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \left(\sqrt{a(k) a^\dagger(k)}\right)^{Q-ord} \tag{154}$$

Moreover, by using (11) and the identities $p_\mu = -\partial_\mu S = (p_0, -p_i)$ and $p^2 = p_i p_i$, it follows that:

$${\omega_k}^2 = \frac{c^2 p^2}{\hbar^2} + \frac{m^2 c^4}{\hbar^2} \left(1 - \frac{\hat{V}_{qu}^{Q-ord}}{mc^2} \right) \tag{155}$$

and, by using (118), being $\lim_{decoherence} \tau_{\mu\nu_{mix}} = 0$ and $\lim_{decoherence} \tau_{\mu\nu_{curv}} = 0$, the macroscopic decoherent CPTD reads:

$$\lim_{decoherence} \widetilde{\Lambda}_{Q} = \widetilde{\Lambda}_{(k)_Q} = -\alpha_{(\hat{V}_{qu0}^{Q-ord})} \left(-\frac{\Delta_{\kappa\kappa(k)}}{4} c^2 g_{\alpha\beta} \frac{k_\alpha k_\beta}{\omega_k \sqrt{1 - \frac{\hat{V}_{qu0}^{Q-ord}}{mc^2}}} \right) \tag{156}$$

whose expectation value reads:

$$< 0_k | \widetilde{\Lambda}_{(k)_Q} | 0_k >= \frac{1}{(2\pi)^2 \hbar c} \int^{\omega_{k_{\max}}} \widetilde{\Lambda}^{(0)dec}{}_{(k)_Q} \sqrt{\frac{\hbar^2 {\omega_k}^2}{c^2} - m^2 c^2 d\omega_k} \tag{157}$$

where $|0_k>$ represents the k-indexed harmonic oscillators of the field in the fundamental state [19].

Moreover, being in the Minkowskian limit $\partial^\mu \left(\sqrt{a(k)a^\dagger(k)}\right)^{Q-ord} = 0$, it follows that:

$$\hat{V}_{qu0}^{Q-ord} = 0 \tag{158}$$

$$\alpha_{(\hat{V}_{qu0}^{Q-ord})} = 0 \tag{159}$$

To

$$\Delta_{\kappa\kappa(k)} = 0 \tag{160}$$

To

$$\widetilde{\Lambda}_{(k)_Q} = 0 \tag{161}$$

To

$$< 0_p|\widetilde{\Lambda}_{(k)_Q}|0_p >= 0 \tag{162}$$

and, finally, from (155), to,

$$\omega_k{}^2 = \frac{c^2 p^2}{\hbar^2} + \frac{m^2 c^4}{\hbar^2} = c^2 k^2 + \frac{m^2 c^4}{\hbar^2}. \tag{163}$$

From the last identity, the standard QFT outputs are warranted at the zero order of approximation.

6. Discussion

The hydrodynamic representation of the KGE field makes it equivalent to a mass distribution $|\psi|^2$ submitted to the non-local quantum potential that leads to the GE (115).

The basic assumption of the presented theory is that such hydrodynamic representation of the KGE field (restricted to irrotational states) owns a physical reality since, as shown by the Aharonov–Bohm effect, the quantum potential can be detected experimentally. Therefore, for the basic principle of general relativity, the (kinetic) energy of the quantum potential contributes to the curvature of space-time. On the basis of this postulate, the quantum-mechanical non-local effects come into the gravity leading to the theoretical appearance of the CPTD Λ_Q in the GE.

In the classical treatment, the Einstein equation for massive particles is not coupled to any field, but just to the energy impulse tensor of classical bodies and does not contain any information about how fields couple with it.

The GE (115) is analytically coupled to the KGE field. The GE–KGE system can be further quantized leading to a quantum gravity theory derived by an analytically field-defined Einstein–Hilbert action.

6.1. Analogy with Brans-Dicke Gravity

The output the work highlights an interesting analogy with Brans-Dicke [13] gravity that solves the problem of the cosmological constant [26] as well as those of the inflaction [27] and dark energy [28]. If we look in detail to the EITD $|\psi|^2 \boldsymbol{\tau}_{\mu\nu_{class}}$ of (115), the macroscopic limit,

$$\lim_{macro} |\psi|^2 \boldsymbol{\tau}_{\mu\nu_{class}} = T_{(\widetilde{k})\mu\nu_{class}} \tag{164}$$

leads to:

$$\begin{aligned} \tfrac{8\pi G}{c^4} T_{\mu\nu_{class}} &= \tfrac{8\pi G}{c^4} |\psi|^2 c^2 \left(\tfrac{\partial S}{\partial t}\right)^{-1} \left(g^{\mu\beta} \tfrac{\partial S}{\partial q^\beta} \tfrac{\partial S}{\partial q^\nu}\right) \\ &= -\tfrac{8\pi G}{c^4} |\psi|^2 \tfrac{\hbar^2 c^2}{4E} \left(g^{\mu\beta} \left(\tfrac{\partial_\beta \psi}{\psi} - \tfrac{\partial_\beta \psi^*}{\psi^*}\right) \left(\tfrac{\partial_\nu \psi}{\psi} - \tfrac{\partial_\nu \psi^*}{\psi^*}\right)\right) \\ &= -\tfrac{8\pi}{c^4} \tfrac{G}{\psi^2} |\psi|^2 \tfrac{\hbar^2 c^2}{4E} \left(g^{\mu\beta} \partial_\beta \psi \partial_\nu \psi + \ldots\ldots\ldots + \left(\tfrac{\psi}{\psi^*}\right)^2 g^{\mu\beta} \partial_\beta \psi^* \partial_\nu \psi^*\right) \end{aligned} \tag{165}$$

showing it to be composed by terms that are of the form of those contained in Brans-Dicke gravity [13] in the absence of external potentials.

If is worth noting that the hydrodynamic gravity, gives theoretical support to the Brans-Dicke effective gravitational constant $G_{eff} = \frac{G}{\psi^2}$.

6.2. The GE and Quantum Gravity

Even if the quantization of the KGE field in the curved space defined by the GE (115) is not treated in this work, the inspection of some features of quantum gravity to the light of the GE (115) deserves a mention.

Since the action of the GE (115) is basically given by the standard Einstein–Hilbert action plus terms stemming from the energy of the non-local quantum potential of massive KGE, the outputs of the quantum "pure gravity" practically remains almost valid.

As shown in [10], one interesting aspect of quantum pure gravity is that the vacuum does not make a transition to the collapsed branched polymer phase, if an even small cosmological constant (i.e., $\Lambda_{class} \neq 0$) is present. With respect to this fact, the presence of the term Λ_Q in the GE (115), in principle, allows to work with the assumption $\Lambda_{class} = \Lambda = 0$ with the GE that reads:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{8\pi G}{c^4}Tr\left(\widetilde{\mathbf{\Lambda}}_Q\right)g_{\mu\nu} = \frac{8\pi G}{c^4}Tr\left(\boldsymbol{\tau}_{\mu\nu_{class}} + \boldsymbol{\Delta\tau}_{\mu\nu_{stress}}\right) \quad (166)$$

Moreover, since matter itself makes space-time curved and, hence, $\alpha_{(V_{qu})} \neq 0$, $\boldsymbol{\Delta\tau}_{\mu\nu_{stress}} \neq 0$ and,

$$\widetilde{\mathbf{\Lambda}}^{Q-ord}_{Q(k>0\nearrow,k<0\searrow)} = -\frac{1}{2}\left(\left(\begin{array}{c} \frac{\frac{e^{\left[\frac{iS_{(k)}}{\hbar}\right]}}{\omega_k}\substack{\nearrow a_k \\ \searrow a_k^\dagger}}{\sum\limits_{k\prime=0}^{\infty}\frac{a_{k\prime}}{\omega_{k\prime}}e^{\left[\frac{iS_{(k\prime)}}{\hbar}\right]}+\frac{a^\dagger{}_{k\prime}}{\omega_{k\prime}}e^{\left[\frac{iS_{(-k\prime)}}{\hbar}\right]}}\left(\alpha\left(\frac{-\frac{\Delta_{\kappa\kappa(k)}}{4}c^2 g_{\alpha\beta}\frac{\partial S_{(k)}}{\partial q_\alpha}\frac{\partial S_{(k)}}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\frac{\partial S_{(k)}}{\partial t}}\right)+\frac{\tau_{(k)\kappa\kappa_{mix}}}{4}\right) \\ +\frac{\frac{e^{\left[-\frac{iS_{(k)}}{\hbar}\right]}}{\omega_k}\substack{\nearrow a_k \\ \searrow a_k^\dagger}}{\sum\limits_{k\prime=0}^{\infty}\frac{a^\dagger{}_{k\prime}}{\omega_{k\prime}}e^{\left[-\frac{iS_{(k\prime)}}{\hbar}\right]}+\frac{a_{k\prime}}{\omega_{k\prime}}e^{\left[-\frac{iS_{(-k\prime)}}{\hbar}\right]}}\left(\alpha\left(\frac{-\frac{\Delta_{\kappa\kappa(k)}}{3}c^2 g_{\alpha\beta}\frac{\partial S_{(k)}}{\partial q_\alpha}\frac{\partial S_{(k)}}{\partial q_\beta}}{\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\frac{\partial S_{(k)}}{\partial t}}-\frac{\tau_{(k)\kappa\kappa_{mix}}}{4}\right)\right) \end{array}\right)+\frac{\tau_{\lambda\lambda_{curv}}}{4}\right)^{Q-ord}\delta_{hk} \quad (167)$$

it follows that matter itself stabilizes the vacuum in the physical strong gravity phase [10] as we perceive it.

On the other hand, under this hypothesis, a perfect Minkowskian vacuum (i.e., without matter) will make transition to the unphysical collapsed branched polymer phase with no sensible continuum limit [10,11], leading to no-space and no-time as we experience.

6.3. Check of the Hydrodynamic GE

If in general relativity the energy-impulse tensor density for classical bodies [22] is defined only with a point-dependence by the mass density, for the electromagnetic (EM) field, the EITD is defined as a function of the EM field itself [22].

On this basis, since the photon is a boson (obeying the a KGE), we can make a direct check of the theory by comparing the known EITD EM expression:

$$T_{\nu\mu\ em} = \frac{1}{4\pi}\left(-F_{\mu\lambda}F_\nu{}^\lambda + \frac{1}{4}F_{\lambda\gamma}F^{\lambda\gamma}g_{\mu\nu}\right) \quad (168)$$

with the EITD (3.6) for a boson field given by the quantum hydrodynamic gravity.

In fact, given the plane wave of the vector potential, in the Minkowskian case, for the photon (e.g., linearly polarized):

$$A_k = A_0 \exp[-ik_\mu q^\mu] = A_{0\gamma} \exp[i \frac{S_{(k)}}{\hbar}] \tag{169}$$

being $|\psi_k|^2$ the number of particles (i.e., photons) per volume, the KGE field ψ_k reads:

$$\psi_k \propto \frac{A_k}{c} \sqrt{\frac{\omega}{\hbar}} \left[l^{-\frac{3}{2}} \right] \tag{170}$$

and, being $V_{qu(k)} = 0$, $\alpha_{(V_{qu(k)})} = 0$, and $L = -g_{\mu\alpha} k^\alpha k^\mu = 0$, we obtain,

$$\boldsymbol{\tau}_{\mu\nu_{class}} = T_{(\widetilde{k})\mu\nu_{class}} \tag{171}$$

$$\boldsymbol{\Lambda}_{\mu\nu_Q} = -\frac{|\psi_k|^2}{2} \left(\frac{\tau_{(k)\kappa\kappa_{mix}}}{4} - \frac{\tau_{(k)\kappa\kappa_{mix}}}{4} \right) g_{\mu\nu} \delta_{hk} = 0 \tag{172}$$

$$\boldsymbol{\Delta\tau}_{\mu\nu_{stress}} = -\frac{|\psi_k|^2}{2} \left(\tau_{s\,(k)_{\mu\nu_{mix}}} - \tau_{s\,(k)_{\mu\nu_{mix}}} \right) \delta_{hk} = 0 \tag{173}$$

from which the EITD for the photon reads:

$$\begin{aligned} T_{\mu\nu} &= T_{(k)\mu\nu_{class}} = |\psi_k|^2 \mathrm{T}_{(k)\mu\nu_{class}} \\ &= |\psi_k|^2 c^2 \left(\frac{\partial S_{(k)}}{\partial t} \right)^{-1} \left(\frac{\partial S_{(k)}}{\partial q^\mu} \frac{\partial S_{(k)}}{\partial q^\nu} - g_{\alpha\beta} \frac{\partial S_{(k)}}{\partial q_\beta} \frac{\partial S_{(k)}}{\partial q^\alpha} g_{\mu\nu} \right) \propto |A|^2 k_\mu k_\nu \end{aligned} \tag{174}$$

that, compared to the output of (168) for the photon [22],

$$T_{\nu\mu\ em} = \frac{|E|^2}{4\pi} \frac{c^2}{\omega^2} k_\mu k_\nu = \frac{|A|^2}{4\pi} k_\mu k_\nu \tag{175}$$

leads to:

$$T_{\mu\nu} \propto 4\pi T_{\nu\mu\ em} \tag{176}$$

6.4. Experimental Tests

The results (161) and (162) basically show that in Minkowskian space-time (i.e., a vacuum very far from particles) the CPTD expectation value is null regardless its zero-point energy density.

The non-zero contribution to the CC will appear at second order in the GE deriving from the first order ψ' of the quantized KGE field. In a qualitative way, the CPTD macroscopic expectation value is vanishing in the region of space-time with Newtonian gravity and it increases at higher gravity as quantum-mechanical corrections. This fact agrees well [19] with the very small value of the observed CC and leads to a scenario where the major contribution to the CC comes from black holes (this is due to the high mass density of a black hole where the matter is so squeezed that the quantum potential energy becomes comparable with the mass energy itself so that $V_{qu} \approx mc^2$ [29] and $\alpha_{(V_{qu(k)})} \approx \max\left\{ \alpha_{(V_{qu(k)})} \right\} \approx 1$, the decoherent quantum-mechanical corrections to Newtonian gravity $\alpha^{(1)}_{(\widetilde{k})} \frac{mc^2}{\gamma} g_{\mu\alpha} u^\alpha_{class} u^\nu_{class}$, $\Lambda^{(1)dec}_{(\widetilde{k})_Q} g_{\mu\nu}$ and $\Delta\tau^{(1)dec}_{\mu\nu_{stress}}$ in (141) must be primarily detectable in the motion of stars around the big black holes at the center of the galaxies., in the rotation of twin neutron stars and in the inter-galactic interaction.

Finally, it is worth mentioning that the interferometric detection of the gravitational waves represents an experimental technique whose angular and frequency-dependent response functions can discriminate among the existing theories of gravity [30].

7. Conclusions

The quantum hydrodynamic representation of the Klein–Gordon equation, describing the evolution of mass density $|\psi|^2$ owing to the hydrodynamic moment $\partial_\mu S = -p_\mu$ and subject to the quantum potential, has been used to derive the correspondent gravity equation, defining the geometry of space-time, by using the minimum action principle. The gravity equation associated to the KGE field takes into account the gravitational effects of the energy of the non-local quantum potential.

The hydrodynamic approach has three main properties:

1. The energy-impulse tensor of the GE is written as a function of the KGE field;
2. In the classical limit, the GE leads to that of Einstein;
3. If we apply the EITD of the GE to the photon field (that is a boson described by the KGE) we obtain the EITD of the EM theory.

The self-generation of the CPTD $Tr\left(\mathbf{\Lambda}_Q\right)$ leads to the attractive hypothesis that the matter itself generates the physical stable vacuum phase in which it is embedded.

The paper shows that the macroscopic CPTD $\widetilde{\Lambda}_{(k)_Q}$ is not null if, and only if, space-time is curvilinear (due to the presence of localized mass) and it tends to zero in the very far flat vacuum regardless the zero-point energy of the vacuum.

The depletion of the CPTD in the vacuum, far from material bodies, lowers its mean value on a cosmological scale so that it can possibly agree with the astronomical observations on the motion of the galaxies.

The GE of the classical KGE field shows that the CPTD $\widetilde{\Lambda}_{(k)_Q}$ and other out-diagonal components of the EITD can be considered as "decoherent quantum-mechanical" gravitational effects generated in highly curved space-time near dense matter such as black holes and neutron stars.

The hydrodynamic gravity model defines a coupling between the boson field of the free KGE and the GE in a form that, to some degree, mimics Brans-Dicke gravity leading to an effective gravitational constant inversely proportional to the field squared.

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References

1. Rugh, S.E.; Zinkernagel, H. The quantum vacuum and the Cosmological Constant Problem. *arXiv* **2000**, arXiv:0012253v1.
2. Barcelò, C. Revisiting the semiclassical gravity scenario for gravitational collapse. *arXiv* **2009**, arXiv:0909.4157v1.
3. Hawking, S.W. Particle creation by black holes. *Commun. Math. Phys.* **1975**, *43*, 199–220. [CrossRef]
4. Parikh, M.K.; Wilczek, F. Hawking radiation as tunneling. *Phys. Rev. Lett.* **2000**, *85*, 5042–5045. [CrossRef] [PubMed]
5. Banerjee, R.; Majhi, B.R. Hawking black body spectrum from tunneling mechanism. *Phys. Lett. B* **2009**, *675*, 243–245. [CrossRef]
6. Corda, C. Quasi-Normal Modes: The "Electrons" of Black Holes as "Gravitational Atoms"? Implications for the Black Hole Information Puzzle. *Adv. High Energy Phys.* **2015**, 1–16. [CrossRef]
7. Ortiz, M.E. Black Holes, the Wheeler-DeWitt equation and the semiclassical Approximation. *arXiv* **1994**, arXiv:9410030v1.
8. Einstein, A. Zum kosmologischen Problem der allgemeinen Relativit¨atstheorie. *Sitzungsber. Preuss. Akad. Wiss.* **1931**, *142*, 235–237.
9. Carroll, S.M.; Press, W.H.; Turner, E.L. The Cosmological Constant. *Ann. Rev. Astron. Astrophys.* **1992**, *30*, 499–542. [CrossRef]
10. Hamber, H.W. Gravitational scaling dimensions. *Phys. Rev. D* **2000**, *61*, 124008. [CrossRef]

11. Hamber, H.W.; Williams, R.M. Higher derivative quantum gravity on a simplicial lattice. *Nucl. Phys. B* **1984**, *248*, 392.
12. Thiemann, T. Introductin to Modern Canonical Quantum General Relativity. *arXiv* **2001**, arXiv:0110034v1.
13. Pakravan, J.; Takook, M.V. Thermodynamics of nonlinearly charged black holes in the Brans-Dicke modified gravity. *J. Theor. Appl. Phys.* **2018**, *12*, 147–157. [CrossRef]
14. Hamber, H.W. Vacuum Condensate Picture of Quantum Gravity. *Symmetry* **2019**, *11*, 87. [CrossRef]
15. Madelung, E.Z. Quantentheorie in hydrodynamischer form. *Zeitschrift für Physik A Hadrons and Nuclei* **1926**, *40*, 322–326. [CrossRef]
16. Bohm, D. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I. *Phys. Rev.* **1952**, *85*, 166. [CrossRef]
17. Birula, I.B.; Cieplak, M.; Kaminski, J. *Theory of Quanta*; Oxford University Press: New York, NY, USA, 1992; pp. 87–115.
18. Jánossy, L. Zum hydrodynamischen Modell der Quantenmechanik. *Z. Phys.* **1962**, *169*, 79. [CrossRef]
19. Chiarelli, P. The hydrodynamic Gravity of the Classical Klein-Gordon Field. *arXiv* **2017**, arXiv:1711.06093.
20. Chiarelli, P. The Classical Mechanics from the Quantum Equation. *Phys. Rev. Res. Int.* **2013**, *3*, 1–9.
21. Chiarelli, P. Can fluctuating quantum states acquire the classical behavior on large scale? *J. Adv. Phys.* **2013**, *2*, 139–163.
22. Landau, L.D.; Lifšits, E.M. Course of Theoretical Physics Italian Edition. *Mir Mosca Editori Riuniti* **1976**, 335–363.
23. Ma, C.-P.; Bertschinger, E. Cosmological Perturbation Theory in the Synchronous and Conformal Newtonian Gauges. *Astrophys. J.* **1995**, *455*, 7. [CrossRef]
24. Bernardeau, F.; Colombi, S.; Gaztanaga, E.; Scoccimarro, R. Large-Scale Structure of the Universe and Cosmological Perturbation Theory. *Phys. Rept.* **2002**, *367*, 1. [CrossRef]
25. Chiarelli, P. The Quantization Relations for the Metric Tensor of Gravitons. *Open Access Libr. J.* **2018**, *5*, e4872. [CrossRef]
26. Klebanov, I.R.; Susskind, L.; Banks, T. Wormholes and the cosmological constant. *Nucl. Phys. B* **1989**, *317*, 665–692. [CrossRef]
27. Herrera, R.; Contreras, C.; del Campo, S. The Starobinsky inflationary model in a Jordan–Brans–Dicke-type theory. *Class. Quantum. Grav.* **1995**, *12*, 1937. [CrossRef]
28. Hrycyna, O.; Szydlowski, M. Brans–Dicke theory and the emergence of K CDM model. *Phys. Rev. D* **2013**, *88*, 064018. [CrossRef]
29. Chiarelli, P. The quantum lowest limit to the black hole mass. *Phys. Sci. Int. J.* **2016**, *9*, 1–25. [CrossRef]
30. Corda, C. Interferometric detection of gravitational waves: the definitive test for General Relativity. *J. Mod. Phys. D* **2009**, *18*, 2275. [CrossRef]