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Statistically and Relatively Modular Deferred-Weighted Summability and Korovkin-Type Approximation Theorems

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Abstract: The concept of statistically deferred-weighted summability was recently studied by Srivastava et al. (Math. Methods Appl. Sci. **41** (2018), 671–683). The present work is concerned with the deferred-weighted summability mean in various aspects defined over a modular space associated with a generalized double sequence of functions. In fact, herein we introduce the idea of relatively modular deferred-weighted statistical convergence and statistically as well as relatively modular deferred-weighted summability for a double sequence of functions. With these concepts and notions in view, we establish a theorem presenting a connection between them. Moreover, based upon our methods, we prove an approximation theorem of the Korovkin type for a double sequence of functions on a modular space and demonstrate that our theorem effectively extends and improves most (if not all) of the previously existing results. Finally, an illustrative example is provided here by the generalized bivariate Bernstein–Kantorovich operators of double sequences of functions in order to demonstrate that our established theorem is stronger than its traditional and statistical versions.

Keywords: statistical convergence; P-convergent; statistically and relatively modular deferred-weighted summability; relatively modular deferred-weighted statistical convergence; Korovkin-type approximation theorem; modular space; convex space; $\mathcal{N}$-quasi convex modular; $\mathcal{N}$-quasi semi-convex modular

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1. Introduction, Preliminaries, and Motivation

The gradual evolution on sequence spaces results in the development of statistical convergence. It is more general than the ordinary convergence in the sense that the ordinary convergence of a sequence requires that almost all elements are to satisfy the convergence condition, that is, every element of the sequence needs to be in some neighborhood (arbitrarily small) of the limit. However, such restriction is relaxed in statistical convergence, where set having a few elements that are not in the neighborhood of the limit is discarded subject to the condition that the natural density of the set is zero, and at the same time the condition of convergence is valid for the other majority of the elements. In the year 1951, Fast [1] and Steinhaus [2] independently studied the term statistical convergence for single real sequences; it is a generalization of the concept of ordinary convergence. Actually, a root of the notion of statistical convergence can be detected by Zygmund (see [3], p. 181), where he used the term "almost convergence", which turned out to be equivalent to the concept of statistical convergence. We also find such concepts in random graph theory (see [4,5]) in the sense that almost convergence means convergence with probability 1, whereas in statistical convergence the probability is not necessarily 1. Mathematically, a sequence of random variables $\{X_n\}$ is statistically convergent (converges in probability) to a random variable X if $\lim_{n\to\infty} P(|Xn - X| \geqq \epsilon) = 0$, for all $\epsilon > 0$ (arbitrarily small); and almost convergent to X if $P(\lim_{n\to\infty} X_n = X) = 1$.

For different results concerning statistical versions of convergence as well as of the summability of single sequences, we refer to References [1,2,6].

Let $\mathbb{N}$ be the set of natural numbers and let $\mathcal{H} \subseteq \mathbb{N}$. Also let

$$\mathcal{H}_n = \{k : k \leqq n, \text{ and } k \in \mathcal{H}\}$$

and suppose that $|\mathcal{H}_n|$ is the cardinality of $\mathcal{H}_n$. Then, the *natural density* of $\mathcal{H}$ is defined by

$$\delta(\mathcal{H}) = \lim_{n\to\infty} \frac{|\mathcal{H}_n|}{n} = \lim_{n\to\infty} \frac{1}{n}\{k : k \leqq n \text{ and } k \in \mathcal{H}\},$$

provided that the limit exists.

A sequence (x_n) is *statistically convergent* to ℓ if for every $\epsilon > 0$,

$$\mathcal{H}_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - \ell| \geqq \epsilon\}$$

has zero natural (asymptotic) density (see [1,2]). That is, for every $\epsilon > 0$,

$$\delta(\mathcal{H}_\epsilon) = \lim_{n\to\infty} \frac{|\mathcal{H}_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leqq n \quad \text{and} \quad |x_k - \ell| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} x_n = \ell.$$

As an extension of statistical versions of convergence, the idea of weighted statistical convergence of single sequences was presented by Karakaya and Chishti [7], and it has been further generalized by various authors (see [8–12]). Moreover, the concept of deferred weighted statistical convergence was studied and introduced by Srivastava et al. [13] (see also [14–19]).

In the year 1900, Pringsheim [20] studied the convergence of double sequences. Recall that a double sequence $(x_{m,n})$ is convergent (or P-convergent) to a number ℓ if for given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_{m,n} - \ell| < \epsilon$, whenever $m, n \geqq n_0$ and is written as $P \lim x_{m,n} = \ell$. Likewise, $(x_{m,n})$ is bounded if there exists a positive number $\mathcal{K}$ such that $|x_{m,n}| \leqq \mathcal{K}$. In contrast to the case of single sequences, here we note that a convergent double sequence is not necessarily bounded. We further recall that, a double sequence $(x_{m,n})$ is non-increasing in *Pringsheim's sense* if $x_{m+1,n} \leqq x_{m,n}$ and $x_{m,n+1} \leqq x_{m,n}$.

Let $\mathcal{H} \subset \mathbb{N} \times \mathbb{N}$ be the set of integers and let $\mathcal{H}(i,j) = \{(m,n) : m \leqq i \text{ and } n \leqq j\}$. The *double natural density* of $\mathcal{H}$ denoted by $\delta(\mathcal{H})$ is given by

$$\delta(\mathcal{H}) = P \lim_{i,j} \frac{1}{ij}|\mathcal{H}(i,j)|,$$

provided the limit exists. A double sequence $(x_{m,n})$ of real numbers is statistically convergent to ℓ in the *Pringsheim sense* if, for each $\epsilon > 0$

$$\delta(\mathcal{H}_\epsilon(i,j)) = 0,$$

where

$$\delta(\mathcal{H}_\epsilon(i,j)) = \frac{1}{ij}\{(m,n) : m \leqq i, n \leqq j \text{ and } |x_{m,n} - \ell| \geqq \epsilon\}.$$

Here, we write

$$\text{stat}^2 \lim_{m,n} x_{m,n} = \ell.$$

Note that every P-convergent double sequence is stat^2-convergent to the same limit, but the converse is not necessarily true.

Example 1. *Suppose we consider a double sequence $x = (x_{m,n})$ as*

$$x_{m,n} = \begin{cases} \sqrt{nm} & (m = k^2,\ n = l^2;\ \forall\, k, l \in \mathbb{N}), \\ \frac{1}{nm} & \textit{otherwise}. \end{cases}$$

It is trivially seen that, in the ordinary sense $(x_{m,n})$ is not P-convergent; however, 0 is its statistical limit.

Let $\mathcal{I} = [0,\infty) \subseteq \mathbb{R}$, and let the Lebesgue measure v be defined over $\mathcal{I}$. Let $\mathcal{I}^2 = [0,\infty) \times [0,\infty)$ and suppose that $X(\mathcal{I}^2)$ is the space of all measurable real-valued functions defined over $\mathcal{I}^2$ equipped with the equality almost everywhere. Also, let $C(\mathcal{I}^2)$ be the space of all continuous real-valued functions and suppose that $C^\infty(\mathcal{I}^2)$ is the space of all functions that are infinitely differentiable on $\mathcal{I}^2$. We recall here that a functional $\omega : X(\mathcal{I}^2) \to [0,\infty)$ is a *modular* on $X(\mathcal{I}^2)$ such that it satisfies the following conditions:

(i) $\omega(f) = 0$ if and only if $f = 0$, almost everywhere in $\mathcal{I}$ $(\forall\, f \in \mathcal{I}')$,
(ii) $\omega(\alpha f + \beta g) \leqq \omega(f) + \omega(g)$, $\forall\, f, g \in X(\mathcal{I}^2)$ and for any $\alpha, \beta \geqq 0$ with $\alpha + \beta = 1$,
(iii) $\omega(-f) = \omega(f)$, for each $f \in X(\mathcal{I}^2)$, and
(iv) ω is continuous on $[0,\infty)$.

Also, we further recall that a modular ω is

- $\mathcal{N}$-*Quasi convex* if there exists a constant $\mathcal{N} \geqq 1$ satisfying

$$\omega(\alpha f + \beta g) \leqq \mathcal{N}\alpha\omega(\mathcal{N} f) + \mathcal{N}\beta\omega(\mathcal{N} g)$$

  for every $f, g \in X(\mathcal{I}^2)$, $\alpha, \beta \geqq 0$ such that $\alpha + \beta = 1$. Also, in particular, for $\mathcal{N} = 1$, ω is simply called *convex*; and
- $\mathcal{N}$-*Quasi semi-convex* if there exists a constant $\mathcal{N} \geqq 1$ such that

$$\omega(\lambda f) \leqq \mathcal{N}\lambda\omega(\mathcal{N} f)$$

  holds for all $f \in X(\mathcal{I}^2)$ and $\lambda \in (0,1]$.

Also, it is trivial that every $\mathcal{N}$-Quasi semi-convex modular is $\mathcal{N}$-Quasi convex. The above concepts were initially studied by Bardaro et al. [21,22].

We now appraise some suitable subspaces of vector space $X(\mathcal{I}^2)$ under the modular ω as follows:

$$L^\omega(\mathcal{I}^2) = \{f \in X(\mathcal{I}^2) : \lim_{\lambda \to 0^+} \omega(\lambda f) = 0\}$$

and

$$E^\omega(\mathcal{I}^2) = \{f \in L^\omega(\mathcal{I}^2) : \omega(\lambda f) < +\infty,\ \forall\ \lambda > 0\}.$$

Here, $L^{\omega}(\mathcal{I}^2)$ is known as the modular space generated by ω and $E^{\omega}(\mathcal{I}^2)$ is known as the space of the finite elements of $L^{\omega}(\mathcal{I}^2)$. Also, it is trivial that whenever ω is $\mathcal{N}$-Quasi semi-convex,

$$\{f \in X(\mathcal{I}^2) : \omega(\lambda f) < +\infty, \ \ \forall \ \lambda > 0\}$$

coincides with $L^{\omega}(\mathcal{I}^2)$. Moreover, for a *convex modular* ω in $X(\mathcal{I}^2)$, the *F-norm* is given by the formula:

$$\|f\|_{\omega} = \inf \left\{ \lambda > 0 : \omega\left(\frac{f}{\lambda}\right) \leqq 1 \right\}.$$

The notion of modular was introduced in [23] and also widely discussed in [22].

In the year 1910, Moore [24] introduced the idea of the relatively uniform convergence of a sequence of functions. Later, along similar lines it was modified by Chittenden [25] for a sequence of functions defined over a closed interval $I = [a,b] \subseteq \mathbb{R}$.

We recall here the definition of uniform convergence relative to a scale function as follows.

A sequence of functions (f_n) defined over $[a,b]$ is *relatively uniformly convergent* to a limit function f if there exists a non-zero scale function σ defined over $[a,b]$, such that for each $\epsilon > 0$ there exists an integer n_ϵ and for every $n > n_\epsilon$,

$$\left| \frac{f_n(x) - f(x)}{\sigma(x)} \right| \leqq \epsilon$$

holds uniformly for all $x \in [a,b] \subseteq \mathbb{R}$.

Now, to see the importance of relatively uniform convergence (ordinary and statistical) over classical uniform convergence, we present the following example.

Example 2. *For all $n \in \mathbb{N}$, we define $f_n : [0,1] \to \mathbb{R}$ by*

$$f_n(x) = \begin{cases} \frac{nx}{1+n^2x^2} & (0 < x \leqq 1), \\ 0 & (x = 0). \end{cases}$$

It is not difficult to see that the sequence (f_n) of functions is neither classically nor statistically uniformly convergent in $[0,1]$; however, it is convergent uniformly to $f = 0$ relative to a scale function

$$\sigma(x) = \begin{cases} \frac{1}{x} & (0 < x \leqq 1) \\ 0 & (x = 0) \end{cases}$$

on $[0,1]$. Here, we write

$$f_n \rightrightarrows f = 0 \ \ ([0,1];\sigma).$$

In the middle of the twentieth century, H. Bohman [26] and P. P. Korovkin [27] established some approximation results by using positive linear operators. Later, some Korovkin-type approximation results with different settings were extended to several functional spaces, such as Banach space and Musielak–Orlicz space etc. Bardaro, Musielak, and Vinti [22] studied generalized nonlinear integral operators in connection with some approximation results over a modular space. Furthermore, Bardaro and Mantellini [28] proved some approximation theorems defined over a modular space by positive linear operators. They also established a conventional Korovkin-type theorem in a multivariate modular function space (see [21]). In the year 2015, Orhan and Demirci [29] established a result on statistical approximation by double sequences of positive linear operators on modular space. Demirci and Burçak [30] introduced the idea of A-statistical relative modular convergence of positive linear operators. Moreover, Demirci and Orhan [31] established some results on statistically relatively approximation on modular spaces. Recently, Srivastava et al. [13] established some approximation results on Banach space by using deferred weighted statistical convergence. Subsequently, they also introduced deferred weighted equi-statistical convergence to prove some approximation theorems (see [17]). Very recently, Md. Nasiruzzaman et al. [32] proved Dunkl-type generalization of Szász-Kantorovich operators via post-quantum calculus, and consequently, Srivastava et al. [33] established the construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ.

Motivated essentially by the above-mentioned results, in this paper we introduce the idea of relatively modular deferred-weighted statistical convergence and statistically as well as relatively modular deferred-weighted summability for double sequences of functions. We also establish an inclusion relation between them. Moreover, based upon our proposed methods, we prove a Korovkin-type approximation theorem for a double sequence of functions defined over a modular space and demonstrate that our result is a non-trivial generalization of some well-established results.

2. Relatively Modular Deferred-Weighted Mean

Let (a_n) and (b_n) be sequences of non-negative integers satisfying the conditions: (i) $a_n < b_n$ $(n \in \mathbb{N})$ and (ii) $\lim_{n\to\infty} b_n = \infty$. Note that (i) and (ii) are the regularity conditions for the proposed deferred weighted mean (see Agnew [34]). Now, for the double sequence $(f_{m,n})$ of functions, we define the deferred weighted summability mean $(N_D(f_{m,n}))$ as

$$N_D(f_{m,n}) = \frac{1}{T_m S_n} \sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v f_{u,v}(x), \tag{1}$$

where (s_n) and (t_n) are the sequences of non-negative real numbers satisfying

$$S_n = \sum_{v=a_n+1}^{b_n} s_v \text{ and } T_m = \sum_{u=a_n+1}^{b_m} t_v.$$

Definition 1. *A double sequence $(f_{m,n})$ of functions belonging to $L^\omega(\mathcal{I}^2)$ is relatively modular deferred weighted $(N_D(f_{m,n}))$-summable to a function f on $L^\omega(\mathcal{I}^2)$ if and only if there exists a non-negative scale function $\sigma \in X(\mathcal{I}^2)$ such that*

$$P \lim_{m,n\to\infty} \omega\left(\lambda\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) = 0 \textit{ for some } \lambda_0 > 0.$$

Here, we write

$$\mathcal{N_D} \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_\omega = 0 \textit{ for some } \lambda_0 > 0.$$

Definition 2. *A double sequence $(f_{m,n})$ of functions belonging to $L^\omega(\mathcal{I}^2)$ is relatively F-norm (locally convex) deferred weighted summable (or relatively strong deferred weighted summable) to f if and only if*

$$P \lim_{m,n\to\infty} \omega\left(\lambda\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) = 0 \textit{ for some } \lambda > 0.$$

Here, we write

$$\mathcal{F}\,\mathcal{N_D} \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_\omega = 0 \textit{ for some } \lambda_0 > 0.$$

It can be promptly seen that, Definitions 1 and 2 are identical if and only if the modular ω fairly holds the Δ_2-condition, that is, there exists a constant $\mathcal{M} > 0$ such that $\omega(2f) \leqq \mathcal{M}\omega(f)$ for every $f \in X(\mathcal{I}^2)$. Precisely, relatively strong summability of the double sequence $(f_{m,n})$ to f is identical to the condition

$$P \lim_{m,n} \omega \left(2^n \lambda \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) = 0,$$

$\forall \quad n \in \mathbb{N}$ and some $\lambda > 0$. Thus, if $(f_{m,n})$ is relatively modular deferred weighted $(N_D(f_{m,n}))$-summable to f, then by Definition 1 there exists a $\lambda > 0$ such that

$$P \lim_{m,n\to\infty} \omega \left(\lambda \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) = 0.$$

Clearly, under Δ_2-condition, we have

$$\omega \left(2^n \lambda \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) \leqq \mathcal{M}^n \omega \left(\lambda \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right).$$

This implies that

$$P \lim_{m,n} \omega \left(2^n \lambda \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) = 0.$$

Definition 3. *A double sequence $(f_{m,n})$ of functions belonging to $L^{\omega}(\mathcal{I}^2)$ is relatively modular deferred-weighted $(N_D(f_{m,n}))$ statistically convergent to a function $f \in L^{\omega}(\mathcal{I}^2)$ if there exists a non-zero scale function $\sigma \in X(\mathcal{I}^2)$ such that, for every $\epsilon > 0$, the following set:*

$$P \lim_{m,n} \frac{1}{T_m S_n} \left\{ (u,v) : u \leqq T_m, v \leqq S_m \text{ and } \omega \left(\lambda_0 \left(\frac{t_u s_v |f_{u,v} - f|}{\sigma} \right) \right) \geqq \epsilon \right\} \text{ for some } \lambda_0 > 0$$

has zero relatively deferred-weighted density, that is,

$$P \lim_{m,n} \frac{1}{T_m S_n} \left| \left\{ (u,v) : u \leqq T_m, v \leqq S_m \text{ and } \omega \left(\lambda_0 \left(\frac{t_u s_v |f_{u,v} - f|}{\sigma} \right) \right) \geqq \epsilon \right\} \right| = 0 \text{ for some } \lambda_0 > 0.$$

Here, we write

$$stat_{N_D} \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_{\omega} = 0.$$

Moreover, $(f_{m,n})$ is relatively F-norm (locally convex) deferred-weighted $(N_D(f_{m,n}))$ statistically convergent (or relatively strong deferred-weighted $(N_D(f_{m,n}))$ statistically convergent) to a function $f \in X(\mathcal{I}^2)$ if and only if

$$P \lim_{m,n} \frac{1}{T_m S_n} \left| \left\{ (u,v) : u \leqq T_m, v \leqq S_m \text{ and } \omega \left(\lambda_0 \left(\frac{t_u s_v |f_{u,v} - f|}{\sigma} \right) \right) \geqq \epsilon \right\} \right| = 0 \text{ for some } \lambda > 0,$$

where $\sigma \in X(\mathcal{I}^2)$ is a non-zero scale function and $\epsilon > 0$.

Here, we write

$$\mathcal{F}\, stat_{N_D} \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_{\omega} = 0.$$

Definition 4. *A double sequence $(f_{m,n})$ of functions belonging to $L^{\omega}(\mathcal{I}^2)$ is statistically and relatively modular deferred-weighted $(N_D(f_{m,n}))$-summable to a function $f \in L^{\omega}(\mathcal{I}^2)$ if there exists a non-zero scale function $\sigma \in X(\mathcal{I}^2)$ such that, for every $\epsilon > 0$, the following set:*

$$P \lim_{m,n} \frac{1}{m,n} \left\{ (u,v) : u \leqq m, v \leqq m \text{ and } \omega \left(\lambda_0 \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) \geqq \epsilon \right\} \text{ for some } \lambda_0 > 0$$

has zero relatively deferred-weighted density, that is,

$$P \lim_{m,n} \frac{1}{mn} \left| \left\{ (u,v) : u \leqq m, v \leqq n \text{ and } \omega \left(\lambda_0 \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) \geqq \epsilon \right\} \right| = 0 \text{ for some } \lambda_0 > 0.$$

Here, we write

$$N_D stat \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_{\omega} = 0.$$

Furthermore, $(f_{m,n})$ is statistically and relatively F-norm (locally convex) deferred-weighted $(N_D(f_{m,n}))$-summable (or statistically and relatively strong deferred-weighted $(N_D(f_{m,n}))$-summable) to a function $f \in X(\mathcal{I}^2)$ if and only if

$$P \lim_{m,n} \frac{1}{m,n} \left| \left\{ (u,v) : u \leqq m, v \leqq n \;\; and \;\; \omega \left(\lambda_0 \left(\frac{N_D(f_{m,n}) - f}{\sigma} \right) \right) \geqq \epsilon \right\} \right| = 0 \;\; for\ some\ \lambda > 0,$$

where $\sigma \in X(\mathcal{I}^2)$ is a non-zero scale function and $\epsilon > 0$.

Here, we write

$$\mathcal{F}\ N_D stat \lim_{m,n} \left\| \frac{f_{m,n} - f}{\sigma} \right\|_{\omega} = 0.$$

Remark 1. *If we put $a_n = 0$, $b_n = n$, $b_m = m$, and $t_m = s_n = 1$ in Definition 3, then it reduces to relatively modular statistical convergence (see [31]).*

Next, for our present study on a modular space we have the assumptions as follows:

- If $\omega(f) \leqq \omega(g)$ for $|f| \leqq |g|$, then ω is monotone;
- If $\chi \in L^{\omega}(\mathcal{I}^2)$ with $\mu(A) < \infty$, where A is a measurable subset of $\mathcal{I}^2$, then ω is finite;
- If ω is finite and for each $\epsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ and $\omega(\lambda \chi_B) < \epsilon$ for any measurable subset $B \subset \mathcal{I}^2$ such that $\mu(B) < \delta$, then ω is absolutely finite;
- If $\chi_{\mathcal{I}^2} \in E^{w}(\mathcal{I}^2)$, then ω is strongly finite;
- If for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\omega(\alpha f \chi_B) < \epsilon$ $(\alpha > 0)$, where B is a measurable subset of $\mathcal{I}^2$ with $\mu(B) < \delta$ and for each $f \in X(\mathcal{I}^2)$ with $\omega(f) < +\infty$, then ω is absolutely continuous.

It is clearly observed from the above assumptions that if a modular ω is finite and monotone, then $C(\mathcal{I}^2) \subset L^{\omega}(\mathcal{I}^2)$. Also, if ω is strongly finite and monotone, then $C(\mathcal{I}^2) \subset E^{\omega}(\mathcal{I}^2)$. Furthermore, if ω is absolutely continuous, monotone, and absolutely finite, then $\overline{C^{\infty}(\mathcal{I}^2)} = L^{\omega}(\mathcal{I}^2)$, where the closure $\overline{C^{\infty}(\mathcal{I}^2)}$ is compact over the modular space.

Now we establish the following theorem by demonstrating an inclusion relation between relatively deferred-weighted statistical convergence and statistically as well as relatively deferred-weighted summability over a modular space.

Theorem 1. *Let ω be a strongly finite, monotone, and $\mathcal{N}$-Quasi convex modular on $L^{\omega}(\mathcal{I}^2)$. If a double sequence $(f_{m,n})$ of functions belonging to $L^{\omega}(\mathcal{I}^2)$ is bounded and relatively modular deferred-weighted statistically convergent to a function $f \in L^{\omega}(\mathcal{I}^2)$, then it is statistically and relatively modular deferred weighted summable to the function f, but not conversely.*

Proof. Assume that $(f_{m,n}) \in L^{\omega}(\mathcal{I}^2) \cap \ell_{\infty}$. Let us set

$$\mathcal{H}_{\epsilon} = \left\{ (u,v) : u \leqq m, v \leqq n \;\text{ and }\; \omega \left(\lambda_0 \left(\frac{f_{u,v} - f}{\sigma} \right) \right) \geqq \epsilon \;\text{ for some }\; \lambda_0 > 0 \right\}$$

and

$$\mathcal{H}_{\epsilon}^{c} = \left\{ (u,v) : u \leqq m, v \leqq n \;\text{ and }\; \omega \left(\lambda_0 \left(\frac{f_{u,v} - f}{\sigma} \right) \right) > \epsilon \;\text{ for some }\; \lambda_0 > 0 \right\}.$$

From the regularity condition of our proposed mean, we have

$$P \lim_{u,v} \frac{1}{T_m S_n} \sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v = 0. \tag{2}$$

Thus, we obtain

$$
\begin{aligned}
\omega\left(\lambda_0\left(N_D\left(\frac{f_{m,n}-f}{\sigma}\right)\right)\right) &= \omega\left(\lambda_0\left(\frac{1}{T_mS_n}\sum_{u,v=a_n+1}^{b_m,b_n} t_us_v\left(\frac{f_{u,v}-f}{\sigma}\right)\right)\right)\\
&\leqq \omega\left(\frac{\lambda_0}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=0,v=b_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\right.\\
&\quad + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=0,\\(u,v)\in\mathcal{H}_\epsilon}}^{\infty,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=b_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{\infty,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\\
&\quad + \omega\left(\frac{\lambda_0}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon^c}}^{b_m,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=0,v=b_n+1,\\(u,v)\in\mathcal{H}_\epsilon^c}}^{b_m,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\right.\\
&\quad + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=0;\\(u,v)\in\mathcal{H}_\epsilon^c}}^{\infty,b_n} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right| + \frac{\lambda_0}{T_mS_n}\sum_{\substack{u=b_m+1,v=b_n+1\\(u,v)\in\mathcal{H}_\epsilon^c}}^{\infty,\infty} t_us_v\left|\frac{f_{u,v}-f}{\sigma}\right|\\
&\quad \left. + \mathcal{K}\left|\frac{1}{T_mS_n}\sum_{u,v=a_n+1}^{\infty,\infty} t_us_v - 1\right|\right),
\end{aligned}
$$

where

$$
\mathcal{K} = \sup_{x,y}\left|\frac{f(x,y)}{\sigma}\right|.
$$

Further, ω being $\mathcal{N}$-Quasi convex modular, monotone, and strongly finite on $L^{\omega}(\mathcal{I}^2)$, it follows that

$$
\begin{aligned}
\omega\left(\lambda_0\left(N_D\left(\frac{f_{m,n}-f}{\sigma}\right)\right)\right) &\leqq 3\omega\left(\frac{9\lambda_0|\mathcal{H}_\epsilon|G}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,b_n} t_us_v\right)\\
&\quad + \epsilon\omega\left(\frac{9\lambda_0|\mathcal{H}_\epsilon|}{T_mS_n}\sum_{\substack{u,v=a_n+1,\\(u,v)\in\mathcal{H}_\epsilon}}^{b_m,b_n} t_us_v\right) + \omega\left(\frac{9\lambda_0Gb_mb_n}{T_mS_n}\sum_{u,v=a_n+1}^{b_m,b_n} t_us_v\right)\\
&\quad + \omega\left(\frac{9\lambda_0Gb_m}{T_mS_n}\sum_{u=0,v=a_n+1}^{b_m,\infty} t_us_v\right) + \omega\left(\frac{9\lambda_0Gb_n}{T_mS_n}\sum_{u=a_n+1,v=0}^{\infty,b_m} t_us_v\right)\\
&\quad + \epsilon\omega\left(\frac{9\lambda_0}{T_mS_n}\sum_{u,v=a_n+1}^{\infty,\infty} t_us_v\right) + \omega\left(\frac{9\lambda_0\mathcal{K}}{T_mS_n}\sum_{u,v=a_n+1}^{\infty,\infty} t_us_v - 1\right),
\end{aligned}
$$

where $G = \max\left|\frac{f_{u,v}-f(x,y)}{\sigma}\right|, \forall\, u,v \in \mathbb{N}$ and $(x,y) \in \mathcal{I}^2$. In the last inequality, considering P limit as $m,n \to \infty$ under the regularity conditions of deferred weighted mean and by using (2), we obtain

$$
P\lim_{m,n}\omega\left(\lambda_0\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) = 0.
$$

This implies that $(f_{m,n})$ is relatively modular deferred weighted $N_D(f_{m,n})$-summable to a function f. Hence,

$$
P\lim_{m,n}\frac{1}{m,n}\left|\left\{(u,v) : u \leqq m, v \leqq m \ \text{and}\ \omega\left(\lambda_0\left(\frac{N_D(f_{m,n})-f}{\sigma}\right)\right) \geqq \epsilon\right\}\right| = 0 \ \text{for some } \lambda_0 > 0.
$$

Next, to see that the converse part of the theorem is not necessarily true, we consider the following example.

Example 3. *Suppose that $\mathcal{I} = [0,1]$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous function with $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$. Let $f \in X(\mathcal{I}^2)$ be a measurable real-valued function, and consider the functional ω^{φ} on $X(\mathcal{I}^2)$ defined by*

$$\omega^{\varphi}(f) = \int_0^1 \int_0^1 \varphi(|f_{m,n}(x,y)|)dxdy \quad (f \in X(\mathcal{I}^2).$$

φ being convex, ω^{φ} is modular convex on $X(\mathcal{I}^2)$, which satisfies the above assumptions. Consider $L^{\omega}_{\varphi}(\mathcal{I}^2)$ as the Orlicz space produced by φ of the form:

$$L^{\omega}_{\varphi}(\mathcal{I}^2) = \{f \in X(\mathcal{I}^2) : \omega^{\varphi}(\lambda(f)) < +\infty \text{ for some } \lambda > 0\}.$$

For all $m,n \in \mathbb{N}$, we consider a double sequence of functions $f_{m,n} : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f_{m,n}(x,y) = \begin{cases} 1, & (m,n) \in \mathfrak{U} \times \mathfrak{U} \text{ and } (x,y) \in (0,\frac{1}{m}] \times (0,\frac{1}{n}], \\ 0, & \{(m,n) \in \mathfrak{V} \times \mathfrak{V} \text{ and } (x,y) \in (\frac{1}{m},0] \times (\frac{1}{n},1]; \\ & (m,n) \in \mathfrak{U} \times \mathfrak{V} \text{ or } (m,n) \in \mathfrak{V} \times \mathfrak{U} \text{ or } (x,y) \in (0,0)\}, \end{cases}$$

where the set of all odd and even numbers are $\mathfrak{U}$ and $\mathfrak{V}$, respectively.

We have

$$\omega\lambda(N_D(f_{m,n})) = \omega\left(\frac{\lambda_0}{S_m T_n} \sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v\right),$$

and this implies

$$\omega\lambda(N_D(f_{m,n})) = \lambda_0 \begin{cases} \int_0^{1/b_m} \int_0^{1/b_n} dxdy, & (m,n) \in \mathfrak{U} \times \mathfrak{U} \text{ and } (x,y) \in (0,\frac{1}{m}] \times (0,\frac{1}{n}], \\ 0, & \{(m,n) \in \mathfrak{V} \times \mathfrak{V} \text{ and } (x,y) \in (\frac{1}{m},0] \times (\frac{1}{n},1]; \\ & (m,n) \in \mathfrak{U} \times \mathfrak{V} \text{ or } (m,n) \in \mathfrak{V} \times \mathfrak{U} \text{ or } (x,y) \in (0,0)\}. \end{cases}$$

Clearly, $(f_{m,n})$ is relatively modular deferred weighted summable to $f = 0$, with respect to a non-zero scale function $\sigma(x,y)$ such that

$$\sigma(x,y) = \begin{cases} 1, & (x,y) = (0,0) \\ \frac{1}{xy}, & (x,y) \in (0,1] \times (0,1]. \end{cases}$$

That is,

$$P \lim_{m,n} \omega\left(\lambda_0\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) = 0 \text{ for some } \lambda_0 > 0.$$

Thus, we have

$$P \lim_{m,n} \frac{1}{m,n} \left|\left\{(u,v) : u \leqq m, v \leqq m \text{ and } \omega\left(\lambda_0\left(\frac{N_D(f_{m,n}) - f}{\sigma}\right)\right) \geqq \epsilon\right\}\right| = 0 \text{ for some } \lambda_0 > 0.$$

On the other hand, it is not relatively modular deferred-weighted statistically convergent to the function $f = 0$, that is,

$$P \lim_{m,n} \frac{1}{T_m S_n} \left| \left\{ (u,v) : u \leqq T_m, v \leqq S_m \ \ and \ \ \omega \left(\lambda_0 \left(\frac{t_u s_v |f_{u,v} - f|}{\sigma} \right) \right) \geqq \epsilon \right\} \right| \neq 0 \ \ for\ some\ \lambda_0 > 0.$$

□

3. A Korovkin-Type Theorem in Modular Space

In this section, we extend here the result of Demirci and Orhan [31] by using the idea of the statistically and relatively modular deferred-weighted summability of a double sequence of positive linear operators defined over a modular space.

Let ω be a finite modular and monotone over $X(\mathcal{I}^2)$. Suppose E is a set such that $C^\infty(\mathcal{I}^2) \subset E \subset L^\omega(\mathcal{I}^2)$. We can construct such a subset E when ω is monotone and finite. We also assume $L = \{\mathcal{L}_{m,n}\}$ as the sequence of positive linear operators from E in to $X(\mathcal{I}^2)$, and there exists a subset $X_L \subset E$ containing $C^\infty(\mathcal{I}^2)$. Let $\sigma \in X(\mathcal{I}^2)$ be an unbounded function with $|\sigma(x,y)| \neq 0$, and R is a positive constant such that

$$N_D\text{stat} \limsup_{m,n} \omega \left(\lambda \left(\frac{\Upsilon_{m,n}(f)}{\sigma} \right) \right) \leqq R\omega(\lambda f) \tag{3}$$

holds for each $f \in X_L, \lambda > 0$ and

$$\Upsilon_{m,n}(f;x,y) = \frac{1}{T_m S_n} \sum_{u,v=a_n+1}^{b_m,b_n} t_u s_v \mathcal{T}_{m,n}(f;x,y).$$

We denote here the value of $\mathcal{L}_{m,n}(f)$ at a point $(x,y) \in \mathcal{I}^2$ by $\mathcal{L}_{m,n}(f(x^*,y^*);x,y)$, or briefly by $\mathcal{L}_{m,n}(f;x,y)$. We now prove the following theorem.

Theorem 2. *Let (a_n) and (b_n) be the sequences of non-negative integers and let ω be an $\mathcal{N}$-Quasi semi-convex modular, absolutely continuous, strongly finite, and monotone on $X(\mathcal{I}^2)$. Assume that $L = \{\mathcal{L}_{m,n}\}$ is a double sequence of positive linear operators from E in to $X(\mathcal{I}^2)$ that satisfy the assumption (3) for every $f \in X_L$ and suppose that $\sigma_i(x,y)$ is an unbounded function such that $|\sigma_i(x,y)| \geqq u_i > 0$ $(i = 0,1,2,3)$. Assume further that*

$$N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f_i;x,y) - f(x,y)}{\sigma} \right\|_\omega = 0 \ \ for\ each\ \ \lambda > 0\ and\ \ i = 0,1,2,3, \tag{4}$$

where

$$f_0(x,y) = 1, \ \ f_1(x,y) = x, \ \ f_2(x,y) = y \ and \ f_3(x,y) = x^2 + y^2.$$

Then, for every $f \in L^\omega(\mathcal{I}^2)$ and $g \in C^\infty(\mathcal{I}^2)$ with $f - g \in X_L$,

$$N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_\omega = 0 \ \ for\ every\ \ \lambda_0 > 0, \tag{5}$$

where $\sigma(x,y) = \max\{|\sigma_i(x,y)| : i = 0,1,2,3\}$.

Proof. First we claim that,

$$N_D\text{stat} \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(g;x,y) - g(x,y)}{\sigma} \right\|_\omega = 0 \ \ \text{for every} \ \ \lambda_0 > 0. \tag{6}$$

In order to justify our claim, we assume that $g \in C(\mathcal{I}^2) \cap E$. Since g is continuous on $\mathcal{I}^2$, for given $\epsilon > 0$, there exists a number $\delta > 0$ such that for every $(x^*,y^*),(x,y) \in \mathcal{I}^2$ with $|x^* - x| < \delta$ and $|y^* - y| < \delta$, we have

$$|g(x^*,y^*) - g(x,y)| < \epsilon. \tag{7}$$

Also, for all $(x^*, y^*), (x, y) \in \mathcal{I}^2$ with $|x^* - x| > \delta$ and $|x^* - x| > \delta$, we have

$$|g(x^*, y^*) - g(x, y)| < \frac{2\mathcal{A}}{\delta^2} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2 \right), \tag{8}$$

where

$$\varphi_1(x^*, x) = (x^* - x), \quad \varphi_2(y^*, y) = (y^* - y), \quad \text{and} \quad \mathcal{A} = \sup_{x,y \in \mathcal{I}^2} |g(x, y)|.$$

From Equations (7) and (8), we obtain

$$|g(x^*, y^*) - g(x, y)| < \epsilon + \frac{2\mathcal{A}}{\delta^2} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2 \right).$$

This implies that

$$-\epsilon - \frac{2\mathcal{A}}{\delta^2} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2 \right) < g(x^*, y^*) - g(x, y) < \epsilon + \frac{2\mathcal{A}}{\delta^2} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2 \right). \tag{9}$$

Now $\mathcal{L}_{m,n}(g_0; x, y)$ being linear and monotone, by applying the operator $\mathcal{L}_{m,n}(g_0; x, y)$ to this inequality (9), we fairly have

$$\begin{aligned} \mathcal{L}_{m,n}(g_0; x, y) \left(-\epsilon - \frac{2\mathcal{A}}{\delta^2} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2 \right) \right) &< \mathcal{L}_{m,n}(g_0; x, y)(g(x^*, y^*) - g(x, y)) \\ &< \mathcal{L}_{m,n}(g_0; x, y) \left(\epsilon + \frac{2\mathcal{A}}{\delta^2} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2 \right) \right). \end{aligned} \tag{10}$$

Note that x, y is fixed, and so also $g(x, y)$ is a constant number. This implies that

$$\begin{aligned} -\epsilon\mathcal{L}_{m,n}(g_0; x, y) - \frac{2\mathcal{A}}{\delta^2} \mathcal{L}_{m,n} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2; x, y \right) &< \mathcal{L}_{m,n}(g; x, y) - g(x, y)\mathcal{L}_{m,n}(g_0; x, y) \\ &< \epsilon\mathcal{L}_{m,n}(g_0; x, y) + \frac{2\mathcal{A}}{\delta^2} \mathcal{L}_{m,n}([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2; x, y). \end{aligned} \tag{11}$$

However,

$$\mathcal{L}_{m,n}(g; x, y) - g(x, y) = [\mathcal{L}_{m,n}(g; x, y) - g(x, y)\mathcal{L}_{m,n}(g_0; x, y)] + g(x, y)[\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)]. \tag{12}$$

Now, using (11) and (12), we have

$$\begin{aligned} |\mathcal{L}_{m,n}(g; x, y) - g(x, y)| \leqq &\left| \epsilon\mathcal{L}_{m,n}(g_0; x, y) + \frac{2\mathcal{A}}{\delta^2} \mathcal{L}_{m,n} \left([\varphi_1(x^*, x)]^2 + [\varphi_2(y^*, y)]^2; x, y \right) \right| \\ &+ \mathcal{A}[\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)]. \end{aligned} \tag{13}$$

Next,

$$\begin{aligned} |\mathcal{L}_{m,n}(g; x, y) - g(x, y)| = \epsilon &+ (\epsilon + \mathcal{A})[\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)] - \frac{4\mathcal{A}}{\delta^2} |g_1(x, y)| [\mathcal{L}_{m,n}(g_1; x, y) - g_1(x, y)] \\ &+ \frac{2\mathcal{A}}{\delta^2} [\mathcal{L}_{m,n}(g_3; x, y) - g_3(x, y)] - \frac{4\mathcal{A}}{\delta^2} |g_2(x, y)| [\mathcal{L}_{m,n}(g_2; x, y) - g_2(x, y)] \\ &+ \frac{2\mathcal{A}}{\delta^2} |g_3(x, y)| [\mathcal{L}_{m,n}(g_0; x, y) - g_0(x, y)]. \end{aligned}$$

Since the choice of ϵ is arbitrarily small, we can easily write

$$
\begin{aligned}
|\mathcal{L}_{m,n}(g;x,y)-g(x,y)| \leqq \epsilon &+ \left(\epsilon+\frac{2\mathcal{A}}{\delta^2}+\mathcal{A}\right)|\mathcal{L}_{m,n}(g_0;x,y)-g_0(x,y)| \\
&+\frac{4\mathcal{A}}{\delta^2}|g_1(x,y)||\mathcal{L}_{m,n}(g_1;x,y)-g_1(x,y)|+\frac{2\mathcal{A}}{\delta^2}|\mathcal{L}_{m,n}(g_3;x,y)-g_3(x,y)| \\
&-\frac{4\mathcal{A}}{\delta^2}|g_2(x,y)||\mathcal{L}_{m,n}(g_2;x,y)-g_2(x,y)|.
\end{aligned} \tag{14}
$$

Now multiplying $\frac{1}{\sigma(x,y)}$ to both sides of (14), we have, for any $\lambda > 0$

$$
\begin{aligned}
\lambda\left|\frac{\mathcal{L}_{m,n}(g;x,y)-g(x,y)}{\sigma(x,y)}\right| \leqq \frac{\lambda\epsilon}{\sigma(x,y)}+\lambda\mathcal{B}\Bigg\{&\left|\frac{\mathcal{L}_{m,n}(g_0;x,y)-g_0(x,y)}{\sigma(x,y)}\right| \\
&+\left|\frac{\mathcal{L}_{m,n}(g_1;x,y)-g_1(x,y)}{\sigma(x,y)}\right|+\left|\frac{\mathcal{L}_{m,n}(g_3;x,y)-g_3(x,y)}{\sigma(x,y)}\right| \\
&-\left|\frac{\mathcal{L}_{m,n}(g_2;x,y)-g_2(x,y)}{\sigma(x,y)}\right|\Bigg\},
\end{aligned} \tag{15}
$$

where $\mathcal{B} = \max\left(\epsilon+\frac{2\mathcal{A}}{\delta^2}+\mathcal{A}, \frac{4\mathcal{A}}{\delta^2}, \frac{2\mathcal{A}}{\delta^2}\right)$ and $g_1(x,y),\ g_2(x,y)$ are constants for $\forall\ (x,y)$.

Next, applying the modular ω to the above inequality, also ω being $\mathcal{N}$-Quasi semi-convex, strongly finite, monotone, and $\sigma(x,y) = \max\{|\sigma_i(x,y)\ \ (i=0,1,2,3)|\}$, we have

$$
\begin{aligned}
\omega\left(\lambda\left(\frac{\mathcal{L}_{m,n}(g;x,y)-g(x,y)}{\sigma(x,y)}\right)\right) \leqq \omega\left(\frac{5\lambda\epsilon}{\sigma(x,y)}\right)&+\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_0;x,y)-g_0(x,y)}{\sigma_0(x,y)}\right)\right) \\
&+\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_1;x,y)-g_1(x,y)}{\sigma_1(x,y)}\right)\right) \\
&+\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_3;x,y)-g_3(x,y)}{\sigma_2(x,y)}\right)\right) \\
&-\omega\left(5\lambda\mathcal{B}\left(\frac{\mathcal{L}_{m,n}(g_2;x,y)-g_2(x,y)}{\sigma_3(x,y)}\right)\right).
\end{aligned} \tag{16}
$$

Now, replacing $\mathcal{L}_{m,n}(f;x,y)$ by

$$
\frac{1}{S_m T_n}\sum_{u,v=a_n+1}^{b_m,b_n} s_u t_v \mathcal{T}_{u,v}(g;x,y) = \Upsilon_{m,n}(f;x,y)
$$

and then by $\Psi(f;x,y)$ in (16), for a given $\kappa > 0$ there exists $\epsilon > 0$, such that $\omega\left(\frac{5\lambda\epsilon}{\sigma}\right) < \kappa$. Then, by setting

$$
\Psi = \left\{(m,n) : \omega\left(\lambda\left(\frac{\Upsilon_{m,n}(g)-g}{\sigma}\right)\right) \geqq \kappa\right\}
$$

and for $i = 0,1,2,$

$$
\Psi_i = \left\{(m,n) : \omega\left(\lambda\left(\frac{\Upsilon_{m,n}(g_i)-g}{\sigma_i}\right)\right) \geqq \frac{\kappa-\omega\left(\frac{5\lambda\epsilon}{\sigma}\right)}{4\mathcal{B}}\right\},
$$

we obtain

$$
\Psi \leqq \sum_{i=0}^{3}\Psi_i.
$$

Clearly,

$$\frac{\|\Psi\|_{\omega}}{mn} \leqq \sum_{i=0}^{3} \frac{\|\Psi_i\|_{\omega}}{mn}. \tag{17}$$

Now, by the assumption under (4) as well as by Definition 4, the right-hand side of (17) tends to zero as $m, n \to \infty$. Clearly, we get

$$\lim_{m,n\to\infty} \frac{\|\Psi\|_{\omega}}{mn} = 0 \ (\kappa > 0),$$

which justifies our claim (6). Hence, the implication (6) is fairly obvious for each $g \in C^{\infty}(\mathcal{I}^2)$.

Now let $f \in L^{\omega}(\mathcal{I}^2)$ such that $f - g \in X_L$ for every $g \in C^{\infty}(\mathcal{I}^2)$. Also, ω is absolutely continuous, monotone, strongly and absolutely finite on $X(\mathcal{I}^2)$. Thus, it is trivial that the space $C^{\infty}(\mathcal{I}^2)$ is modularly dense in $L^{\omega}(\mathcal{I}^2)$. That is, there exists a sequence $(g_{i,j}) \in C^{\infty}(\mathcal{I}^2)$ provided that $\omega(3\lambda_0^* g) < +\infty$ and

$$P \lim_{i,j} \omega(3\lambda_0^*(g_{i,j} - f)) = 0 \ \text{ for some } \ \lambda_0^*. \tag{18}$$

This implies that for each $\epsilon > 0$ there exist two positive integers $\bar{i}$ and $\bar{j}$ such that

$$\omega(3\lambda_0^*(g_{i,j} - f)) < \epsilon \ \text{ whenever } \ i \geqq \bar{i} \text{ and } j \geqq \bar{j}.$$

Further, since the operators $\Upsilon_{m,n}$ are positive and linear, we have that

$$\begin{aligned}\lambda_0^* |\Upsilon_{m,n}(f; x, y) - f(x, y)| \leqq \lambda_0^* |\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}}; x, y)| + \lambda_0^* |\Upsilon_{m,n}(g_{\bar{i},\bar{j}}; x, y) - g_{\bar{i},\bar{j}}(x, y)| \\ + \lambda_0^* |g_{\bar{i},\bar{j}}(x, y) - f(x, y)|\end{aligned}$$

holds true for each $m, n \in \mathbb{N}$ and $x, y \in \mathcal{I}$. Applying the monotonicity of modular ω and further multiplying $\frac{1}{\sigma(x,y)}$ to both sides of the above inequality, we have

$$\begin{aligned}\omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f; x, y) - f(x, y)}{\sigma}\right)\right) &\leqq \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}})}{\sigma}\right)\right) \\ &+ \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma}\right)\right) + \omega\left(3\lambda_0^*\left(\frac{g_{\bar{i},\bar{j}} - f}{\sigma}\right)\right).\end{aligned}$$

Thus, for $|\sigma(x, y)| \geqq M > 0$ $(M = \max\{M_i : i = 0, 1, 2, 3\})$, we can write

$$\begin{aligned}\omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f) - f}{\sigma}\right)\right) &\leqq \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}})}{\sigma}\right)\right) \\ &+ \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma}\right)\right) + \omega\left(\frac{3\lambda_0^*}{M}\left(g_{\bar{i},\bar{j}} - f\right)\right).\end{aligned} \tag{19}$$

Then, it follows from (18) and (19) that

$$\omega\left(\lambda_0^*\left(\frac{\Upsilon_{m,n}(f) - f}{\sigma}\right)\right) \leqq \epsilon + \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(f - g_{\bar{i},\bar{j}})}{\sigma}\right)\right) + \omega\left(3\lambda_0^*\left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma}\right)\right). \tag{20}$$

Now, taking statistical limit superior as $m, n \to \infty$ on both sides of (20) and also using (3), we deduce that

$$P \limsup_{m,n} \omega \left(\lambda_0^* \left(\frac{\Upsilon_{m,n}(f) - f}{\sigma} \right) \right) \leqq \epsilon + R\omega \left(3\lambda_0^* (f - g_{\bar{i},\bar{j}}) \right) + P \limsup_{m,n} \omega \left(3\lambda_0^* \left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma} \right) \right).$$

Thus, it implies that

$$P \limsup_{m,n} \omega \left(\lambda_0^* \left(\frac{\Upsilon_{m,n}(f) - f}{\sigma} \right) \right) \leqq \epsilon + \epsilon R + P \limsup_{m,n} \omega \left(3\lambda_0^* \left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma} \right) \right). \tag{21}$$

Next, by (4), for some $\lambda_0^* > 0$, we obtain

$$P \limsup_{m,n} \omega \left(3\lambda_0^* \left(\frac{\Upsilon_{m,n}(g_{\bar{i},\bar{j}}) - g_{\bar{i},\bar{j}}}{\sigma} \right) \right) = 0. \tag{22}$$

Clearly from (21) and (22), we get

$$P \limsup_{m,n} \omega \left(\lambda_0^* \left(\frac{\Upsilon_{m,n}(f) - f}{\sigma} \right) \right) \leqq \epsilon(1 + R).$$

Since $\epsilon > 0$ is arbitrarily small, the right-hand side of the above inequality tends to zero. Hence,

$$P \limsup_{m,n} \omega \left(\lambda_0^* \left(\frac{\Upsilon_{m,n}(f) - f}{\sigma} \right) \right) = 0,$$

which completes the proof. □

Next, one can get the following theorem as an immediate consequence of Theorem 2 in which the modular ω satisfies the Δ_2-condition.

Theorem 3. *Let $(\mathcal{L}_{m,n})$, (a_n), (b_n), σ and ω be the same as in Theorem 2. If the modular ω satisfies the Δ_2-condition, then the following assertions are identical:*

(a) N_D*stat* $\lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f_i;x,y) - f(x,y)}{\sigma} \right\|_\omega = 0$ *for each* $\lambda > 0$ *and* $i = 0, 1, 2, 3;$

(b) N_D*stat* $\lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_\omega = 0$ *for each* $\lambda > 0$ *such that any function* $f \in L^\omega(\mathcal{I}^2)$ *provided that* $f - g \in X_L$ *for each* $g \in C^\infty(\mathcal{I}^2)$.

Next, by using the definitions of relatively modular deferred-weighted statistical convergence given in Definition 3 and statistically as well as relatively modular deferred-weighted summability given in Definition 4, we present the following corollaries in view of Theorem 2.

Let $a_n = 0$ and $b_n = n$, $b_m = m$, then Equation (3) reduces to

$$\text{stat}_N \limsup_{m,n} \omega \left(\lambda \left(\frac{\mathfrak{L}_{m,n}(f)}{\sigma} \right) \right) \leqq R\omega(\lambda f) \tag{23}$$

for each $f \in X_L$ and $\lambda > 0$, where R is a constant.

Moreover, if we replace stat_N limit by Nstat limit, then Equation (3) reduces to

$$N\text{stat} \limsup_{m,n} \omega \left(\lambda \left(\frac{\Omega_{m,n}(f)}{\sigma} \right) \right) \leqq R\omega(\lambda f). \tag{24}$$

Corollary 1. *Let ω be an $\mathcal{N}$-Quasi semi-convex modular, strongly finite, monotone, and absolutely continuous on $X(\mathcal{I}^2)$. Also, let $(\mathfrak{L}_{m,n})$ be a double sequence of positive linear operators from E in to $X(\mathcal{I}^2)$ satisfying the* *assumption* (23) *for every* X_L *and* $\sigma_i(x,y)$ *be an unbounded function such that* $|\sigma_i(x,y)| \geqq u_i > 0 \ \ (i = 0,1,2,3)$. *Suppose that*

$$stat_N \lim_{m,n} \left\| \frac{\mathfrak{L}_{m,n}(f_i;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for each } \lambda > 0 \text{ and } i = 0,1,2,3,$$

where

$$f_0(x,y) = 1, \ \ f_1(x,y) = x, \ \ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2.$$

Then, for every $f \in L^{\omega}(\mathcal{I}^2)$ *and* $g \in C^{\infty}(\mathcal{I}^2)$ *with* $f - g \in X_L$,

$$stat_N \lim_{m,n} \left\| \frac{\mathfrak{L}_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for each } \lambda_0 > 0,$$

where

$$\sigma(x,y) = \max\{|\sigma_i(x,y)| : i = 0,1,2,3\}. \tag{25}$$

Corollary 2. *Let* ω *be an* $\mathcal{N}$*-Quasi semi-convex modular, absolutely continuous, monotone, and strongly finite on* $X(\mathcal{I}^2)$. *Also, let* $\Omega_{m,n}$ *be a double sequence of positive linear operators from E in to* $X(\mathcal{I}^2)$ *satisfying the assumption* (24) *for every* X_L *and* $\sigma_i(x,y)$ *be an unbounded function such that* $|\sigma_i(x,y)| \geqq u_i > 0 \ \ (i = 0,1,2,3)$. *Suppose that*

$$Nstat \lim_{m,n} \left\| \frac{\Omega_{m,n}(f_i;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for each } \lambda > 0 \text{ and } i = 0,1,2,3,$$

where

$$f_0(x,y) = 1, \ \ f_1(x,y) = x, \ \ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2.$$

Then, for every $f \in L^{\omega}(\mathcal{I}^2)$ *and* $g \in C^{\infty}(\mathcal{I}^2)$ *with* $f - g \in X_L$,

$$Nstat \lim_{m,n} \left\| \frac{\Omega_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \text{ for every } \lambda_0 > 0,$$

where σ *is given by* (25).

Note that for $a_n = 0$, $b_n = n$, $b_m = m$, and $s_m = 1 = t_n$, Equation (3) reduces to

$$\text{stat} \limsup_{m,n} \omega\left(\lambda\left(\mathfrak{L}^*_{m,n}(f)\right)\right) \leqq R\omega(\lambda f) \tag{26}$$

for each $f \in X_L$ and $\lambda > 0$, where R is a positive constant.

Also, if we replace statistically convergent limit by the statistically summability limit, then Equation (3) reduces to

$$\text{stat} \limsup_{m,n} \omega\left(\lambda\left(\Lambda_{m,n}(f)\right)\right) \leqq R\omega(\lambda f). \tag{27}$$

Now, we present the following corollaries in view of Theorem 2 as the generalization of the earlier results of Demirci and Orhan [31].

Corollary 3. *Let* ω *be an* $\mathcal{N}$*-Quasi semi-convex modular, absolutely continuous, monotone, and strongly finite on* $X(\mathcal{I}^2)$. *Also, let* $(\mathfrak{L}^*_{m,n})$ *be a double sequence of positive linear operators from E in to* $X(\mathcal{I}^2)$ *satisfying the assumption* (26) *for every* X_L *and* $\sigma_i(x,y)$ *be an unbounded function such that* $|\sigma_i(x,y)| \geqq u_i > 0 \ \ (i = 0,1,2,3)$. *Suppose that*

$$stat \lim_{m,n} \left\| \frac{\mathfrak{L}^*_{m,n}(f_i; x, y) - f(x, y)}{\sigma} \right\|_\omega = 0 \text{ for every } \lambda > 0 \text{ and } i = 0, 1, 2, 3,$$

where

$$f_0(x, y) = 1, \ \ f_1(x, y) = x, \ \ f_2(x, y) = y \text{ and } f_3(x, y) = x^2 + y^2.$$

Then, for every $f \in L^\omega(\mathcal{I}^2)$ *and* $g \in C^\infty(\mathcal{I}^2)$ *with* $f - g \in X_L$,

$$stat \lim_{m,n} \left\| \frac{\mathfrak{L}^*_{m,n}(f; x, y) - f(x, y)}{\sigma} \right\|_\omega = 0 \text{ for every } \lambda_0 > 0,$$

where σ *is given by* (25).

Corollary 4. *Let* ω *be an* $\mathcal{N}$*-Quasi semi-convex modular, monotone, absolutely continuous, and strongly finite on* $X(\mathcal{I}^2)$*. Also, let* $(\Lambda_{m,n})$ *be a double sequence of positive linear operators from E in to* $X(\mathcal{I}^2)$ *satisfying the assumption* (27) *for every* X_L *and* $\sigma_i(x, y)$ *be an unbounded function such that* $|\sigma_i(x, y)| \geqq u_i > 0$ $(i = 0, 1, 2, 3)$*. Suppose that*

$$stat \lim_{m,n} \left\| \frac{\Lambda_{m,n}(f_i; x, y) - f(x, y)}{\sigma} \right\|_\omega = 0 \text{ for every } \lambda > 0 \text{ and } i = 0, 1, 2, 3,$$

where

$$f_0(x, y) = 1, \ \ f_1(x, y) = x, \ \ f_2(x, y) = y \text{ and } f_3(x, y) = x^2 + y^2.$$

Then, for every $f \in L^\omega(\mathcal{I}^2)$ *and* $g \in C^\infty(\mathcal{I}^2)$ *with* $f - g \in X_L$,

$$stat \lim_{m,n} \left\| \frac{\Lambda_{m,n}(f; x, y) - f(x, y)}{\sigma} \right\|_\omega = 0 \text{ for every } \lambda_0 > 0,$$

where σ *is given by* (25).

4. Application of Korovkin-Type Theorem

In this section, by presenting a further example, we demonstrate that our proposed Korovkin-type approximation results in modular space are stronger than most (if not all) of the previously existing results in view of the corollaries provided in this paper.

Let $\mathcal{I} = [0, 1]$ and φ, ω^φ, and $L^\omega_\varphi(\mathcal{I}^2)$ be as given in Example 3. Also, recall the *bivariate Bernstein–Kantorovich* operators (see [35]), $\mathbb{B} = \{B_{m,n}\}$ on the space $L^\omega_\varphi(\mathcal{I}^2)$ given by

$$B_{m,n}(f; x, y) = \sum_{i,j=0}^{m,n} p^{(m,n)}_{i,j}(x, y)(m+1)(n+1) \times \int_{\frac{i}{m+1}}^{\frac{i+1}{m+1}} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(s, t) ds dt \tag{28}$$

for $x, y \in \mathcal{I}$ and

$$p^{(m,n)}_{i,j}(x, y) = \binom{m}{i}\binom{n}{j} x^i y^j (1-x)^{m-i}(1-y)^{n-j}.$$

Also, we have

$$\sum_{i,j=0}^{m,n} p^{(m,n)}_{i,j}(x, y) = 1. \tag{29}$$

Clearly, we observe that

$$B_{m,n}(1;x,y) = 1,$$
$$B_{m,n}(s;x,y) = \frac{mx}{m+1} + \frac{1}{2(m+1)},$$
$$B_{m,n}(t;x,y) = \frac{ny}{n+1} + \frac{1}{2(n+1)}$$

and

$$B_{m,n}(t^2+s^2;x,y) = \frac{m(m-1)x^2}{(m+1)^2} + \frac{2mx}{(m+1)^2} + \frac{1}{3(m+1)^2}\frac{n(n-1)y^2}{(n+1)^2} + \frac{2ny}{(n+1)^2} + \frac{1}{3(n+1)^2}.$$

It is further observed that $B_{m,n} : L_{\varphi}^{\omega}(\mathcal{I}^2) \to L_{\varphi}^{\omega}(\mathcal{I}^2)$. Recall [28] (Lemma 5.1) and [29] (Example 1). Now because of (29), we have from *Jensen inequality*, for each $f \in L_{\varphi}^{\omega}(\mathcal{I}^2)$ and $m,n \in \mathbb{N}$, there exists a constant M such that

$$\omega^{\varphi}\left(\frac{B_{m,n}(f;x,y)}{\sigma}\right) \leqq M\omega^{\varphi}(f).$$

We now present an illustrative example for the validity of the operators $(\mathcal{L}_{m,n})$ for our Theorem 2.

Example 4. *Let $\mathcal{L}_{m,n} : L^{\omega}(\mathcal{I}^2) \to L^{\omega}(\mathcal{I}^2)$ be defined by*

$$\mathcal{L}_{m,n}(f;x,y) = (1+f_{m,n})B_{m,n}(f;x,y), \tag{30}$$

where $(f_{m,n})$ is a sequence defined as in Example 3. Then, we have

$$\mathcal{L}_{m,n}(1;x,y) = 1 + f_{m,n}(x,y),$$
$$\mathcal{L}_{m,n}(1;x,y) = 1 + f_{m,n}(x,y) \cdot \left[\frac{mx}{m+1} + \frac{1}{2(m+1)}\right],$$
$$\mathcal{L}_{m,n}(1;x,y) = 1 + f_{m,n}(x,y) \cdot \left[\frac{ny}{n+1} + \frac{1}{2(n+1)}\right]$$

and

$$\mathcal{L}_{m,n}(1;x,y) = 1 + f_{m,n}(x,y) \cdot \left[\frac{m(m-1)x^2}{(m+1)^2} + \frac{2mx}{(m+1)}^2 + \frac{1}{3(m+1)^2}\frac{n(n-1)y^2}{(n+1)^2} + \frac{2ny}{(n+1)^2} + \frac{1}{3(n+1)^2}\right].$$

We thus obtain

$$N_D stat \lim_{m,n} \left\|\frac{\mathcal{L}_{m,n}(1;x,y)-1}{\sigma}\right\|_{\omega} = 0,$$
$$N_D stat \lim_{m,n} \left\|\frac{\mathcal{L}_{m,n}(s;x,y)-s}{\sigma}\right\|_{\omega} = 0,$$
$$N_D stat \lim_{m,n} \left\|\frac{\mathcal{L}_{m,n}(t;x,y)-t}{\sigma}\right\|_{\omega} = 0,$$
$$N_D stat \lim_{m,n} \left\|\frac{\mathcal{L}_{m,n}(s^2+t^2;x,y)-s^2+t^2}{\sigma}\right\|_{\omega} = 0.$$

This means that the operators $\mathcal{L}_{m,n}(f;x,y)$ fulfil the conditions (4). Hence, by Theorem 2 we have

$$N_D stat \lim_{m,n} \left\| \frac{\mathcal{L}_{m,n}(f;x,y) - f(x,y)}{\sigma} \right\|_{\omega} = 0 \ \textit{for every} \ \lambda_0 > 0.$$

However, since $(f_{m,n})$ is not relatively modular weighted statistically convergent, the result of Demirci and Orhan ([31], p. 1173, Theorem 1) is not fairly true under the operators defined by us in (30). Furthermore, since $(f_{m,n})$ is statistically and relatively modular deferred-weighted summable, we therefore conclude that our Theorem 2 works for the operators which we have considered here.

5. Concluding Remarks and Observations

In the concluding section of our study, we put forth various supplementary remarks and observations concerning several outcomes which we have established here.

Remark 2. *Let $(f_{m,n})_{m,n\in\mathbb{N}}$ be a sequence of functions given in Example 3. Then, since*

$$N_D stat \lim_{m\to\infty} f_{m,n} = 0 \ \textit{on} \ [0,1]\times[0,1],$$

we have

$$N_D stat \lim_{m\to\infty} \|\mathcal{L}_{m,n}(f_i;x,y) - f_i(x,y)\|_{\omega} = 0 \quad (i = 0,1,2,3). \tag{31}$$

Thus, we can write (by Theorem 2)

$$N_D stat \lim_{m\to\infty} \|\mathcal{L}_{m}(f;x,y) - f(x,y)\|_{\omega} = 0, \ (i = 0,1,2,3), \tag{32}$$

where

$$f_0(x,y) = 1, \ f_1(x,y) = x, \ f_2(x,y) = y \ \textit{and} \ f_3(x,y) = x^2 + y^2.$$

Moreover, as $(f_{m,m})$ is not classically convergent it therefore does not converge uniformly in modular space. Thus, the traditional Korovkin-type approximation theorem will not work here under the operators defined in (30). Therefore, this application evidently demonstrates that our Theorem 2 is a non-trivial extension of the conventional Korovkin-type approximation theorem (see [27]).

Remark 3. *Let $(f_{m,n})_{m,n\in\mathbb{N}}$ be a sequence as considered in Example 3. Then, since*

$$N_D stat \lim_{m\to\infty} f_{m,n} = 0 \ \textit{on} \ [0,1]\times[0,1],$$

(31) fairly holds true. Now under condition (31) and by applying Theorem 2, we have that the condition (32) holds true. Moreover, since $(f_{m,n})$ is not relatively modular statistically Cesàro summable, Theorem 1 of Demirci and Orhan (see [31], p. 1173, Theorem 1) does not hold fairly true under the operators considered in (30). Hence, our Theorem 2 is a non-trivial generalization of Theorem 1 of Demirci and Orhan (see [31], p. 1173, Theorem 1) (see also [29]). Based on the above facts, we conclude here that our proposed method has effectively worked for the operators considered in (30), and therefore it is stronger than the traditional and statistical versions of the Korovkin-type approximation theorems established earlier in References [27,29,31].

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