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The Extended Minimax Disparity RIM Quantifier Problem

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Abstract: An interesting regular increasing monotone (RIM) quantifier problem is investigated. Amin and Emrouznejad [Computers & Industrial Engineering 50(2006) 312–316] have introduced the extended minimax disparity OWA operator problem to determine the OWA operator weights. In this paper, we propose a corresponding continuous extension of an extended minimax disparity OWA model, which is the extended minimax disparity RIM quantifier problem, under the given orness level and prove it analytically.

Keywords: fuzzy sets; RIM quantifier; extended minimax disparity; OWA model; RIM quantifier problem

1. Introduction

One of the important topic in the theory of ordered weighted averaging (OWA) operators is the determination of the associated weights. Several authors have suggested a number of methods for obtaining associated weights in various areas such as decision making, approximate reasoning, expert systems, data mining, fuzzy systems and control [1–18]. Researchers can easily see most of OWA papers in the recent bibliography published in Emrouznejad and Marra [5]. Yager [16] proposed RIM quantifiers as a method for finding OWA weight vectors through fuzzy linguistic quantifiers. Liu [19] and Liu and Da [20] gave solutions to the maximum-entropy RIM quantifier model when the generating functions are differentiable. Liu and Lou [21] studied the equivalence of solutions to the minimax ratio and maximum-entropy RIM quantifier models, and the equivalence of solutions to the minimax disparity and minimum-variance RIM quantifier problems. Hong [22,23] gave the proof of the minimax ratio RIM quantifier problem and the minimax disparity RIM quantifier model when the generating functions are absolutely continuous. He also gave solutions to the maximum-entropy RIM quantifier model and the minimum-variance RIM quantifier model when the generating functions are Lebesgue integrable. Liu [24] proposed a general RIM quantifier determination model, proved it analytically using the optimal control method and investigated the solution equivalence to the minimax problem for the RIM quantifier. However, Hong [11] recently provided a modified model for the general RIM quantifier model and the correct formulation of Liu's result.

Amin and Emrouznejad [1] have introduced the following the extended minimax disparity OWA operator model to determine the OWA operator weights:

$$\begin{aligned} & \text{Minimize} && \max_{i \in \{1, \dots, n-1\}, j \in \{i+1, \dots, n\}} |w_i - w_j| \\ & \text{subject to} && \text{orness}(W) = \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\ & && w_1 + \dots + w_n = 1, 0 \leq w_i, i = 1, \dots, n. \end{aligned}$$

In this paper, we propose a corresponding extended minimax disparity model for RIM quantifier determination under given orness level and prove it analytically. This paper is organized as follows: Section 2 presents the preliminaries and Section 3 reviews some models for the RIM quantifier problems and propose the extended minimax disparity model for the RIM quantifier problem. In Section 4, we prove the extended minimax disparity model problem mathematically for the case in which the generating functions are Lebesgue integrable functions.

2. Preliminaries

Yager [15] introduced a new aggregation technique based on the OWA operators. An OWA operator of dimension n is a function $F : R^n \rightarrow R$ that has an associated weighting vector $W = (w_1, \dots, w_n)^T$ of having the properties $0 \leq w_i \leq 1$, $i = 1, \dots, n$, $w_1 + \dots + w_n = 1$, and such that

$$F(a_1, \dots, a_n) = \sum_{i=1}^n w_i b_i,$$

where b_j is the j th largest element of the collection of the aggregated objects $\{a_1, \dots, a_n\}$. In [15], Yager defined a measure of “orness” associated with the vector W of an OWA operator as

$$\text{orness}(W) = \sum_{i=1}^n \frac{n-i}{n-1} w_i,$$

and it characterizes the degree to which the aggregation is like an *or* operation.

The RIM quantifiers was introduced by Yager [16] as a method for obtaining the OWA weight vectors via fuzzy linguistic quantifiers. The RIM quantifiers can provide information aggregation procedures guided by a dimension independent description and verbally expressed concepts of the desired aggregation.

Definition 1 ([14]). A fuzzy subset Q is called a **RIM quantifier** if $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \geq Q(y)$ for $x > y$.

The quantifier *for all* is represented by the fuzzy set

$$Q_*(r) = \begin{cases} 1, & x = 1, \\ 0, & x \neq 1. \end{cases}$$

The quantifier *there exist, not none*, is defined as

$$Q^*(r) = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0. \end{cases}$$

Both of these are examples of RIM quantifier. To analyze the relationship between OWA and RIM quantifier, a generating function representation of RIM quantifier was proposed.

Definition 2. For $f(t)$ on $[0, 1]$ and a RIM quantifier $Q(x)$, $f(t)$ is called **generating function** of $Q(x)$, if it satisfies

$$Q(x) = \int_0^x f(t) dt$$

where $f(t) \geq 0$ and $\int_0^1 f(t) dt = 1$.

If $Q(x)$ is an absolutely continuous function, then $f(x)$ is a Lebesgue integrable function; moreover, $f(x)$ is unique in the sense of “almost everywhere” in abbreviated form, *a.e.*

Yager extended the *orness* measure of OWA operator, and defined the *orness* of a RIM quantifier [16].

$$\text{orness}(Q) = \int_0^1 Q(x)dx = \int_0^1 (1-t)f(t)dt.$$

As the RIM quantifier can be seen as the continuous form of OWA operator with generating function, OWA optimization problem is extended to the RIM quantifier case.

The definitions of *essential supremum* and *essential infimum* [21] of f are as follows:

$$\text{ess sup } f = \inf \{t : |\{x \in [0,1] : f(x) > t\}| = 0\},$$

$$\text{ess inf } f = \sup \{t : |\{x \in [0,1] : f(x) < t\}| = 0\},$$

where $|E|$ is the Lebesgue measure of the Lebesgue measurable set E .

3. Models for the RIM Quantifier Problems

Fullér and Majlender [8] proposed the minimum variance model, which minimizes the variance of OWA operator weights under a given level of orness. Their method requires the proof of the following mathematical programming problem:

$$\begin{aligned} \text{Minimize } D(W) &= \frac{1}{n} \sum_{i=1}^{n-1} \left(w_i - \frac{1}{n}\right)^2 \\ \text{subject to } \text{orness}(W) &= \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\ w_1 + \dots + w_n &= 1, \quad 0 \leq w_i, i = 1, \dots, n. \end{aligned}$$

Liu [19,24] extended the minimum variance problem for OWA operator to the RIM quantifier problem case:

$$\begin{aligned} \text{Minimize } D_f &= \int_0^1 f^2(r)dr - 1 \\ \text{subject to } \int_0^1 r f(r)dr &= 1 - \alpha, \quad 0 < \alpha < 1, \\ \int_0^1 f(r)dr &= 1, \quad f(r) \geq 0. \end{aligned}$$

Wang and Parkan [13] proposed the minimax disparity problem as follows:

$$\begin{aligned} \text{Minimize } \max_{i \in \{1, \dots, n-1\}} |w_i - w_{i+1}| \\ \text{subject to } \text{orness}(W) &= \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\ w_1 + \dots + w_n &= 1, \quad 0 \leq w_i, i = 1, \dots, n. \end{aligned}$$

Similar to the minimax disparity OWA operator problem, Hong [11] proposed the minimax disparity RIM quantifier problem as follows:

$$\begin{aligned} \text{Minimize } \text{ess sup}_{t \in [0,1]} |f'(t)| \\ \text{subject to } \int_0^1 r f(r)dr &= 1 - \alpha, \quad 0 < \alpha < 1, \\ \int_0^1 f(r)dr &= 1, \quad \text{absolutely continuous } f(r) \geq 0. \end{aligned}$$

Wang et al. [14] have introduced the following least squares deviation (LSD) method as an alternative approach to determine the OWA operator weights.

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{n-1} (w_i - w_{i-1})^2 \\ \text{subject to} \quad & \text{orness}(W) = \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\ & w_1 + \dots + w_n = 1, \quad 0 \leq w_i, i = 1, \dots, n. \end{aligned}$$

Hong [25] proposed the following corresponding least squares disparity RIM quantifier problem under a given orness level:

$$\begin{aligned} \text{Minimize} \quad & D_f = \int_0^1 (f')^2(r) dr \\ \text{subject to} \quad & \int_0^1 (1-r)f(r) dr = \alpha, \quad 0 < \alpha < 1, \\ & \int_0^1 f(r) dr = 1, \\ & f(r) > 0. \end{aligned}$$

Recently, Amin and Emrouznejad [1] proposed a problem of minimizing the maximum disparity of any distinct pairs of weights instead of adjacent weights. that is:

$$\begin{aligned} \text{Minimize} \quad & \max_{i \in \{1, \dots, n-1\}, j \in \{i+1, \dots, n\}} |w_i - w_j| \quad (1) \\ \text{subject to} \quad & \text{orness}(W) = \sum_{i=1}^n \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\ & w_1 + \dots + w_n = 1, \quad 0 \leq w_i, i = 1, \dots, n. \end{aligned}$$

We consider the following easy important fact.

Note

$$\max_{i \in \{1, \dots, n-1\}, j \in \{i+1, \dots, n\}} |w_i - w_j| = \max w_i - \min w_i.$$

For this, first it is trivial that

$$\max_{i \in \{1, \dots, n-1\}, j \in \{i+1, \dots, n\}} |w_i - w_j| \leq \max w_i - \min w_i.$$

Next, suppose that $\max w_i = w_{i_0}$, $\min w_i = w_{j_0}$. If $i_0 < j_0$, then

$$\begin{aligned} \max w_i - \min w_i &= w_{i_0} - w_{j_0} \\ &= |w_{i_0} - w_{j_0}| \\ &\leq \max_{i \in \{1, \dots, n-1\}, j \in \{i_0+1, \dots, n\}} |w_i - w_j| \end{aligned}$$

If $i_0 > j_0$, then

$$\begin{aligned} \max w_i - \min w_i &= w_{i_0} - w_{j_0} \\ &= |w_{j_0} - w_{i_0}| \\ &\leq \max_{i \in \{1, \dots, n-1\}, j \in \{j_0+1, \dots, n\}} |w_i - w_j|. \end{aligned}$$

and hence the equality holds.

Then the corresponding extended minimax disparity model for RIM quantifier problem with given orness level can be proposed as follows:

$$\begin{aligned} & \text{Minimize} && \text{ess sup } f - \text{ess inf } f && (2) \\ & \text{subject to} && \int_0^1 r f(r) dr = 1 - \alpha, \quad 0 < \alpha < 1, \\ & && \int_0^1 f(r) dr = 1, \quad f(r) \geq 0. \end{aligned}$$

4. Relation of Solutions between OWA Operator Model and RIM Quantifier Model

The following result is the solution of the extended minimax OWA operator problem given by Hong [26].

Theorem 1 ($n = 2k$:even). *An optimal weight for the constrained optimization problem (2) for a given level of $\alpha = \text{orness}(W)$ should satisfy the following equation:*

$$H(\alpha) = \text{Minimize} \left\{ \max_{i \in \{1, \dots, n-1\}, j \in \{i+1, \dots, n\}} |w_i - w_j| \right\} = \left| \frac{(1 - 2\alpha)(n - 1)}{(n - m)m} \right|$$

$$w_1^* = w_2^* = \dots = w_m^*, \quad w_{k+1}^* = w_{k+2}^* = \dots = w_n^*,$$

where

$$w_1^* = \frac{m - (1 - 2\alpha)(n - 1)}{nm}$$

and

$$w_{m+1}^* = \frac{n - m - (2\alpha - 1)(n - 1)}{n(n - m)}.$$

Here m satisfies the following:

$$m = \begin{cases} \lceil (1 - 2\alpha)(n - 1) \rceil, & \text{if } 0 \leq \alpha \leq \frac{n-2}{4(n-1)}, \\ k, & \text{if } \frac{n-2}{4(n-1)} \leq \alpha \leq \frac{3n-2}{4(n-1)}, \\ n - \lceil (2\alpha - 1)(n - 1) \rceil, & \text{if } \frac{3n-2}{4(n-1)} \leq \alpha \leq 1. \end{cases}$$

where $\lceil x \rceil = m + 1 \iff m < x \leq m + 1$ for any integer m .

Can we get a hint about the solution of the extended minimax Rim quantifier problem? Here, we suggest an idea.

For a given associated weighting vector $W_n = (w_1, \dots, w_n)$ of having the property $w_1 + \dots + w_n = 1, 0 \leq w_i \leq 1, i = 1, \dots, n$, we define a generating function $f(t)$

$$f_{W_n}(x) = nw_i, \quad x \in \left[\frac{i}{n}, \frac{i+1}{n} \right), \quad i = 0, 1, \dots, n - 1,$$

having the property $\int_0^1 f_{W_n}^n(x) dx = 1$ and let

$$f^*(x) = \lim_{n \rightarrow \infty} f_{W_n}(x).$$

Can this function $f^*(x)$ be a solution of the corresponding extended minimax Rim quantifier problem? Maybe, yes! Let's try to follow this idea.

For given $W_n^* = (w_1^*, \dots, w_n^*)$ from above Theorem 1, we have for $0 < \alpha \leq \frac{1}{4}$,

$$f_{W_n^*}(x) = \begin{cases} \frac{\lceil (1-2\alpha)(n-1) \rceil - (1-2\alpha)(n-1)}{\lceil (1-2\alpha)(n-1) \rceil}, & \text{if } x \in \left[0, \frac{\lceil (1-2\alpha)(n-1) \rceil}{n}\right) \\ \frac{n - \lceil (1-2\alpha)(n-1) \rceil - (2\alpha-1)(n-1)}{n - \lceil (1-2\alpha)(n-1) \rceil}, & \text{if } x \in \left[\frac{\lceil (1-2\alpha)(n-1) \rceil}{n}, 1\right]. \end{cases}$$

for $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$,

$$f_{W_n^*}(x) = \begin{cases} \frac{n/2 - (1-2\alpha)(n-1)}{n/2}, & \text{if } x \in \left[0, \frac{1}{2}\right) \\ \frac{n/2 - (2\alpha-1)(n-1)}{(n/2)}, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

for $3/4 \leq \alpha \leq 1$,

$$f_{W_n^*}(x) = \begin{cases} \frac{n - \lceil (2\alpha-1)(n-1) \rceil - (1-2\alpha)(n-1)}{n - \lceil (2\alpha-1)(n-1) \rceil}, & \text{if } x \in \left[0, 1 - \frac{\lceil (1-2\alpha)(n-1) \rceil}{n}\right) \\ \frac{\lceil (2\alpha-1)(n-1) \rceil - (2\alpha-1)(n-1)}{\lceil (2\alpha-1)(n-1) \rceil}, & \text{if } x \in \left[1 - \frac{\lceil (1-2\alpha)(n-1) \rceil}{n}, 1\right]. \end{cases}$$

Let $\lim_{n \rightarrow \infty} f_{W_n^*}(x) = f^*(x)$, then

1. for $0 < \alpha \leq \frac{1}{4}$,

$$f^*(r) = \begin{cases} 0, & \text{if } r \in [0, 1 - 2\alpha), \\ \frac{1}{2\alpha}, & \text{if } r \in [1 - 2\alpha, 1]. \end{cases}$$

2. for $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$,

$$f^*(r) = \begin{cases} 4\alpha - 1, & \text{if } r \in \left[0, \frac{1}{2}\right), \\ 3 - 4\alpha, & \text{if } r \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

3. for $\frac{3}{4} < \alpha \leq 1$,

$$f^*(r) = \begin{cases} \frac{1}{2(1-\alpha)}, & \text{if } r \in [0, 2\alpha], \\ 0, & \text{elsewhere.} \end{cases}$$

In the following section, we will show that f^* can be the solution of the extended minimax RIM quantifier problem.

5. Proof of the Extended Minimax RIM Quantifier Problem

In this section, we prove the following main result.

Theorem 2. The optimal solution for problem (2) for given orness level α is the weighting function f^* such that

1. for $0 < \alpha \leq \frac{1}{4}$,

$$f^*(r) = \begin{cases} 0 \text{ a.e.}, & \text{if } r \in [0, 1 - 2\alpha), \\ \frac{1}{2\alpha} \text{ a.e.}, & \text{if } r \in [1 - 2\alpha, 1]. \end{cases}$$

2. for $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$,

$$f^*(r) = \begin{cases} 4\alpha - 1 \text{ a.e.}, & \text{if } r \in \left[0, \frac{1}{2}\right), \\ 3 - 4\alpha \text{ a.e.}, & \text{if } r \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

3. for $\frac{3}{4} < \alpha \leq 1$,

$$f^*(r) = \begin{cases} \frac{1}{2(1-\alpha)} \text{ a.e.,} & \text{if } r \in [0, 2\alpha], \\ 0 \text{ a.e.,} & \text{elsewhere.} \end{cases}$$

and

$$H(\alpha) = \text{Minimize } |\text{ess sup } f - \text{ess inf } f| = \begin{cases} \frac{1}{2\alpha} & \text{if } 0 < \alpha \leq \frac{1}{4}, \\ 4|(1 - 2\alpha)| & \text{if } \frac{1}{4} \leq \alpha \leq \frac{3}{4}, \\ \frac{1}{2\alpha} & \text{if } \frac{3}{4} < \alpha \leq 1. \end{cases}$$

We need the following two lemma's to prove the main result. We denote $D_f(x) = \int_0^x f(t)dt$, $0 \leq x \leq 1$ and $E(f) = \int_0^1 rf(r)dr$.

The following result is known.

Lemma 1. $E(f) = \int_0^1 (1 - D_f(t))dt$.

Lemma 2. Let $\text{ess inf } f = \beta_0 \geq 0$ and $\text{ess sup } f = \beta_1 > 0$ such that $\int_0^1 f(r)dr = 1$ and define a function f_0 as

$$f_0(r) = \begin{cases} \beta_0 \text{ a.e.,} & \text{if } r \in [0, c_0), \\ \beta_1 \text{ a.e.,} & \text{if } r \in [c_0, 1]. \end{cases}$$

for some $c_0 \in (0, 1)$ such that $\int_0^1 f_0(r)dr = 1$. Then we have $E(f) \leq E(f_0)$ and the equality holds iff $f = f_0$ a.e.

Proof. The result follows immediately from Lemma 1 if we show that $D_{f_0}(x) \leq D_f(x), x \in [0, 1]$. It is clear that $D_{f_0}(x) \leq D_f(x), x \in [0, c_0]$. Suppose that there exists a point $t_0 \in (c_0, 1)$ such that $D_{f_0}(t_0) > D_f(t_0)$. Then

$$\int_{t_0}^1 \beta_1 dr = \int_{t_0}^1 f_0(r)dr = 1 - D_{f_0}(t_0) < 1 - D_f(t_0) = \int_{t_0}^1 f(r)dr$$

which implies $\text{ess sup}_{(t_0,1)} f > \beta_1$. It is a contradiction. \square

Proof of Theorem 2. If $\alpha = \frac{1}{2}$, we clearly have the optimal solution is $f^*(r) = 1$ a.e. for $r \in [0, 1]$. Note that $\text{ess inf } f^* < 1 < \text{ess sup } f^*$ for $\alpha \in (0, \frac{1}{2})$. Without loss of generality, we can assume that $\alpha \in (0, \frac{1}{2})$, since if a weighting function $f^*(r)$ is optimal to problem (2) for some given level of preference $\alpha \in (0, \frac{1}{2}]$, then $f^*(1 - r)$ is optimal to the problem (2) for a given level of preference $1 - \alpha$. Indeed, since $D_f = D_{f^R}, \int_0^1 f(r)dr = \int_0^1 f^R(r)dr$ and $E(f^R) = 1 - E(f)$, where $f^R(r) = f(1 - r)$ hence for $\alpha > \frac{1}{2}$, we can consider problem (2) for the level of preference with index $1 - \alpha$, and then take the reverse of that optimal solution. We can easily check that the weighting functions, f^* , given above are feasible for problem (2). We show that f^* is the unique optimal solution for a given α . Let nonnegative function f satisfy $1 = \int_0^1 f(r)dr$ and $E(f) = \int_0^1 rf(r)dr = 1 - \alpha$. Let $\text{ess inf } f = \beta_0$ and $\text{ess sup } f = \beta_1$.

Case (A): $\alpha \in (0, \frac{1}{4}]$.

We note that $\text{ess inf } f^* - \text{ess inf } f^* = \frac{1}{2\alpha}$. We will show that $\beta_1 - \beta_0 \geq \frac{1}{2\alpha}$. To show this, we define a function f_0 as

$$f_0(r) = \begin{cases} \beta_0 & \text{if } r \in [0, x_0), \\ \beta_1 & \text{if } r \in [x_0, 1], \end{cases}$$

for some $x_0 \in (0, 1)$ such that $\int_0^1 f_0(r) dr = 1$. Then by Lemma 2, $E(f) \leq E(f_0)$. Suppose that $\beta_1 - \beta_0 < \frac{1}{2\alpha}$ and define another function f_0^* as

$$f_0^*(r) = \begin{cases} \beta_0 & \text{if } r \in [0, x_0^*), \\ \beta_0 + \frac{1}{2\alpha} & \text{if } r \in [x_0^*, 1], \end{cases}$$

for some $x_0^* \in (0, 1)$ such that $\int_0^1 f_0^*(r) dr = 1$. Then $E(f_0) < E(f_0^*)$. We note that $1 = \beta_0 x_0^* + (1 - x_0^*)(\beta_0 + \frac{1}{2\alpha})$. Then

$$x_0^* = 2\alpha\beta_0 + 1 - 2\alpha. \quad (3)$$

We know that

$$\begin{aligned} E(f_0^*) &= \beta_0 \int_0^{x_0^*} x dx + \left(\beta_0 + \frac{1}{2\alpha}\right) \int_{x_0^*}^1 x dx \\ &= \frac{\beta_0}{2} + \frac{1}{4\alpha} - \frac{x_0^{*2}}{4\alpha} \end{aligned}$$

and

$$E(f^*) = \frac{1}{2\alpha} \int_{1-2\alpha}^1 x dx = 1 - \alpha.$$

And we have

$$\begin{aligned} E(f^*) - E(f_0^*) &= \frac{1}{2} \frac{x_0^{*2}}{2\alpha} - \frac{1}{2} \frac{(1-2\alpha)^2}{2\alpha} - \frac{\beta_0}{2} \\ &= \frac{1}{2} \left[\frac{1}{2\alpha} x_0^{*2} - \frac{(1-2\alpha)^2}{2\alpha} - \beta_0 \right] \\ &= \frac{1}{2} \left[\frac{1}{2\alpha} (2\alpha\beta_0 + 1 - 2\alpha)^2 - \frac{(1-2\alpha)^2}{2\alpha} - \beta_0 \right] \\ &= \frac{\beta_0}{2} [2\alpha\beta_0 + 2(1-2\alpha) - 1] \\ &\geq 0 \end{aligned}$$

where the third equality comes from (3) and the last inequality comes from the facts that $1 - 2\alpha \geq \frac{1}{2}$, $\beta_0 \geq 0$ and $\alpha > 0$. This proves $E(f) < E(f_0^*) \leq E(f^*) = 1 - \alpha$, which is a contradiction. Hence f^* is an optimal solution for the case of $\alpha \in \left(0, \frac{1}{4}\right]$.

Case (B): $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

We note that $\text{ess inf } f^* - \text{ess inf } f^* = 4(1 - 2\alpha)$. We will show that $\beta_1 - \beta_0 \geq 4(1 - 2\alpha)$. As in the Case (A), we define a function f_0 as

$$f_0(r) = \begin{cases} \beta_0 & \text{if } r \in [0, x_0), \\ \beta_1 & \text{if } r \in [x_0, 1], \end{cases}$$

for some $x_0 \in (0, 1)$ such that $\int_0^1 f_0(r) dr = 1$. Then by lemma 2, $E(f) \leq E(f_0)$. Suppose that $\beta_1 - \beta_0 < \frac{1}{2\alpha}$ and define another function f_1^* as

$$f_1^*(r) = \begin{cases} \beta_0 & \text{if } r \in [0, x_1^*), \\ \beta_0 + 4(1 - 2\alpha) & \text{if } r \in [x_1^*, 1], \end{cases}$$

for some $x_1^* \in (0, 1)$ such that $\int_0^1 f_1^*(r) dr = 1$. Then, since $x_0 < x_1^*$, by lemma 2 $E(f_0) < E(f_1^*)$. We note that $1 = \beta_0 x_1^* + (1 - x_1^*)(\beta_0 + 4(1 - 2\alpha))$. Then

$$x_1^* = 1 + \frac{\beta_0 - 1}{4(1 - 2\alpha)}$$

and

$$x_1^{*2} = 1 + \frac{\beta_0 - 1}{2(1 - 2\alpha)} + \frac{(\beta_0 - 1)^2}{16(1 - 2\alpha)^2} \quad (4)$$

We know that

$$\begin{aligned} E(f_0^*) &= \beta_0 \int_0^{x_1^*} x dx + (\beta_0 + 4(1 - 2\alpha)) \int_{x_1^*}^1 x dx \\ &= \frac{1}{2}[\beta_0 + 4(1 - 2\alpha)] - 2(1 - 2\alpha)x_1^{*2} \end{aligned}$$

and

$$\begin{aligned} E(f^*) &= (4\alpha - 1) \int_0^{\frac{1}{2}} x dx + (3 - 4\alpha) \int_{\frac{1}{2}}^1 x dx \\ &= 1 - \alpha. \end{aligned}$$

Then we have that

$$\begin{aligned} E(f^*) - E(f_1^*) &= 3\alpha - 1 - \frac{\beta_0}{2} + 2(1 - 2\alpha)x_1^{*2} \\ &= \frac{(\beta_0 - 1)^2}{8(1 - 2\alpha)} + \frac{\beta_0}{2} - \alpha \\ &= \frac{[\beta_0 - (4\alpha - 1)]^2}{8(1 - 2\alpha)} \\ &\geq 0 \end{aligned}$$

where the second equality comes from (4) and hence $E(f) < E(f_1^*) \leq E(f^*) = 1 - \alpha$, which is a contradiction. This completes the proof. \square

6. Conclusions

Previous studies have suggested a number of methods for obtaining optimal solution of the RIM quantifier problem. This paper proposes the extended minimax disparity RIM quantifier problem under a given orness level. We completely prove it analytically.

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