symmetry

MDPI

Article

Covering-Based Spherical Fuzzy Rough Set Model Hybrid with TOPSIS for Multi-Attribute Decision-Making

Shouzhen Zeng [1,2], Azmat Hussain [3,*], Tahir Mahmood [3], Muhammad Irfan Ali [4], Shahzaib Ashraf [5,*] and Muhammad Munir [3]

1 School of Managment, Fudan University, Shanghai 200433, China; zszzxl@163.com
2 School of Business, Ningbo University, Ningbo 315211, China
3 Department of Mathematics and Statistics, Faculty of Basic and Applied Sciences, International Islamic University, Islamabad 44000, Pakistan; tahirbakhat@iiu.edu.pk (T.M.); munir.phdma78@iiu.edu.pk (M.M.)
4 Department of Mathematics, Islamabad Model College for Girls F-6/2, Islamabad 44000, Pakistan; mirfanali13@yahoo.com
5 Department of Mathmatics, Abdul Wali Khan University, Mardan 23200, Pakistan
* Correspondence: azmat.phdma66@iiu.edu.pk (A.H.); shahzaibashraf@awkum.edu.pk (S.A.)

Received: 28 February 2019; Accepted: 10 April 2019; Published: 16 April 2019

Abstract: In real life, human opinion cannot be limited to yes or no situations as shown in an ordinary fuzzy sets and intuitionistic fuzzy sets but it may be yes, abstain, no, and refusal as treated in Picture fuzzy sets or in Spherical fuzzy (SF) sets. In this article, we developed a comprehensive model to tackle decision-making problems, where strong points of view are in the favour; neutral; and against some projects, entities, or plans. Therefore, a new approach of covering-based spherical fuzzy rough set (CSFRS) models by means of spherical fuzzy β-neighborhoods (SF β-neighborhoods) is adopted to hybrid spherical fuzzy sets with notions of covering the rough set. Then, by using the principle of TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) to present the spherical fuzzy, the TOPSIS approach is presented through CSFRS models by means of SF β-neighborhoods. Via the SF-TOPSIS methodology, a multi-attribute decision-making problem is developed in an SF environment. This model has stronger capabilities than intuitionistic fuzzy sets and picture fuzzy sets to manage the vague and uncertainty. Finally, the proposed method is demonstrated through an example of how the proposed method helps us in decision-making problems.

Keywords: fuzzy sets; spherical fuzzy sets; spherical fuzzy β-covering; spherical fuzzy β-covering neighborhoods; covering based spherical fuzzy rough set; spherical fuzzy TOPSIS methodology

1. Introduction

The dominant notion of q Fuzzy set by Zadeh [1] plays a vital role in the field of mathematics. This theory brought a revolution not only in the field mathematics but also in science and technology. Different direct and indirect generalizationw of this theory have been made which are successfully applied to solve the problems of real situations. Authors studied the different generalizations of fuzzy sets, and one of the most significant and useful generalizations of this theory is the Intuitionistic fuzzy set (IFS) initiated by Atanassove [2]. IFSs is a significant tool to tackle uncertainties and vague data by defining the membership grade (MG) and nonmembership grade (NMG) with the condition that the of sum of these two MG and NMG must belong to the unit of closed intervals of 0 and 1. However, a fascinating scenario emerges when the sum of the MG and NMG of an object is given from the unit interval $[0,1]$ but their sum exceeds 1. Ordinary IFSs fail to handle such situations. Therefore, some more comprehensive model is required for such situations.

Yager enquired this scenario in References [3,4] and initiated the notion of a Pythagorean fuzzy set (PytFS). This concept became more favorable among the scholars and was considered a significant generalization of IFSs. The main difference between IFSs and PytFSs is that, in the case of PytFSs, the sum of MG and NMG is greater than 1, but their squares sum belong to the unit interval $[0,1]$.

From the above analysis, it has been observed that all these notions are applied in different areas with successful results, but in real life, there are some situations that cannot be accurately tackle by IFSs and PytFSs because the most significant tool of neutral grade (NG) is missing in IFS and PytFS theories. In some situations of real life, a decision maker requires more answers of the type like yes, neutral/abstain, no, and refusal degrees which are not accurately handled with ordinary IFSs and PytFSs. A new extension WASPAS method based on the use of intuitionistic fuzzy numbers is proposed by Stanujkić and Karabašević [5]. In order to tackle such situations, a more comprehensive model was needed to cover this space. Thus, Cuong [6] originated the notion of Picture fuzzy set (PFS) which was consider as a successful extension of IFSs by put together the ideas of the MG, NG, and NMG of an object with the condition that the sum of these three grades belong to the unit closed interval $[0,1]$; for details of the study, see References [7–9]. Joshi and kumar [10] studied the concept of multi-criteria decision-making (MCDM) in PFS. Ashraf et al. [11] presented different approaches to MCDM in a picture fuzzy environment. In Reference [12], Zeng et al. proposed the exponential jensen picture fuzzy divergence measure and discussed their applications. In some situations of real life, a decision maker assigns the values to MG, NG, and NMG of an object from the unit interval $[0,1]$, but their sum exceed 1. In this case, ordinary PFSs failed to handle such situations. To tackle this situations, Ashraf et al. [13] presented the new idea of spherical fuzzy set (SFS). The main difference between PFSs and SFSs is that, in the later case, the sum of MG, NG, and NMG are greater than 1, but the sum of their squares belong to the unit interval $[0,1]$. Ashraf and saleem [14] proposed the spherical fuzzy aggregation operators using t-norm and t-conorm and, in Reference [15], proposed the concept of spherical linguistic fuzzy set and introduced the GRA method to deal with spherical fuzzy information. Similarly, Mahmood et al. [16] studied the idea of SFSs and presented their applications in a medical diagnosis; for details of the study, see References [17,18]. TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) is one of the standard decision-making methods, having simple mathematical calculations. Liu et al. [19] presented a model for evaluating and selecting a transport service provider based on a single valued neutrosophic number. Hwang and Yoon [20] proposed that TOPSIS could handle the problems related MCDM, where the target was to get the ideal solutions for each alternative. Ashraf et al. [21] proposed the logarithmic aggregation operators for single-valued neutrosophic information.

The basic idea of rough sets was initiated by Pawalk [22]. Dubios and Prade [23] originated the combine study of fuzzy sets and rough sets to get the notions of rough fuzzy and fuzzy rough sets. Liu and Lin [24] studied roughness in IFSs on the bases of conflict distance. Hussain et al. [25] initiated the concept of rough Pythagorean fuzzy ideals in semigroups; for details of the study of rough sets, see References [26,27]. Nowadays, many researchers are working on covering-based rough sets (CRSs) models. Zakowski [28] presented the idea of CRSs, which is a extension of Pawlak rough sets. Xu and Zhang [29] put forward a new CRS models which is based on the measure of roughness and discussed the properties of this measure. Liu and Sai [30] made the comparison between different CRS models presented in References [29,31,32]. Wang et al. [33] studied and improved the attribute reduction scheme with the help of CRSs. Many researchers have studied covering-based fuzzy rough sets (CFRS). D'eer et al. [34,35] presented the idea of fuzzy neighborhoods and fuzzy β-neighborhoods. Ma [36] introduced the generalized structure of CFRSs.

The Motivation and Organization of Our Research

Spherical fuzzy sets have their own importance in circumstances where an opinion is not only constrained to yes or no but there is some sort of abstinence or refusal. A good example of a spherical fuzzy set could be decision-making, such as when four decision makers have four different types of opinion about a candidate, which could possibly be in the form of yes, abstain, no, or refusal. Another example could be of voting where four types of voters vote in favor or vote against or refuse to vote or abstain.

Similarly in many situation of real life, there exists many cases where people have quite different strong points of view about certain situations, projects, plans, or entities. These points of view are diverse and opposite to each other. For example, in a certain country, a government starts a project to portray his performance. Leaders of the ruling party may rate their project highly in favor of the project by giving a membership grade about 0.7, whereas the opposition considers the same project as a waste of money and try to defame it by providing strongly opposite points of view. Therefore, a nonmembership grade suggested may be 0.6, while a third party remains neutral. Therefore, the neutral/abstain degree may be 0.1. In this case, $0.75 + 0.1 + 0.65 > 1$, but in the case of SPS, their square sum is $(0.75)^2 + (0.1)^2 + (0.65)^2 < 1$.

In order to tackle such situations, the need for a more comprehensive model was felt. To cope with such circumstances, the notion of SFS has been initiated in References [13,16]. Here, the square sum of MG, NG, and NMG is a real number between 0 and 1. The concept of SFS gives more space and freedom to the decision makers for the selection of MG, NG, and NMG.

According to the best of our knowledge, there does not exist any notion of SF rough sets via SF β-neighborhood systems in SF environments. To fulfill this space in research, the current paper was motivated to study CSFRS models via SF β-neighborhood systems. Therefore, a new approach is adopted to hybrid spherical fuzzy sets with notions of covering rough set and TOPSIS, and their application is presented in multi-attribute decision making. In real life, the CSFRSs model is a significant tools to cope with complexities and uncertainties. The idea of CSFRSs model via SF β-neighborhoods has been investigated from the hybridization of the prominent concepts of CRSs, SFSs, and FRSs. Further, it has been observed that the CSFRSs is an important generalization of cover-based intuitionistic fuzzy rough sets, cover-based Pythagorean fuzzy rough sets by adjusting the value of the NG to zero, and cover-based picture fuzzy rough sets by adjusting $0 \leq MG + NG + NMG \leq 1$. This show that CSFRS models have stronger capabilities than IFS, PytFS, and PFS in managing the uncertainty.

The arrangement of the manuscript is summarized as follows: Section 2 presents the basic notions of IFS and their generalization. Section 3 consists of the notion of covering-based spherical fuzzy set (CSFRS) models based on spherical fuzzy β-neighborhoods (SF β-neighborhoods). In Section 4, based on the analysis of CSFRS models, we introduce the spherical fuzzy TOPSIS (SF-TOPSIS) method to solve the problems of MCDM by applying SFSs. Further, Section 4.2 is devoted for the illustrated example in which we demonstrate how SF-TOPSIS methodology works in decision-making problems by using the concept of CSFRS models based on SF β-neighborhoods.

2. Preliminaries

This section consists of a brief review of IFSs and their generalizations such as PytFSs, PFSs, and SFSs. Also in the same section, we present a brief study of the fuzzy covering approximation space through fuzzy β-neighborhood.

Definition 1 ([2])**.** *Let us consider a universal set $\mathcal{X}$. An IFS $\mathcal{A}$ on a set $\mathcal{X}$ consists of two mappings which are defined as*

$$\mathcal{A} = \{< r, \psi_{\mathcal{A}}(r), \lambda_{\mathcal{A}}(r) > / r \in \mathcal{X}\}$$

such that the mapping $\psi_{\mathcal{A}} : \mathcal{X} \to [0,1]$ represents the MG and the mapping $\lambda_{\mathcal{A}} : \mathcal{X} \to [0,1]$ represents the NMG of $r \in \mathcal{X}$ to the set $\mathcal{A}$, with the condition that $0 \le \psi_{\mathcal{A}}(r) + \lambda_{\mathcal{A}}(r) \le 1$. Furthermore, $\pi_{\mathcal{A}}(r) = 1 - (\psi_{\mathcal{A}}(r) + \lambda_{\mathcal{A}}(r))$ is known to be the degree of indeterminacy.

Definition 2 ([3,4]). *Consider a universe set $\mathcal{X}$. A PytFS $\mathcal{P} = \{< r, \psi_{\mathcal{P}}(r), \lambda_{\mathcal{P}}(r) > / r \in \mathcal{X}\}$ where $\psi_{\mathcal{P}} : \mathcal{X} \to [0,1]$ represents the MG and $\lambda_{\mathcal{P}} : \mathcal{X} \to [0,1]$ represents the NMG for $r \in \mathcal{X}$ to the set $\mathcal{P}$, with the condition that $0 \le (\psi_{\mathcal{P}}(r))^2 + (\lambda_{\mathcal{P}}(r))^2 \le 1$. Furthermore, $\pi_{\mathcal{P}}(r) = \sqrt{1 - \{(\psi_{\mathcal{P}}(r))^2 + (\lambda_{\mathcal{P}}(r))^2\}}$ is said to be the degree of indeterminacy.*

Definition 3 ([6]). *Suppose a universal set $\mathcal{X}$ and an PFS $\mathcal{P}$ in $\mathcal{X}$ is an object represented by the following form:*

$$\mathcal{P} = \{< r, \psi_{\mathcal{P}}(r), \eta_{\mathcal{P}}(r), \lambda_{\mathcal{P}}(r) > / r \in \mathcal{X}\},$$

such that $\psi_{\mathcal{P}} : \mathcal{X} \to [0,1]$ represents the MG, $\eta_{\mathcal{P}} : \mathcal{X} \to [0,1]$ represents the NG, and $\lambda_{\mathcal{P}} : \mathcal{X} \to [0,1]$ represents the NMG of an object $r \in \mathcal{X}$ to the set $\mathcal{P}$, with the condition that $0 \le \psi_{\mathcal{P}}(r) + \eta_{\mathcal{P}}(r) + \lambda_{\mathcal{P}}(r) \le 1$. Moreover, $\pi(r) = 1 - (\psi_{\mathcal{P}}(r) + \eta_{\mathcal{P}}(r) + \lambda_{\mathcal{P}}(r))$ is said to be the refusal degree of $r \in \mathcal{X}$ in $\mathcal{P}$. The triplet $(\psi_{\mathcal{P}}, \eta_{\mathcal{P}}, \lambda_{\mathcal{P}})$ is called a picture fuzzy number. The collection of PFSs on $\mathcal{X}$ is represented by $PFS(\mathcal{X})$.

Definition 4 ([13]). *Suppose a universal set $\mathcal{X}$ and a spherical fuzzy set (SFS) $\Im$ in $\mathcal{X}$ consist of three mappings which are defined as*

$$\Im = \{< r, \psi_{\Im}(r), \eta_{\Im}(r), \lambda_{\Im}(r) > / r \in \mathcal{X}\},$$

such that $\psi_{\Im} : \mathcal{X} \to [0,1]$ represents the MG, $\eta_{\Im} : \mathcal{X} \to [0,1]$ represents the NG, and $\lambda_{\Im} : \mathcal{X} \to [0,1]$ represents the NMG of an object $r \in \mathcal{X}$ to the set $\Im$, with the condition that $0 \le (\psi_{\Im}(r))^2 + (\eta_{\Im}(r))^2 + (\lambda_{\Im}(r))^2 \le 1$ and $\pi(r) = \sqrt{1 - \{(\psi_{\Im}(r))^2 + (\eta_{\Im}(r))^2 + (\lambda_{\Im}(r))^2\}}$ is said to be the degree of refusal of $r \in \mathcal{X}$ in $\Im$. The collection of SFSs on $\mathcal{X}$ is represented by $SFS(\mathcal{X})$.
Let $\alpha = (\psi_{\Im}, \eta_{\Im}, \lambda_{\Im})$, then α is called a spherical fuzzy number (SFN), where $0 \le (\psi_{\Im})^2 + (\eta_{\Im})^2 + (\lambda_{\Im})^2 \le 1$.

From the above definitions of SFS, it is clear that IFS, PytFS, and PFS are the special cases of SFS because the notion of SFS provides a huge space for the decision makers by assigning their preference domain from $[0,1]$.

Definition 5. *Let $\alpha_1 = (\psi_{\Im_1}, \eta_{\Im_1}, \lambda_{\Im_1})$ and $\alpha_2 = (\psi_{\Im_2}, \eta_{\Im_2}, \lambda_{\Im_2})$ be the two SFNs. Then $\alpha_1 \le \alpha_2$, if and only if $\psi_{\Im_1} \le \psi_{\Im_2}, \eta_{\Im_1} \ge \eta_{\Im_2}$ and $\lambda_{\Im_1} \ge \lambda_{\Im_2}$.*

Let $\Im_1$=$\{< r, \psi_{\Im_1}(r), \eta_{\Im_1}(r), \lambda_{\Im_1}(r) > / r \in \mathcal{X}\}$ and $\Im_2 = \{< r, \psi_{\Im_2}(r), \eta_{\Im_2}(r), \lambda_{\Im_2}(r) > / r \in \mathcal{X}\}$ be the two SFSs.Then, Ashraf et al. [13] defined the basic operation on them as follows:

- i $\Im_1 \cup \Im_2 = \{< r, \max(\psi_{\Im_1}(r), \psi_{\Im_2}(r)), \min(\eta_{\Im_1}(r), \eta_{\Im_2}(r)), \min(\lambda_{\Im_1}(r), \lambda_{\Im_2}(r)) > / r \in \mathcal{X}\}$;
- ii $\Im_1 \cap \Im_2 = \{< r, \min(\psi_{\Im_1}(r), \psi_{\Im_2}(r)), \min(\eta_{\Im_1}(r), \eta_{\Im_2}(r)), \max(\lambda_{\Im_1}(r), \lambda_{\Im_2}(r)) > / r \in \mathcal{X}\}$;
- iii $\Im_1 \subseteq \Im_2$ if and only if $\psi_{\Im_1}(r) \le \psi_{\Im_2}(r), \eta_{\Im_1}(r) \le \eta_{\Im_2}(r), \lambda_{\Im_1}(r) \ge \lambda_{\Im_2}(r)$ for all $r \in \mathcal{X}$;
- iv $\Im_1^c = \{< r, \lambda_{\Im_1}(r), \eta_{\Im_1}(r), \psi_{\Im_1}(r) > / r \in \mathcal{X}\}$;
- v $\Im_1 = \Im_2$ if and only if $\psi_{\Im_1}(r) = \psi_{\Im_2}(r), \eta_{\Im_1}(r) = \eta_{\Im_2}(r), \lambda_{\Im_1}(r) = \lambda_{\Im_2}(r)$ for all $r \in \mathcal{X}$;

Definition 6 ([31]). *Consider a universal set $\mathcal{X}$ and K= $\{\mathcal{A} / \mathcal{A} \subseteq \mathcal{X}\}$ are a collection of non-empty subsets of $\mathcal{X}$ such that $\underset{\mathcal{A} \in K}{\cup} \mathcal{A} = \mathcal{X}$. Then, K is known to be a covering of $\mathcal{X}$, and $(\mathcal{X}, K)$ is known to be a covering approximation space (short CASp).*

Definition 7 ([31]). *Consider the CASp $(\mathcal{X}, K)$. Then, $\mathbb{N}_K(r) = \cap\{\mathcal{A} / \mathcal{A} \in$K and $r \in \mathcal{A}\}$ is known as the neighborhood of $r \in \mathcal{X}$ with respect to $(\mathcal{X}, K)$.*

Definition 8 ([37]). *For a universal set $\mathcal{X}$, $\mathcal{I}(\mathcal{X})$ represents the collection of fuzzy power set of $\mathcal{X}$. Let $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_m\}$ with $\mathcal{L}_i \in \mathcal{I}(\mathcal{X})\ (i = 1, ..., m)$, then $\mathcal{L}$ is known to be a fuzzy covering of $\mathcal{X}$, if $\bigvee_{\mathcal{L}_i \in \mathcal{L}} \mathcal{L}_i(r) = 1$ for each $r \in \mathcal{X}$. Then, the pair $(\mathcal{X}, \mathcal{L})$ is known to be a fuzzy approximation space based on fuzzy covering.*

Definition 9 ([36]). *Suppose that a universal set $\mathcal{X}$. For any $\beta \in (0, 1]$ and $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_m\}$ with $\mathcal{L}_i \in \mathcal{I}(\mathcal{X})\ (i = 1, ..., m)$, then $\mathcal{L}$ is known to be a fuzzy β-covering of $\mathcal{X}$, if $\bigcup_{i=1}^{m} \mathcal{L}_i(r) \succcurlyeq \beta$ for each $r \in \mathcal{X}$ and the pair $(\mathcal{X}, \mathcal{L})$ called fuzzy β-CASp.*

Definition 10 ([36]). *Let us consider $(\mathcal{X}, \mathcal{L})$ as a fuzzy CASp. For some $\beta \in (0, 1]$, consider a fuzzy β-covering $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_m\}$ of a set $\mathcal{X}$. Then, $\mathbb{N}_r^{\beta} = \cap\{\mathcal{L}_i \in \mathcal{L} / \mathcal{L}_i(r) \succcurlyeq \beta (i = 1, 2, ..., m)\}$ is said to be a fuzzy β-neighborhood of $r \in \mathcal{X}$.*

Definition 11 ([36]). *Let us consider a fuzzy CASp $(\mathcal{X}, \mathcal{L})$. For some $\beta \in (0, 1]$, consider a fuzzy β-covering $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_m\}$ of a set $\mathcal{X}$. Then, $\overset{*}{\mathbb{N}}{}_y^{\beta} = \{r \in \mathcal{X} / \mathbb{N}_y^{\beta}(r) \succcurlyeq \beta\}$ is said to be β-neighborhood of y for each $y \in \mathcal{X}$.*

3. Covering-Based Spherical Fuzzy Rough Set

This section is devoted to the new notion of covering-based spherical fuzzy rough sets. According to the best of our knowledge, there does not exist any notion of SF rough sets via SF β-neighborhoods system in the literature. To fulfil this space in research, the current work was motivated to study CSFRS models through SF β-neighborhood systems. The idea of CSFRS models through SF β-neighborhoods has been investigated from the hybridization of the prominent concepts of CRSs, SFSs, and FRSs. Further, it has been observed that the CSFRSs is an important generalization of cover-based intuitionistic fuzzy rough sets, cover-based Pythagorean fuzzy rough sets by adjusting the value of the NG to zero, and cover-based picture fuzzy rough sets by adjusting $0 \leq MG + NG + NMG \leq 1$. This shows that CSFRS models have stronger capabilities than IFS, PytFS, and PFS to cope with complexities and uncertainties.

Definition 12.
(1) *Suppose that a universal set $\mathcal{X}$ and $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m\}$ where $\mathcal{C}_i \in PFS(\mathcal{X})$ for $i = 1, 2, ..., m$. Now, for any picture fuzzy number (PFN) $\beta = (\psi_{\Im}, \eta_{\Im}, \lambda_{\Im})$, then $\mathcal{C}$ is said to be a picture fuzzy β-covering (PF β-covering) of $\mathcal{X}$, if $(\bigcup_{i=1}^{m} \mathcal{C}_i)(r) \succcurlyeq \beta\ \forall r \in \mathcal{X}$. Then, the pair $(\mathcal{X}, \mathcal{C})$ is called a picture fuzzy CASp (PFCAS).*
(2) *Suppose $(\mathcal{X}, \mathcal{C})$ is a PFCAS. Now, for some $\beta = (\psi_{\Im}, \eta_{\Im}, \lambda_{\Im})$, $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m\}$ is a PF β-covering of $\mathcal{X}$. Then, $\mathbb{N}_{\mathcal{C}(r)}^{\beta} = \cap\{\mathcal{C}_j \in \mathcal{C} / \mathcal{C}_j(r) \succcurlyeq \beta, j = 1, 2, ..., m\}$ is said to be a PF β-covering neighborhood of $\mathcal{X}$.*
(3) *Let $\mathbb{N}_{\mathcal{C}}^{\beta} = \{\mathbb{N}_{\mathcal{C}(r)}^{\beta} / r \in \mathcal{X}\}$ represent a PF β-neighborhood system induced by a PF β-covering $\mathcal{C}$. With the help of a picture fuzzy matrix, the following represents a PF β-neighborhood system.*

$$\mathbb{M}_{\mathcal{C}}^{\beta} = \left[\mathbb{N}_{\mathcal{C}(r)}^{\beta}(r)\right]_{(r_i, r_j) \in \mathcal{X} \times \mathcal{X}}$$

Proposition 1.
(1) *By taking $\beta = (1, 0, 0)$, then PF β-coverings degenerate into a crisp covering.*
(2) *By taking $\beta = (1, 0, 0)$, then PF β-neighborhoods degenerate into a crisp neighborhood.*
(3) *By taking $\beta = (a, 0, 0)$ where $0 < a < 1$, then PF β-coverings degenerate into a fuzzy covering.*
(4) *By taking $\beta = (a, 0, 0)$, then PF β-neighborhoods degenerate into a fuzzy β-neighborhood.*

Definition 13. *Suppose that a universal set $\mathcal{X}$ and $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,...,\mathcal{C}_m\}$ where $\mathcal{C}_i \in SFS(\mathcal{X})$ for $i=1,2,...,m$. Now, for any SFN $\beta=(\psi_{\Im},\eta_{\Im},\lambda_{\Im})$, then $\mathcal{C}$ is said to be a spherical fuzzy β-covering (SF β-covering) of $\mathcal{X}$, if $(\bigcup_{i=1}^{m}\mathcal{C}_i)(r) \succcurlyeq \beta\ \forall r \in \mathcal{X}$. Then, the pair $(\mathcal{X},\mathcal{C})$ is called a spherical fuzzy CASp (SFCAS).*

Definition 14.
(1) *Suppose $(\mathcal{X},\mathcal{C})$ is a SFCAS. Now, for some $\beta=(\psi_{\Im},\eta_{\Im},\lambda_{\Im})$, $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,...,\mathcal{C}_m\}$ is an SF β-covering of $\mathcal{X}$. Then, $\mathbb{N}^{\beta}_{\mathcal{C}(r)}=\cap\{\mathcal{C}_j \in \mathcal{C}/\mathcal{C}_j(r) \succcurlyeq \beta,\ j=1,2,...,m\}$ is said to be an SF β-covering neighborhood of $\mathcal{X}$.*
(2) *Let $\mathbb{N}^{\beta}_{\mathcal{C}}=\{\mathbb{N}^{\beta}_{\mathcal{C}(r)}/r \in \mathcal{X}\}$ represent an SF β-neighborhood system induced by an SF β-covering $\mathcal{C}$. With the help of a spherical fuzzy matrix, the following represents an SF β-neighborhood system as follows.*

$$\mathbb{M}^{\beta}_{\mathcal{C}}=\left[\mathbb{N}^{\beta}_{\mathcal{C}(r)}(r)\right]_{(r_i,r_j)\in\mathcal{X}\times\mathcal{X}}$$

Proposition 2.
(1) *By taking $\beta=(1,0,0)$, then SF β-coverings degenerate into a crisp covering.*
(2) *By taking $\beta=(1,0,0)$, then SF β-neighborhoods degenerate into a crisp neighborhood.*
(3) *By taking $\beta=(a,0,0)$ where $0<a<1$, then SF β-coverings degenerate into a fuzzy covering.*
(4) *By taking $\beta=(a,0,0)$, then SF β-neighborhoods degenerate into a fuzzy β-neighborhood.*

Proof.
(1) Let us consider that $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,...,\mathcal{C}_m\}$ is an SF β-covering. Then, by definition $(\bigcup_{i=1}^{m}\mathcal{C}_i)(r) \succcurlyeq \beta$, $\forall r \in \mathcal{X}$. If $\beta=(1,0,0)$, then there exists at least an SFN $\alpha=(\psi_{\Im},\eta_{\Im},\lambda_{\Im})=(1,0,0)$ such that $(1,0,0)=\mathcal{C}_j(r)$, (for some $j=1,2,...,m$) for $r \in \mathcal{X}$. Thus, $\bigcup_{\mathcal{C}_i\in\mathcal{C}}\mathcal{C}_i=\mathcal{X}$. Therefore, if $\beta=(1,0,0)$, then SF β-coverings degenerate into crisp cover.
(2) Consider $\mathbb{N}^{\beta}_{\mathcal{C}(r)}=\cap\{\mathcal{C}_j \in \mathcal{C}/\mathcal{C}_j(r) \succcurlyeq \beta,\ j=1,2,...,m\}$ is an SF β-covering neighborhood of $\mathcal{X}$. If $\beta=(1,0,0)$, then there exists at least an SFN $\alpha=(\psi_{\Im},\eta_{\Im},\lambda_{\Im})=(1,0,0)=\mathcal{C}_i(r)$, such that $\alpha=\mathcal{C}_i(r) \succcurlyeq \beta$, for $r \in \mathcal{X}$. Then, each $\mathbb{N}^{\beta}_{\mathcal{C}(r)}$ contains at least an SFN $\alpha=(\psi_{\Im},\eta_{\Im},\lambda_{\Im})=(1,0,0)$ for $r \in \mathcal{X}$. Thus, $\mathbb{N}^{\beta}_{r}=\cap\{\mathcal{C}_j/\mathcal{C}_j \in \mathcal{C}$ and $r \in \mathcal{C}_j,\ j=1,2,...,m\}$. Therefore, if $\beta=(1,0,0)$, then SFβ-covering neighborhoods degenerate into crisp neighborhood.

Proofs of (3) and (4) are similar to (1) and (2). □

Definition 15. *For any SFN $\alpha=(\psi_{\Im},\eta_{\Im},\lambda_{\Im})$, the score function of α is denoted and defined as*

$$S(\alpha)=\psi^2_{\Im}-\eta^2_{\Im}-\lambda^2_{\Im},\quad S(\alpha)\in[-1,1].$$

The larger the value of score function, the better the score function is.

Example 1. *Let us consider that $(\mathcal{X},\mathcal{C})$ is an SFCAS and $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_5\}$ is the collection of SFSs of $\mathcal{X}$ such that $\mathcal{X}=\{r_1,r_2,...,r_6\}$ with $\beta=(0.5,0.3,0.6)$ as follows from Table 1,*

Table 1. A tabular representation of SF β-covering $\mathcal{C}$ in Example 1.

$\mathcal{X}/\mathcal{C}$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
r_1	(0.4, 0.3, 0.5)	(0.8, 0.3, 0.5)	(0.7, 0.5, 0.4)	(0.6, 0.4, 0.5)	(0.9, 0.3, 0.2)
r_2	(0.7, 0.3, 0.6)	(0.9, 0.2, 0.3)	(0.7, 0.3, 0.4)	(0.9, 0.4, 0.1)	(0.7, 0.5, 0.5)
r_3	(0.5, 0.4, 0.6)	(0.8, 0.1, 0.5)	(0.8, 0.3, 0.4)	(0.5, 0.4, 0.7)	(0.5, 0.4, 0.6)
r_4	(0.6, 0.6, 0.2)	(0.8, 0.3, 0.5)	(0.7, 0.6, 0.3)	(0.7, 0.3, 0.6)	(0.9, 0.1, 0.4)
r_5	(0.6, 0.3, 0.7)	(0.9, 0.2, 0.3)	(0.5, 0.6, 0.6)	(0.7, 0.2, 0.6)	(0.4, 0.3, 0.5)
r_6	(0.6, 0.5, 0.4)	(0.6, 0.3, 0.7)	(0.5, 0.3, 0.4)	(0.7, 0.6, 0.2)	(0.6, 0.2, 0.5)

Hence, $\mathcal{C}$ is an SF β-covering of $\mathcal{X}$. Then $\mathbb{N}^{(0.5,0.3,0.6)}_{\mathcal{C}(r_1)} = \mathcal{C}_2 \cap \mathcal{C}_5$, $\mathbb{N}^{(0.5,0.3,0.6)}_{\mathcal{C}(r_2)} = \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$, $\mathbb{N}^{(0.5,0.3,0.6)}_{\mathcal{C}(r_3)} = \mathcal{C}_2 \cap \mathcal{C}_3$, $\mathbb{N}^{(0.5,0.3,0.6)}_{\mathcal{C}(r_4)} = \mathcal{C}_2 \cap \mathcal{C}_4 \cap \mathcal{C}_5$, $\mathbb{N}^{(0.5,0.3,0.6)}_{\mathcal{C}(r_5)} = \mathcal{C}_2 \cap \mathcal{C}_4$, *and* $\mathbb{N}^{(0.5,0.3,0.6)}_{\mathcal{C}(r_6)} = \mathcal{C}_3 \cap \mathcal{C}_5$

From $\mathbb{N}^{\beta}_{\mathcal{C}} = \{\mathbb{N}^{\beta}_{\mathcal{C}(r)} / r \in \mathcal{X}\}$, Table 2 is obtained.

Table 2. A tabular representation of $\mathbb{N}^{(0.5,0.3,0.6)}_{\mathcal{C}}$ in Example 1.

$\mathbb{N}^{\beta}_{\mathcal{C}}$	r_1	r_2	r_3	r_4	r_5	r_6
r_1	$(0.8, 0.3, 0.5)$	$(0.7, 0.2, 0.5)$	$(0.5, 0.1, 0.6)$	$(0.8, 0.1, 0.5)$	$(0.4, 0.2, 0.5)$	$(0.6, 0.2, 0.7)$
r_2	$(0.4, 0.3, 0.5)$	$(0.7, 0.2, 0.6)$	$(0.5, 0.1, 0.6)$	$(0.6, 0.3, 0.5)$	$(0.5, 0.2, 0.7)$	$(0.5, 0.3, 0.7)$
r_3	$(0.7, 0.3, 0.5)$	$(0.7, 0.2, 0.4)$	$(0.8, 0.1, 0.5)$	$(0.7, 0.3, 0.5)$	$(0.5, 0.2, 0.6)$	$(0.5, 0.3, 0.7)$
r_4	$(0.6, 0.3, 0.5)$	$(0.7, 0.2, 0.5)$	$(0.5, 0.1, 0.7)$	$(0.7, 0.1, 0.6)$	$(0.4, 0.2, 0.6)$	$(0.6, 0.2, 0.7)$
r_5	$(0.6, 0.3, 0.5)$	$(0.9, 0.2, 0.3)$	$(0.5, 0.1, 0.7)$	$(0.7, 0.3, 0.6)$	$(0.7, 0.2, 0.6)$	$(0.6, 0.3, 0.7)$
r_6	$(0.7, 0.3, 0.7)$	$(0.7, 0.3, 0.5)$	$(0.5, 0.3, 0.6)$	$(0.7, 0.1, 0.4)$	$(0.4, 0.3, 0.6)$	$(0.5, 0.2, 0.5)$

Therefore

$$\mathbb{M}^{(0.5,0.3,0.6)}_{\mathcal{C}} = \begin{pmatrix} r_1 & (0.8,0.3,0.5) & (0.7,0.2,0.5) & (0.5,0.1,0.6) & (0.8,0.1,0.5) & (0.4,0.2,0.5) & (0.6,0.2,0.7) \\ r_2 & (0.4,0.3,0.5) & (0.7,0.2,0.6) & (0.5,0.1,0.6) & (0.6,0.3,0.5) & (0.5,0.2,0.7) & (0.5,0.3,0.7) \\ r_3 & (0.7,0.3,0.5) & (0.7,0.2,0.4) & (0.8,0.1,0.5) & (0.7,0.3,0.5) & (0.5,0.2,0.6) & (0.5,0.3,0.7) \\ r_4 & (0.6,0.3,0.5) & (0.7,0.2,0.5) & (0.5,0.1,0.7) & (0.7,0.1,0.6) & (0.4,0.2,0.6) & (0.6,0.2,0.7) \\ r_5 & (0.6,0.3,0.5) & (0.9,0.2,0.3) & (0.5,0.1,0.7) & (0.7,0.3,0.6) & (0.7,0.2,0.6) & (0.6,0.3,0.7) \\ r_6 & (0.7,0.3,0.7) & (0.7,0.3,0.5) & (0.5,0.3,0.6) & (0.7,0.1,0.4) & (0.4,0.3,0.6) & (0.5,0.2,0.5) \end{pmatrix}$$

Definition 16. *Suppose that $(\mathcal{X}, \mathcal{C})$ is an SFCAS and $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m\}$ is a set of SF β-coverings of $\mathcal{X}$ for some $\beta = (\psi_{\Im}, \eta_{\Im}, \lambda_{\Im})$ and $\mathcal{X} = \{r_1, ..., r_n\}$. Consider that the neighborhood system $\mathbb{N}^{\beta}_{\mathcal{C}} = \left\{\mathbb{N}^{\beta}_{\mathcal{C}(r)} / r \in \mathcal{X}\right\}$ induced by SF β-covering of $\mathcal{C}$ such that $\mathbb{N}^{\beta}_{\mathcal{C}(r_i)} = \left\{< r_i, \psi_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i, r_j), \eta_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i, r_j), \lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_j)}}(r_i, r_j) > / j = 1, ...m\right\}$ for all $i = 1, ..., n$. Now, for any $\Im \in SFS(\mathcal{X})$ where $\Im = \{ < \psi_{\Im}(r_j), \eta_{\Im}(r_j), \lambda_{\Im}(r_j) > / j = 1, ..., m\}$, the lower and upper approximations of $\Im$ with respect to $\mathbb{N}^{\beta}_{\mathcal{C}(r)}$ is denoted and defined by*

$$\mathbb{N}^{\beta}_{\mathcal{C}}(\Im) = \left(\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im), \overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im)\right)$$

where

$$\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im) = \left\{< r_i, \psi_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im)}(r_i), \eta_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im)}(r_i), \lambda_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im)}(r_i) > / i = 1, ..., n\right\} \quad (1)$$

$$\text{and } \overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im) = \left\{< r_i, \psi_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im)}(r_i), \eta_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im)}(r_i), \lambda_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\Im)}(r_i) > / i = 1, ..., n\right\} \quad (2)$$

such that

$$\begin{aligned}
\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)}(r_i) &= \overset{m}{\underset{j=1}{\wedge}}\left\{\psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i,r_j)\wedge\psi_{\Im}(r_j)\right\} \\
\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)}(r_i) &= \overset{m}{\underset{j=1}{\wedge}}\left\{\eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i,r_j)\wedge\eta_{\Im}(r_j)\right\} \\
\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)}(r_i) &= \overset{m}{\underset{j=1}{\vee}}\left\{\lambda_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i,r_j)\vee\lambda_{\Im}(r_j)\right\} \\
\text{and } \psi_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)}(r_i) &= \overset{m}{\underset{j=1}{\vee}}\left\{\psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i,r_j)\vee\psi_{\Im}(r_j)\right\} \\
\eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)}(r_i) &= \overset{m}{\underset{j=1}{\wedge}}\left\{\eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i,r_j)\wedge\eta_{\Im}(r_j)\right\} \\
\lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)}(r_i) &= \overset{m}{\underset{j=1}{\wedge}}\left\{\lambda_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i,r_j)\wedge\lambda_{\Im}(r_j)\right\}
\end{aligned}$$

Therefore, the operators $\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im), \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)$ *:SFS*$(\mathcal{X})$ $\rightarrow$*SFS*$(\mathcal{X})$ *are said to be lower and upper spherical fuzzy rough approximation operators (SFRAOs) with respect to* $\mathbb{N}_{\mathcal{C}}^{\beta}$.
Hence, the covering-based spherical fuzzy rough set (CSFRS) is the pair $\mathbb{N}_{\mathcal{C}}^{\beta}(\Im) = \left(\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im), \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)\right)$, *whenever* $\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im) \neq \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)$.

Remark 1.
(a) : *The notion of CSFRS is the generalized structure of intuitionistic fuzzy covering rough sets model given in Reference [38], if* $0 \leq \psi_{\Im} + \lambda_{\Im} \leq 1$ *and* $\eta_{\Im} = 0$.
(b) : *The notion of CSFRS is the generalized structure of Pythagorean fuzzy covering rough sets model given in Reference [39], if* $0 \leq (\psi_{\Im})^2 + (\lambda_{\Im})^2 \leq 1$ *and* $\eta_{\Im} = 0$.
(c) : *The notion of CSFRS is the generalized structure of picture fuzzy covering rough sets model, if* $0 \leq \psi_{\Im} + \eta_{\Im} + \lambda_{\Im} \leq 1$, *as defined in Definition 12.*

Example 2. Consider that $\Im \in SFS(\mathcal{X})$, that is $\Im = \{< r_1, 0.9, 0.2, 0.3 >, < r_2, 0.8, 0.3, 0.5 >, < r_3, 0.7, 0.4, 0.5 >, < r_4, 0.5, 0.3, 0.7 >, < r_5, 0.7, 0.2, 0.6 >, < r_6, 0.8, 0.4, 0.3 >\}$, *and if we consider* $\mathbb{M}_{\mathcal{C}}^{\beta} = \left[\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}(r_j)\right]_{(r_i,r_j)\in\mathcal{X}\times\mathcal{X}}$ *as from Example 1, where* $\beta = (0.5, 0.3, 0.6)$. *Then, from Equations (1) and (2), we have*
$\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im) = \{< r_1, 0.4, 0.1, 0.7 >, < r_2, 0.4, 0.1, 0.7 >, < r_3, 0.5, 0.1, 0.7 >, < r_4, 0.4, 0.1, 0.7 >, < r_5, 0.5, 0.1, 0.7 >, < r_6, 0.4, 0.1, 0.7 >\}$
$\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im) = \{< r_1, 0.9, 0.1, 0.3 >, < r_2, 0.9, 0.1, 0.3 >, < r_3, 0.9, 0.1, 0.3 >, < r_4, 0.9, 0.1, 0.3 >, < r_5, 0.9, 0.1, 0.3 >, < r_6, 0.9, 0.1, 0.3 >\}$.

Definition 17. *Let us consider that* $\Im_1 = (\psi_{\Im_1}, \eta_{\Im_1}, \lambda_{\Im_1})$ *and* $\Im_2 = (\psi_{\Im_2}, \eta_{\Im_2}, \lambda_{\Im_2})$ *are two SFSs. Then, the distance between* $\Im_1$ *and* $\Im_2$ *are define as follows:*

$$\mathcal{D}(\Im_1, \Im_2) = \sqrt{\frac{1}{2n}\sum_{r\in\mathcal{X}}\left\{\left|\psi_{\Im_1}^2 - \psi_{\Im_2}^2\right| + \left|\eta_{\Im_1}^2 - \eta_{\Im_2}^2\right| + \left|\lambda_{\Im_1}^2 - \lambda_{\Im_2}^2\right|\right\}}. \tag{3}$$

Theorem 1. *Let us consider* $(\mathcal{X}, \mathcal{C})$ *to be an SFCAS. Now, for some* $\beta = (\psi_{\Im}, \eta_{\Im}, \lambda_{\Im})$, $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m\}$ *is an SF* β*-covering of set* $\mathcal{X}$. *Consider a neighborhood system* $\mathbb{N}_{\mathcal{C}}^{\beta} = \left\{\mathbb{N}_{\mathcal{C}(r)}^{\beta} / r \in \mathcal{X}\right\}$ *induced by SF* β*-covering* $\mathcal{C}$. *Now, for any* $\Im_1, \Im_2 \in SFS(\mathcal{X})$ *and from Equations (1) and (2), we have the following:*

i $\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im) \subseteq \Im \subseteq \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im)$;

ii If $\Im_1 \subseteq \Im_2$, then $\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1) \subseteq \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)$ and $\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1) \subseteq \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)$;

iii $\sim \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1) = \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\sim \Im_1)$ and $\sim \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1) = \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\sim \Im_1)$

iv $\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2) = \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1) \cap \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)$;

v $\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cup \Im_2) \supseteq \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1) \cup \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)$;

vi $\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1 \cup \Im_2) = \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1) \cup \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)$;

vii $\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1 \cap \Im_2) \subseteq \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1) \cap \overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)$.

Proof. Proofs of **i:** to **iii:** are straightforward and follows from Definition 16.

iv: As we know that

$$\begin{aligned}
\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2) &= \left\{ \begin{array}{c} < r_i, \psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i), \eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i), \lambda_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) > / \\ i = 1, ..., n \end{array} \right\} \\
\text{and } \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1) &= \left\{ < r_i, \psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i), \eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i), \lambda_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i) > / i = 1, ..., n \right\} \\
\underline{\mathcal{N}}_{\mathcal{C}}^{\beta}(\Im_2) &= \left\{ < r_i, \psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)}(r_i), \eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i), \lambda_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)}(r_i) > / i = 1, ..., n \right\}
\end{aligned}$$

In order to show $\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2) = \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1) \cap \underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)$, we have to prove

$$\begin{aligned}
\psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) &= \psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i) \wedge \psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)}(r_i) \\
\eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) &= \eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i) \wedge \eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)}(r_i) \\
\text{and } \lambda_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) &= \lambda_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i) \vee \lambda_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)}(r_i)
\end{aligned}$$

Consider

$$\begin{aligned}
\psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) &= \mathop{\wedge}_{j=1}^{m} \left\{ \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \psi_{(\Im_1 \cap \Im_2)}(r_j) \right\} \\
&= \mathop{\wedge}_{j=1}^{m} \left\{ \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \left\{ \psi_{(\Im_1)}(r_j) \wedge \psi_{(\Im_2)}(r_j) \right\} \right\} \\
&= \mathop{\wedge}_{j=1}^{m} \left\{ \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \psi_{(\Im_1)}(r_j) \right\} \wedge \mathop{\wedge}_{j=1}^{m} \left\{ \begin{array}{c} \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \\ \psi_{(\Im_2)}(r_j) \end{array} \right\} \\
\psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) &= \psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i) \wedge \psi_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)}(r_i)
\end{aligned}$$

Now,

$$\begin{aligned}
\eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) &= \mathop{\wedge}_{j=1}^{m} \left\{ \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \eta_{(\Im_1 \cap \Im_2)}(r_j) \right\} \\
&= \mathop{\wedge}_{j=1}^{m} \left\{ \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \left\{ \eta_{(\Im_1)}(r_j) \wedge \eta_{(\Im_2)}(r_j) \right\} \right\} \\
&= \mathop{\wedge}_{j=1}^{m} \left\{ \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \eta_{(\Im_1)}(r_j) \right\} \wedge \mathop{\wedge}_{j=1}^{m} \left\{ \begin{array}{c} \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \\ \eta_{(\Im_2)}(r_j) \end{array} \right\} \\
\eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1 \cap \Im_2)}(r_i) &= \eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_1)}(r_i) \wedge \eta_{\underline{\mathbb{N}}_{\mathcal{C}}^{\beta}(\Im_2)}(r_i)
\end{aligned}$$

Next,

$$\begin{aligned}
\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cap\Im_2)}(r_i) &= \bigvee_{j=1}^{m}\left\{\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee\lambda_{(\Im_1\cap\Im_2)}(r_j)\right\}\\
&= \bigvee_{j=1}^{m}\left\{\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee\left\{\lambda_{(\Im_1)}(r_j)\vee\lambda_{(\Im_2)}(r_j)\right\}\right\}\\
&= \bigvee_{j=1}^{m}\left\{\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee\lambda_{(\Im_1)}(r_j)\right\}\vee\bigvee_{j=1}^{m}\left\{\begin{array}{c}\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee\\ \lambda_{(\Im_2)}(r_j)\end{array}\right\}\\
\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cap\Im_2)}(r_i) &= \lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)}(r_i)\vee\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)}(r_i)
\end{aligned}$$

Therefore,

$$\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cap\Im_2)=\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)\cap\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)$$

v: Next, to prove

$$\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)\supseteq\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)\cup\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)$$

we have to show $r_i\in\mathcal{X}$

$$\begin{aligned}
\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &\geq \psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)}(r_i)\vee\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)}(r_i)\\
\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &\geq \eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)}(r_i)\wedge\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)}(r_i)\\
\text{and }\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &\leq \lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)}(r_i)\wedge\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)}(r_i)
\end{aligned}$$

Consider

$$\begin{aligned}
\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &= \bigwedge_{j=1}^{m}\left\{\psi_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\psi_{(\Im_1\cup\Im_2)}(r_j)\right\}\\
&= \bigwedge_{j=1}^{m}\left\{\psi_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\left\{\psi_{(\Im_1)}(r_j)\vee\psi_{(\Im_2)}(r_j)\right\}\right\}\\
&\geq \bigwedge_{j=1}^{m}\left\{\psi_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\psi_{(\Im_1)}(r_j)\right\}\vee\bigwedge_{j=1}^{m}\left\{\begin{array}{c}\psi_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\\ \psi_{(\Im_2)}(r_j)\end{array}\right\}\\
\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &\geq \psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)}(r_i)\vee\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)}(r_i)
\end{aligned}$$

Now,

$$\begin{aligned}
\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &= \bigwedge_{j=1}^{m}\left\{\eta_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\eta_{(\Im_1\cup\Im_2)}(r_j)\right\}\\
&= \bigwedge_{j=1}^{m}\left\{\eta_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\left\{\eta_{(\Im_1)}(r_j)\wedge\eta_{(\Im_2)}(r_j)\right\}\right\}\\
&= \bigwedge_{j=1}^{m}\left\{\eta_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\psi_{(\Im_1)}(r_j)\right\}\wedge\bigwedge_{j=1}^{m}\left\{\begin{array}{c}\eta_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\wedge\\ \eta_{(\Im_2)}(r_j)\end{array}\right\}\\
\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &= \eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)}(r_i)\wedge\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)}(r_i)
\end{aligned}$$

Further,

$$\begin{aligned}
\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &= \bigvee_{j=1}^{m}\left\{\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee\lambda_{(\Im_1\cup\Im_2)}(r_j)\right\} \\
&= \bigvee_{j=1}^{m}\left\{\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee\left\{\lambda_{(\Im_1)}(r_j)\wedge\lambda_{(\Im_2)}(r_j)\right\}\right\} \\
&\leq \bigvee_{j=1}^{m}\left\{\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee\lambda_{(\Im_1)}(r_j)\right\}\wedge\bigvee_{j=1}^{m}\left\{\begin{array}{c}\lambda_{\mathbb{N}^{\beta}_{\mathcal{C}(r_i)}}(r_i,r_j)\vee \\ \lambda_{(\Im_2)}(r_j)\end{array}\right\} \\
\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)}(r_i) &\leq \lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)}(r_i)\wedge\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)}(r_i)
\end{aligned}$$

Hence,

$$\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1\cup\Im_2)\supseteq\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_1)\cap\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\Im_2)$$

Thus concludes, the proofs of **vi** and **vii** are similar to **iv** and **v**. □

4. A New Proposal for Multi-Attribute Decision-Making Using Spherical Fuzzy Rough Sets Hybrid with TOPSIS

In this section, a new technique for MADM is proposed. Here, concepts of CSFRS model will be employed, which are stated in Section 3. Major steps for this decision-making method and its associated algorithms are presented in the following.

In real life situations, MADM has an important role and an intelligent decision approach to solve the complex and uncertain decisions under senior experts. The basic concepts of this proposed method for MADM are given as follows. Let $\mathcal{X}=\{r_1,r_2,...,r_n\}$ be any set of n feasible alternatives and $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,...,\mathcal{C}_m\}$ be the finite set of attributes. If all the attributes $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,...,\mathcal{C}_m\}$ are of same type, then there is no need for normalization. Conversely, if these contain different scales and/or units, then there is need to transform them all to the same scale and/or unit. Let us consider two types of attributes, namely (a), the cost type, and (b), the benefit type. Considering their natures, a benefit attribute (the bigger the values the better it is) and cost attribute (the smaller the values the better it is) are of rather opposite types. In such cases, we need to first transform the attribute values of cost type into the attribute values of benefit type. So, transform the Spherical fuzzy decision matrix $\mathcal{M}=\left[\alpha_{ij}\right]_{m\times n}$, into a normalized decision matrix $\mathcal{M}^*=\left[\gamma^*_{ij}\right]_{m\times n}$ where $\alpha_{ij}=(\psi_{\Im_{ij}},\eta_{\Im_{ij}},\lambda_{\Im_{ij}})$ and

$$\gamma^*_{ij}=\begin{cases}\alpha_{ij}=(\psi_{\Im_{ij}},\eta_{\Im_{ij}},\lambda_{\Im_{ij}}) & \text{for benefit attribute, } i=1,2,...,m \text{ and } j=1,2,...n\\ (\alpha_{ij})^c=(\lambda_{\Im_{ij}},\eta_{\Im_{ij}},\psi_{\Im_{ij}}) & \text{for cost attribute, } i=1,2,...,m \text{ and } j=1,2,...n\end{cases}$$

where $(\alpha_{ij})^c$ is the complement of (α_{ij}). Let $\omega=(\omega_1,\omega_2,...,\omega_m)^T$ be the weight vector of all attributes such that $0\leq\omega_i\leq 1$ and $\sum_{i=1}^{m}\omega_i=1$ with $i=1,....m$. Decision makers $\mathcal{D}_{mem}$, $\mathcal{D}_{nuet.\ mem}$, and $\mathcal{D}_{non-mem}$ put forward the assessment values of all the alternatives $r_i(i=1,...n)$ corresponding to the set of attributes $\mathcal{C}_{ij}(i=1,2,...,n$ and $j=1,2,...,m)$ which is defined by a mapping $\mathcal{F}=\{f(r_i,\mathcal{C}_{ij})/i=1,2,...,n$ and $j=1,2,...,m\}$ where $\mathcal{C}_{ij}=(\psi_{ij},\eta_{ij},\lambda_{ij})$ from $\mathcal{X}$ to $\mathcal{C}$. This means that the decision maker $\mathcal{D}_{mem}$ provides MG ψ_{ij} to the alternative r_i according to the attribute $\mathcal{C}_j$, the decision maker $\mathcal{D}_{nuet.\ mem}$ provides NG η_{ij} to the alternative r_i according to the attribute $\mathcal{C}_j$, and the decision maker $\mathcal{D}_{non-mem}$ provides NMG λ_{ij} to the alternative r_i according to the attribute $\mathcal{C}_j$.

By the principle of the SF-TOPSIS method, to get the best SF decision making object (SFDMO) D^+ and the worst SFDMO D^-, a new SFS $D=(\psi_D,\eta_D,\lambda_D)=(\xi_{\mathcal{D}^+},\xi_{\mathcal{D}^-})$ will be constructed. Hence, a multi-attribute spherical fuzzy decision making information system (MASFDMIS) ($\mathcal{X}$,$\mathcal{C}$,$\Im$, $\mathcal{D}$) is established. Then, by the preference evaluations, the rank of all objects of the decision-making problem is determined.

Here, we will first suggest the SF-TOPSIS method and, in this method, first calculate the most favorable/best and worst SFDMOs. The most favorable/best and worst SFDMOs of the universe according to the decision maker $\mathcal{D}$ w.r.t. the attribute $\mathcal{C}$ are defined as follows:

$\mathcal{D}^+ = \{(r_i, \max\{\omega_i \cdot f(r_i, \psi_{ij})\}, \min\{\omega_i \cdot f(r_i, \eta_{ij})\}, \min\{\omega_i \cdot f(r_i, \lambda_{ij})\})/r \in \mathcal{X}, i = 1, 2, ..., n$ and $j = 1, 2, ..., m\}$

$\mathcal{D}^- = \{(r_i, \min\{\omega_i \cdot f(r_i, \psi_{ij})\}, \min\{\omega_i \cdot f(r_i, \eta_{ij})\}, \max\{\omega_i \cdot f(r_i, \lambda_{ij})\})/r \in \mathcal{X}, i = 1, 2, ..., n$ and $j = 1, 2, ..., m\}$

Therefore, we get the two new SFS $\mathcal{D}^+ = (\psi_{\mathcal{D}^+}, \eta_{\mathcal{D}^+}, \lambda_{\mathcal{D}^+})$ and $\mathcal{D}^- = (\psi_{\mathcal{D}^-}, \eta_{\mathcal{D}^-}, \lambda_{\mathcal{D}^-})$.

Now, we are going to define the t-norm and t-conorm which will help us to finding the ranking of alternatives.

Definition 18. *A mapping $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$ is known to be a triangular norm (briefly t-norm), if it is commutative, associative, increasing, and satisfies the boundary condition that is $\mathcal{T}(r,1,1) = r \ \forall r \in [0,1]$ At the same time mapping $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$ is known to be a triangular conorm (briefly t-conorm), if it is commutative, associative, increasing, and satisfies the boundary condition that is $\mathcal{T}(r,0,0) = r \ \forall r \in [0,1]$. Here, in this paper, t-norm and t-conorm are used for the MADM problem.*

$$\mathcal{T}_{\Im}(r_1, r_2, r_3) = \frac{r_1 r_2 r_3}{\sqrt{1 + (1 - r_1^2)(1 - r_2^2)(1 - r_3^2)}} \text{ and } \mathcal{T}_{\Im}(r_1, r_2, r_3) = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2}{1 + r_1^2 r_2^2 r_3^2}}$$

Further, by the use of Definition 16, to find the lower and upper approximations of best and worst SFDMOs under the consistency consensus threshold $\alpha (0 < \alpha \leq 1)$, the following are presented.

$$\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^-)}(r_i) = \bigwedge_{j=1}^{m} \left\{ \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \psi_{\mathcal{D}^-}(r_j) \right\} \quad (4)$$

$$\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^-)}(r_i) = \bigwedge_{j=1}^{m} \left\{ \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \eta_{\mathcal{D}^-}(r_j) \right\} \quad (5)$$

$$\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^-)}(r_i) = \bigvee_{j=1}^{m} \left\{ \lambda_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \vee \lambda_{\mathcal{D}^-}(r_j) \right\} \quad (6)$$

$$\psi_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^-)}(r_i) = \bigvee_{j=1}^{m} \left\{ \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \vee \psi_{\mathcal{D}^-}(r_j) \right\} \quad (7)$$

$$\eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^-)}(r_i) = \bigwedge_{j=1}^{m} \left\{ \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \eta_{\mathcal{D}^-}(r_j) \right\} \quad (8)$$

$$\lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^-)}(r_i) = \bigwedge_{j=1}^{m} \left\{ \lambda_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \lambda_{\mathcal{D}^-}(r_j) \right\} \quad (9)$$

and

$$\psi_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^+)}(r_i) = \bigvee_{j=1}^{m} \left\{ \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \vee \psi_{\mathcal{D}^+}(r_j) \right\} \quad (10)$$

$$\eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^+)}(r_i) = \bigwedge_{j=1}^{m} \left\{ \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \eta_{\mathcal{D}^+}(r_j) \right\} \quad (11)$$

$$\lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^+)}(r_i) = \bigwedge_{j=1}^{m} \left\{ \lambda_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \lambda_{\mathcal{D}^+}(r_j) \right\} \quad (12)$$

$$\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i) = \bigwedge_{j=1}^{m} \left\{ \psi_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \psi_{\mathcal{D}^{+}}(r_j) \right\} \tag{13}$$

$$\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i) = \bigwedge_{j=1}^{m} \left\{ \eta_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \wedge \eta_{\mathcal{D}^{+}}(r_j) \right\} \tag{14}$$

$$\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i) = \bigvee_{j=1}^{m} \left\{ \lambda_{\mathbb{N}_{\mathcal{C}(r_i)}^{\beta}}(r_i, r_j) \vee \lambda_{\mathcal{D}^{+}}(r_j) \right\} \tag{15}$$

In the final step, we use the principle of ranking through the lower and upper approximations of the best and worst SFDMOs for all alternatives of the universal set $\mathcal{X}$ under the consistency consensus threshold $\alpha(0 < \alpha \leq 1)$.

Now, we define the ranking function of the MASFDM problem for any alternative of the universal set $\mathcal{X}$.

Definition 19. *Suppose the MASFDMIS is $(\mathcal{X}, \mathcal{C}, \Im, \mathcal{D})$. For the best and worst SFDMOs $\mathcal{D}^{+} = (\psi_{\mathcal{D}^{+}}, \eta_{\mathcal{D}^{+}}, \lambda_{\mathcal{D}^{+}})$ and $\mathcal{D}^{-} = (\psi_{\mathcal{D}^{-}}, \eta_{\mathcal{D}^{-}}, \lambda_{\mathcal{D}^{-}}) \in SFS(\mathcal{X})$ represents the preference information of the decision maker $\mathcal{D}$ under the consistency consensus threshold $\alpha(0 < \alpha \leq 1)$. Then, we define the ranking function of alternative $r_i (i = 1, ..., n)$ w.r.t. $\mathcal{D}^{+}$ and $\mathcal{D}^{-}$ as*

$$\begin{aligned} \xi_{\mathcal{D}^{-}}(r_i) = \quad & \alpha \mathcal{T}_{\Im}\left(\mu_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i), \eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i), \lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i) \right) + \\ & (1-\alpha) \cdot \mathcal{T}_{\Im}\left(\mu_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i), \eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i), \lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i) \right) \end{aligned} \tag{16}$$

and

$$\begin{aligned} \xi_{\mathcal{D}^{+}}(r_i) = \quad & \alpha \mathcal{T}_{\Im}\left(\mu_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i), \eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i), \lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i) \right) + \\ & (1-\alpha) \cdot \mathcal{T}_{\Im}\left(\mu_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i), \eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i), \lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i) \right) \end{aligned} \tag{17}$$

From the definition of ranking function, it is cleared that $0 \leq \xi_{\mathcal{D}^{-}}(r_i), \xi_{\mathcal{D}^{+}}(r_i) \leq 1$, $r_i \in \mathcal{X}$, $i = 1, 2, ..., n$.

Definition 20. *Finally, we put forward the optimal object for the MASFDM problem with the help of ranking function $\xi(r_i)$ for all alternatives; $r_i \in \mathcal{X}, (i = 1, ..., n)$ can be calculated as*

$$\xi(r_i) = \frac{1}{2}\{\xi_{\mathcal{D}^{+}}(r_i) + \xi_{\mathcal{D}^{-}}(r_i)\}, r_i \in \mathcal{X}, (i = 1, ..., n) \tag{18}$$

From the definition of ranking function, it is cleared that $0 \leq \xi(r_i) \leq 1$.

4.1. Algorithm of Decision Making Problem Based on CSFRS

In this subsection, we presented the step wise algorithm of the proposed method for solving the MADM approach based on CSFRS under the SF environment. The practical utility of the proposed approach is demonstrated through a numerical example given in Section 4.2. With the help of the above interpretation, the algorithm of the proposed approach based on CSFRS consist of the following steps:
input: MASFDMIS $(\mathcal{X}, \mathcal{C}, \Im, \mathcal{D})$;
output: The sort ordering for all alternatives;

step i Calculate the best and worst SFDMOs $\mathcal{D}^{+} = (\psi_{\mathcal{D}^{+}}, \eta_{\mathcal{D}^{+}}, \lambda_{\mathcal{D}^{+}})$ and $\mathcal{D}^{-} = (\psi_{\mathcal{D}^{-}}, \eta_{\mathcal{D}^{-}}, \lambda_{\mathcal{D}^{-}}) \in$SFS$(\mathcal{X})$,

Step ii Next by using Equations (4) to (15) to find the lower and upper approximations $\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i)$, $\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i)$, $\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i)$, $\psi_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i)$, $\eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i)$, $\lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{-})}(r_i)$ and $\psi_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i)$, $\eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i)$, $\lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i)$, $\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i)$, $\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i)$, $\lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})}(r_i)$;

step iii Determine the ranking function $\xi_{\mathcal{D}^{-}}(r_i)$, $\xi_{\mathcal{D}^{+}}(r_i)$ and $\xi(r_i)$, $r_i \in \mathcal{X}$, $(i = 1, ..., n)$ by using Equations (16) and (17);

step iv Finally, present the ranking order of all the alternatives through a ranking function by using Equation (18) to get the optimal alternative.

4.2. Illustrative Example

Here, in this section, we will present the proposed method of MADM based on CSFRS models which relates the assessment and rank of heavy rainfall in the Lasbella district and adjoining areas of the Baluchistan, Pakistan. Then, SF-TOPSIS will provide the desired ranking.

A recent storm caused a spell of heavy rainfall in the Lasbella district, and adjoining areas of Baluchistan, Pakistan were hit with unprecedented flash floods in February 2019. A large number of roads, which connect the Lasbella district with other parts of Baluchistan had been destroyed in this flood. In this context, the Pakistan government has to take a considerable number of road building projects either to preserve the roads already built or to undertake the new roads.

These projects have been carried out by a limited number of the well-established contractors, and the selection process has been on the basis of bid price alone. In recent years, the increased project complexity, technical capability, higher performance, and safety and financial requirements have been demanding the use of multi-attribute decision making methods. For this, Pakistan government has issued a notice in the newspapers, and one construction company take the responsibility of selecting the best construction company out of a set of six possible alternatives, $\mathcal{X} = \{r_1 = Ahmed\ Construction$, $r_2 = Matracon\ Pakistan\ Private(Pvt)\ Limited(Ltd)$, $r_3 = Eastern\ Highway\ Company$, $r_4 = Banu\ Mukhtar\ Concrete\ Pvt.\ Ltd.$,$r_5 = Khyber\ Grace\ Pvt.\ Ltd.$, $r_6 = Experts\ Engineering\ Services\}$ on the basis of the attributes, $\mathcal{C}_1 = technical\ capability$, $\mathcal{C}_2 = higher\ performance$, $\mathcal{C}_3 = safety$, $\mathcal{C}_4 = financial\ requirements$, $\mathcal{C}_5 = time\ saving$, that is bid for these projects, and all the criterion are of the benefit type, so there is no need to normalized it. Then, the objective of the Government is to choose the best construction company among them for the task. In order to fulfill it, let

$$\omega_1 = 0.2,\ \omega_2 = 0.18,\ \omega_3 = 0.22,\ \omega_4 = 0.25,\ \omega_5 = 0.15$$

be the weight vector corresponding to the six attributes such that they have evaluated each company and gave their preferences in terms of spherical fuzzy information and, hence, constructed the following decision matrices given in Table 3 as shown below:

Table 3. A tabular representation of SFSs for $\mathcal{C}$.

$\mathcal{X}/\mathcal{C}$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
r_1	(0.9, 0.1, 0.2)	(0.8, 0.2, 0.5)	(0.7, 0.3, 0.5)	(0.8, 0.2, 0.5)	(0.9, 0.1, 0.3)
r_2	(0.8, 0.2, 0.4)	(0.3, 0.4, 0.5)	(0.7, 0.5, 0.3)	(0.6, 0.2, 0.1)	(0.5, 0.6, 0.2)
r_3	(0.9, 0.3, 0.1)	(0.6, 0.3, 0.5)	(0.7, 0.2, 0.4)	(0.3, 0.4, 0.1)	(0.5, 0.4, 0.6)
r_4	(0.8, 0.1, 0.5)	(0.6, 0.2, 0.7)	(0.5, 0.3, 0.4)	(0.7, 0.3, 0.2)	(0.8, 0.4, 0.3)
r_5	(0.6, 0.5, 0.2)	(0.9, 0.4, 0.3)	(0.5, 0.3, 0.7)	(0.3, 0.2, 0.1)	(0.8, 0.5, 0.2)
r_6	(0.8, 0.3, 0.5)	(0.6, 0.3, 0.1)	(0.9, 0.3, 0.2)	(0.6, 0.2, 0.3)	(0.5, 0.1, 0.4)

Step i According to the steps of the algorithm, first to calculate the best and worst SFDMOs $\mathcal{D}^+$ and $\mathcal{D}^- \in$SFS$(\mathcal{X})$,

$$\mathcal{D}^+ = \left\{ \begin{array}{c} \frac{(0.2,0.015,0.04)}{r_1}, \frac{(0.16,0.04,0.025)}{r_2}, \frac{(0.18,0.044,0.02)}{r_3}, \frac{(0.175,0.02,0.045)}{r_4}, \\ \frac{(0.162,0.05,0.025)}{r_5}, \frac{(0.198,0.015,0.018)}{r_6} \end{array} \right\}$$
$$\mathcal{D}^- = \left\{ \begin{array}{c} \frac{(0.135,0.015,0.125)}{r_1}, \frac{(0.054,0.04,0.09)}{r_2}, \frac{(0.075,0.044,0.09)}{r_3}, \frac{(0.108,0.02,0.126)}{r_4}, \\ \frac{(0.075,0.05,0.154)}{r_5}, \frac{0.075,0.015,0.1}{r_6} \end{array} \right\}$$

Consider the computation of the SF β- neighborhood of r in $\mathcal{X}$, and where $\beta = (0.6, 0.5, 0.4)$. Then

$\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}(r_1)} = \mathcal{C}_1 \cap \mathcal{C}_5$ $\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}(r_2)} = \mathcal{C}_1 \cap \mathcal{C}_3 \cap \mathcal{C}_4$ $\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}(r_3)} = \mathcal{C}_1 \cap \mathcal{C}_3$
$\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}(r_4)} = \mathcal{C}_4 \cap \mathcal{C}_5$ $\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}(r_5)} = \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_5$ $\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}(r_6)} = \mathcal{C}_2 \cap \mathcal{C}_3 \cap \mathcal{C}_4$

From $\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}} = \left\{ \mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}(r)} / r \in \mathcal{X} \right\}$, Table 4 is obtained.

Table 4. A tabular representation of $\mathbb{N}^{(0.6,0.5,0.4)}_{\mathcal{C}}$.

$\mathbb{N}^{\beta}_{\mathcal{C}}$	r_1	r_2	r_3	r_4	r_5	r_6
r_1	(0.9, 0.1, 0.3)	(0.5, 0.2, 0.4)	(0.5, 0.3, 0.6)	(0.8, 0.1, 0.5)	(0.6, 0.5, 0.2)	(0.5, 0.1, 0.5)
r_2	(0.7, 0.1, 0.5)	(0.6, 0.2, 0.4)	(0.3, 0.2, 0.4)	(0.5, 0.1, 0.5)	(0.3, 0.2, 0.7)	(0.6, 0.2, 0.5)
r_3	(0.7, 0.1, 0.5)	(0.7, 0.2, 0.4)	(0.7, 0.2, 0.4)	(0.5, 0.1, 0.5)	(0.5, 0.3, 0.7)	(0.8, 0.3, 0.5)
r_4	(0.8, 0.1, 0.5)	(0.5, 0.2, 0.2)	(0.3, 0.4, 0.6)	(0.7, 0.3, 0.3)	(0.3, 0.2, 0.2)	(0.5, 0.1, 0.4)
r_5	(0.9, 0.1, 0.5)	(0.3, 0.2, 0.5)	(0.5, 0.3, 0.6)	(0.6, 0.1, 0.7)	(0.6, 0.4, 0.3)	(0.5, 0.1, 0.5)
r_6	(0.7, 0.2, 0.5)	(0.3, 0.2, 0.5)	(0.3, 0.2, 0.5)	(0.5, 0.2, 0.7)	(0.3, 0.2, 0.7)	(0.6, 0.2, 0.3)

Step ii Next to compute the lower and upper approximation, that is $\psi_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)}(r_i)$, $\eta_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)}(r_i)$, $\lambda_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)}(r_i)$, $\psi_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)}(r_i)$, $\eta_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)}(r_i)$, $\lambda_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)}(r_i)$ and $\psi_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^+)}(r_i)$, $\eta_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^+)}(r_i)$, $\lambda_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^+)}(r_i)$, $\psi_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^+)}(r_i)$, $\eta_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^+)}(r_i)$, $\lambda_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^+)}(r_i)$, use Equations (4)–(15);

$$\begin{aligned}
\psi_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)} &= \frac{0.054}{r_1}, \frac{0.054}{r_2}, \frac{0.054}{r_3}, \frac{0.054}{r_4}, \frac{0.054}{r_5}, \frac{0.054}{r_6} \\
\eta_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)} &= \frac{0.015}{r_1}, \frac{0.015}{r_2}, \frac{0.015}{r_3}, \frac{0.015}{r_4}, \frac{0.015}{r_5}, \frac{0.015}{r_6} \\
\lambda_{\underline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)} &= \frac{0.6}{r_1}, \frac{0.7}{r_2}, \frac{0.7}{r_3}, \frac{0.6}{r_4}, \frac{0.7}{r_5}, \frac{0.7}{r_6} \\
\psi_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)} &= \frac{0.9}{r_1}, \frac{0.7}{r_2}, \frac{0.8}{r_3}, \frac{0.8}{r_4}, \frac{0.9}{r_5}, \frac{0.7}{r_6} \\
\eta_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)} &= \frac{0.015}{r_1}, \frac{0.015}{r_2}, \frac{0.015}{r_3}, \frac{0.015}{r_4}, \frac{0.015}{r_5}, \frac{0.015}{r_6} \\
\lambda_{\overline{\mathbb{N}^{\beta}_{\mathcal{C}}}(\mathcal{D}^-)} &= \frac{0.09}{r_1}, \frac{0.09}{r_2}, \frac{0.09}{r_3}, \frac{0.09}{r_4}, \frac{0.09}{r_5}, \frac{0.09}{r_6}
\end{aligned}$$

and

$$\psi_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})} = \frac{0.16}{r_1}, \frac{0.16}{r_2}, \frac{0.16}{r_3}, \frac{0.16}{r_4}, \frac{0.16}{r_5}, \frac{0.16}{r_6}$$
$$\eta_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})} = \frac{0.015}{r_1}, \frac{0.015}{r_2}, \frac{0.015}{r_3}, \frac{0.015}{r_4}, \frac{0.015}{r_5}, \frac{0.015}{r_6}$$
$$\lambda_{\underline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})} = \frac{0.6}{r_1}, \frac{0.7}{r_2}, \frac{0.7}{r_3}, \frac{0.6}{r_4}, \frac{0.7}{r_5}, \frac{0.7}{r_6}$$
$$\psi_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})} = \frac{0.9}{r_1}, \frac{0.7}{r_2}, \frac{0.8}{r_3}, \frac{0.8}{r_4}, \frac{0.9}{r_5}, \frac{0.7}{r_6}$$
$$\eta_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})} = \frac{0.015}{r_1}, \frac{0.015}{r_2}, \frac{0.015}{r_3}, \frac{0.015}{r_4}, \frac{0.015}{r_5}, \frac{0.015}{r_6}$$
$$\lambda_{\overline{\mathbb{N}_{\mathcal{C}}^{\beta}}(\mathcal{D}^{+})} = \frac{0.018}{r_1}, \frac{0.018}{r_2}, \frac{0.018}{r_3}, \frac{0.018}{r_4}, \frac{0.018}{r_5}, \frac{0.018}{r_6}$$

Step iii Further, from Equations (16) and (17), determine the ranking functions $\xi_{\mathcal{D}^-}(r_i)$ and $\xi_{\mathcal{D}^+}(r_i)$, and consider the risk preference threshold $\alpha = 0.8$, where $(0 < \alpha \leq 1)$,;

$$\xi_{\mathcal{D}^-} = \frac{0.00052655}{r_1}, \frac{0.00052315}{r_2}, \frac{0.00055456}{r_3}, \frac{0.00048902}{r_4}, \frac{0.00059210}{r_5}, \frac{0.00052315}{r_6}$$
$$\xi_{\mathcal{D}^+} = \frac{0.00094868}{r_1}, \frac{0.0011293}{r_2}, \frac{0.0011356}{r_3}, \frac{0.00094117}{r_4}, \frac{0.0011431}{r_5}, \frac{0.0011293}{r_6}$$

Next, calculate the optimal object for the MASFDM problem with the help of the ranking function given in Equation (18);

$$\xi(r_i) = \frac{0.00073762}{r_1}, \frac{0.00082623}{r_2}, \frac{0.00084508}{r_3}, \frac{0.00071510}{r_4}, \frac{0.0008676}{r_5}, \frac{0.00082623}{r_6}$$

Step iv Finally, we are able to present the best optimal alternative sort of the well-established construction company according to the values of the ranking function. Therefore, we can rank all the alternative in order as

$$r_5 > r_3 > r_2 \approx r_6 > r_1 > r_4$$

Hence, by the process of decision-making, finally, we get the optimal selection by the use of CSFRS model based on the MADM method. Therefore, from the numerical calculation, it is clear that the 5th construction company is the best optimal decision making.

4.3. Comparative Analysis

From the above analysis, it is clear that the proposed approach is better than IFSs, PFSs, and PytFSs. The advantages of the proposed method with the existing literature is given below.
Advantages:

(a) If $NG = 0$ and $0 \leq MG + NMG \leq 1$, then the covering-based spherical fuzzy rough set (CSFRS) model was reduced to a covering-based intuitionistic fuzzy rough set model (CIFRS) initiated in Reference [40].
(b) If $NG = 0$ and $0 \leq (MG)^2 + (NMG)^2 \leq 1$, then the CSFRS model was reduced to a covering-based Pythagorean fuzzy rough set model (CPytFRS) presented in Reference [39].
(c) If $0 \leq MG + NG + NMG \leq 1$, then CSFRS model was reduced to a covering-based picture fuzzy rough set (CPFRS) model defined in Definition 12. Therefore, it is clear that the CIFRS, CPytFRS, and CPFRS models are the special cases of CSFRS.

Now, the comparative study of the proposed method with the existing literature is given in Table 5 by considering the above Illustrative Example of Section 4.2.

The main difference of the proposed method with the existing methods given in Table 5 is that the study of CIFRS [40] consists of MG and NMG with the condition that $0 \leq MG + NMG \leq 1$ and has no information about NG. So, due to lake information about NG, it failed to handle the example of Section 4.2. Similarly, the study of CPytFRS [39] consists of MG and NMG with the condition that $0 \leq (MG)^2 + (NMG)^2 \leq 1$ and has no information about NG. So, due to lake information about NG in CPytFRS, it failed to handle the example of Section 4.2. Furthermore, in the case of CPFRSs, it consists of all the three degrees, that is MG, NG, and NMG, with the condition that $0 \leq MG + NG + NMG \leq 1$. Here, if we assign values to MG, NG, and NMG as $(\psi, \eta, \lambda) = (0.8, 0.3, 0.5)$, then in this case, their sum is $0.8 + 0.3 + 0.5 = 1.6 > 1$, but the sum of their square is $0.8^2 + 0.3^2 + 0.5^2 = 0.98 < 1$. So, in this case ordinary PFSs and CPFRS failed to tackle the situation. Thus, from the comparative study, it is clear that the proposed method is more superior and provides more freedom to the decision makers for the selection of MG, NG, and NMG as compared to existing literature.

Table 5. The comparative analysis of the proposed method with the existing literature.

Methods	Score Values	Ranking
CIFRS [40]	Failed to handle	×
CPytFRS [39]	Failed to handle	×
CPFRS (def. 12)	Failed to handle	×
CSFRS	$\frac{0.00073762}{r_1}, \frac{0.00082623}{r_2}, \frac{0.00084508}{r_3}, \frac{0.00071510}{r_4}, \frac{0.0008676}{r_5}, \frac{0.00082623}{r_6}$	$r_5 > r_3 > r_2 \approx r_6 > r_1 > r_4$

5. Conclusions

In real life, the CSFRS model is a significant tool to handle uncertainties. The aim of this paper is to develop a comprehensive model to tackle decision-making problems where strong points of view are in the favour; neutral; and against some projects, entities, or plans. Therefore, a new approach is adopted to hybrid spherical fuzzy sets with notions of covering rough set to presented the new approach of SFCRS through SF β-neighborhoods. By using the proposed approach of SFCRS via SF β-neighborhoods, the existing approach of TOPSIS is generalized to SF-TOPSIS to MADM. An algorithm for the proposed method is given. The main difference of the presented method with existing literature is given in Section 4.3. Proposed model is very useful in decision making problems where decision makers have contradictory views about certain plan or proposal. For example, if we assign values to MG, NG and NMG as $(\psi, \eta, \lambda) = (0.75, 0.1, 0.65)$, then the existing CIFRS [40], CPytFRS [39] and CPFRS failed to handle the situations because the notion of CIFRS and CPytFRS have no information about NG. Further in CPFRS sum of MG, NG and NMG belongs to $[0, 1]$ but in this case $\psi + \eta + \lambda = 0.75 + 0.1 + 0.65 > 1$, so CPFRS failed to cope the situation. Our proposed method have the ability to cope this scenario easily, that is $\psi + \eta + \lambda = 0.75 + 0.1 + 0.65 > 1$, but in case of CSFRS their square sum $(0.75)^2 + (0.1)^2 + (0.65)^2 < 1$. Thus our proposed model has stronger capability than the existing CIFRS, CPytFRS and also CPFRS to manage the uncertainty. An example is given to demonstrate how the proposed method helps us in decision making problems.

Future Work:

In the future, we intend to further discuss these following topics:

- **i** The investigation of SF-entropy of CSFRS models.
- **ii** The investigation of SF soft sets and its applications in MADM.
- **iii** Applying other decision-making methodology based CSFRS models to MASFDM problem.
- **iv** The discussions of other applied methods in information systems.
- **v** The applications of CSFRS models to management sciences and big data processing technique.

Author Contributions: Conceptualization, A.H., T.M., M.I.A., S.A. and M.M.; methodology, A.H.; software, S.Z.; validation, S.Z., T.M. and M.I.A.; formal analysis, A.H.; investigation, S.A. and A.H.; writing—original draft preparation, A.H.; writing—review and editing, S.Z., A.H. and S.A.; visualization, T.M. and M.I.A.; supervision, M.I.A.; funding acquisition, S.Z.

Funding: This paper is supported by Philosophy and Social Science Planning Projects of Zhejiang (No. 16ZJQN022YB), Zhejiang Province Natural Science Foundation (No. LY18G010007), Zhejiang Public Technology Applied Research Projects (No. LGG18F020001), Major Humanities and Social Sciences Research Projects in Zhejiang Universities (No. 2018QN058) and K. C. Wong Magna Fund in Ningbo University.

Acknowledgments: The authors would like to thank the editor in chief, associate editor and the anonymous referees for detailed and valuable comments which helped to improve this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–356. [CrossRef]
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Yager, R.R. Pythagorean membership grades in multicriteria decision making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 958–965. [CrossRef]
4. Yager, R.R.; Abbasov, A.M. Pythagorean membership grades, complex numbers, and decision making. *Int. J. Intell. Syst.* **2013**, *28*, 436–452. [CrossRef]
5. Stanujkić, D.; Karabašević, D. An extension of the WASPAS method for decision-making problems with intuitionistic fuzzy numbers: A case of website evaluation. *Oper. Res. Eng. Sci. Theory Appl.* **2018**, *1*, 29–39. [CrossRef]
6. Cuong, B.C. *Picture Fuzzy Sets First Results. Part 1. Seminar Neuro-Fuzzy Systems with Applications*; Institute of Mathematics: Hanoi, Vitnam, 2013; Volume 3.
7. Cuong, B.C. Picture fuzzy sets. *J. Comput. Sci. Cybern.* **2014**, *30*, 409–420.
8. Cuong, B.C.; Kreinovich, V. Picture fuzzy sets-a new concept for computational intelligence problems. In Proceedings of the Third World Congress on Information and Communication Technologies (WICT), Hanoi, Vietnam, 15–18 December 2013; pp. 1–6.
9. Cuong, B.C.; Van Hai, P. Some fuzzy logic operators for picture fuzzy sets. In Proceedings of the Seventh International Conference onKnowledge and Systems Engineering (KSE), Ho Chi Minh City, Vietnam, 8–10 Octopber 2015; pp. 132–137.
10. Joshi, D.; Kumar, S. An approach to multi-criteria decision making problems using dice similarity measure for picture fuzzy sets. In Proceedings of the International Conference on Mathematics and Computing, Varanasi, India, 9–11 January 2018; pp. 135–140.
11. Ashraf, S.; Mahmood, T.; Abdullah, S.; Khan, Q. Different approaches to aulti-criteria group decision making problems for picture fuzzy environment. *Bull. Braz. Math. Soc.* **2018**, 1–25. [CrossRef]
12. Zeng, S.; Asharf, S.; Arif, M.; Abdullah, S. Application of exponential jensen picture fuzzy divergence measure in multi-criteria group decision making. *Mathematics* **2019**, *7*, 191. [CrossRef]
13. Ashraf, S.; Abdullah, S.; Mahmood, T.; Ghani, F.; Mahmood, T. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J. Intell. Fuzzy Syst.* **2019**, *36*, 2829–2844. [CrossRef]
14. Ashraf, S.; Abdullah, S. Spherical aggregation operators and their application in multiattribute group decision-making. *Int. J. Intell. Syst.* **2019**, *34*, 493–523.
15. Ashraf, S.; Abdullah, S.; Mahmood, T. GRA method based on spherical linguistic fuzzy Choquet integral environment and its application in multi-attribute decision-making problems. *Math. Sci.* **2018**, *12*, 263–275. [CrossRef]
16. Mahmood, T.; Ullah, K.; Khan, Q.; Jan, N. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Comput. Appl.* **2018**, 1–13. [CrossRef]
17. Ullah, K.; Hassan, N.; Mahmood, T.; Jan, N.; Hassan, M. Evaluation of investment policy based on multi-attribute decision-making using interval valued T-spherical fuzzy aggregation operators. *Symmetry* **2019**, *11*, 357. [CrossRef]
18. Liu, P.; Khan, Q.; Mahmood, T.; Hassan, N. T-spherical fuzzy power muirhead mean operator based on novel operational laws and their application in multi-attribute group decision making. *IEEE Access* **2019**, *7*, 22613–22632. [CrossRef]

19. Liu, F.; Aiwu, G.; Lukovac, V.; Vukic, M. A multicriteria model for the selection of the transport service provider: A single valued neutrosophic DEMATEL multicriteria model. Decision Making. *Appl. Manag. Eng.* **2018**, *1*, 121–130. [CrossRef]
20. Yoon, K.P.; Hwang, C.L. *Multiple Attribute Decision Making an Introduction*; Sage Publications: Thousand Oaks, CA, USA, 1995.
21. Ashraf, S.; Abdullah, S.; Smarandache, F. Logarithmic Hybrid Aggregation Operators Based on Single Valued Neutrosophic Sets and Their Applications in Decision Support Systems. *Symmetry* **2019**, *11*, 364. [CrossRef]
22. Pawlak, Z. Rough sets. *Int. J. Comput. Inf. Sci.* **1982**, *11*, 341–356. [CrossRef]
23. Dubois, D.; Prade, H. Rough fuzzy sets and fuzzy rough sets. *Int. J. Gen. Syst.* **1989**, *17*, 191–209. [CrossRef]
24. Liu, Y.; Lin, Y. Intuitionistic fuzzy rough set model based on conflict distance and applications. *Appl. Soft Comput.* **2015**, *31*, 266–273. [CrossRef]
25. Hussain, A.; Mahmood, T.; Ali, M.I. Rough Pythagorean fuzzy ideals in semigroups. *Comput. Appl. Math.* **2019**, *38*, 67. [CrossRef]
26. Mahmood, T.; Ali, M.I.; Hussain, A. Generalized roughness in fuzzy filters and fuzzy ideals with thresholds in ordered semigroups. *Comput. Appl. Math.* **2018**, *37*, 2013–5033. [CrossRef]
27. Ali, M.I.; Mahmood, T.; Hussain, A. A study of generalized roughness in-fuzzy filters of ordered semigroups. *J. Taibah Univ. Sci.* **2018**, *12*, 163–172. [CrossRef]
28. Zakowski, W. Approximations in the space (U, π). *Demonstr. Math.* **1983**, *16*, 761–769. [CrossRef]
29. Xu, W.H.; Zhang, W.X. Measuring roughness of generalized rough sets induced by a covering. *Fuzzy Sets Syst.* **2007**, *158*, 2443–2455. [CrossRef]
30. Liu, G.; Sai, Y. A comparison of two types of rough sets induced by coverings. *Int. J. Approx. Reason.* **2009**, *50*, 521–528. [CrossRef]
31. Zhu, W. Topological approaches to covering rough sets. Information sciences, ership grades, complex numbers, and decision making. *Int. J. Intell. Syst.* **2013**, *28*, 436–452.
32. Zhu, W.; Wang, F.Y. On three types of covering based rough sets. *IEEE Trans. Knowl. Data Eng.* **2007**, *19*, 1131–1144. [CrossRef]
33. Wang, C.; Shao, M.; Sun, B.; Hu, Q. An improved attribute reduction scheme with covering based rough sets. *Appl. Soft Comput.* **2015**, *26*, 235–243. [CrossRef]
34. D'eer, L.; Restrepo, M.; Cornelis, C.; Gómez, J. Neighborhood operators for covering-based rough sets. *Inf. Sci.* **2016**, *336*, 21–44. [CrossRef]
35. D'eer, L.; Cornelis, C.; Godo, L. Fuzzy neighborhood operators based on fuzzy coverings. *Fuzzy Sets Syst.* **2017**, *312*, 17–35. [CrossRef]
36. Ma, L. Two fuzzy covering rough set models and their generalizations over fuzzy lattices. *Fuzzy Sets Syst.* **2016**, *294*, 1–17. [CrossRef]
37. Wu, M.F.; Han, H.H.; Si, Y.F. Properties and axiomatization of fuzzy rough sets based on fuzzy coverings. In Proceedings of the International Conference on Machine Learning and Cybernetics, Xi'an, China, 15–17 July 2012; Volume 1, pp. 184–189.
38. Huang, B.; Guo, C.; Zhang, Y.; Li, H.; Zhou, X. An intuitioistic fuzzy graded covering rough sets. *Knowl.-Based Syst.* **2016**, *107*, 155–178. [CrossRef]
39. Hussain, A.; Ali, M.I.; Mahmood, T. Covering based q-rung orthopair fuzzy rough set model hybrid with TOPSIS for multi-attribute decision making. *J. Int. Fuzzy Syst.* **2019**, accepted.
40. Zhan, J.; Sun, B. Covering-based intuitionistic fuzzy rough sets and applications in multi-attribute decision-making. *Artif. Int. Rev.* **2018**, 1–31. [CrossRef]