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On a New type of Tensor on Real Hypersurfaces in Non-Flat Complex Space Forms

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Abstract: In this paper the notion of *-Weyl curvature tensor on real hypersurfaces in non-flat complex space forms is introduced. It is related to the *-Ricci tensor of a real hypersurface. The aim of this paper is to provide two classification theorems concerning real hypersurfaces in non-flat complex space forms in terms of *-Weyl curvature tensor. More precisely, Hopf hypersurfaces of dimension greater or equal to three in non-flat complex space forms with vanishing *-Weyl curvature tensor are classified. Next, all three dimensional real hypersurfaces in non-flat complex space forms, whose *-Weyl curvature tensor vanishes identically are classified. The used methods are based on tools from differential geometry and solving systems of differential equations.

Keywords: real hypersurfaces; non-flat complex space forms; *-Ricci tensor; *-Weyl curvature tensor

1. Introduction

A Kahler manifold $\tilde{N}$ is a complex manifold of complex dimension n and real dimension 2n, which is equipped with

- a complex structure J defined $J : T\tilde{N} \to \tilde{N}$, where $T\tilde{N}$ is the tangent space of $\tilde{N}$, satisfying relations $J^2 = -Id$ and $\tilde{\nabla} J = 0$, i.e., J is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{N}$
- and a Riemanian metric G that is compatible with J, i.e., $G(JX, JY) = G(X,Y)$ for all tangent X, Y on $\tilde{N}$.

The pair (J, G) is called *Kahler structure*. A Kahler manifold of constant holomorphic sectional curvature c is called *complex space form*. Complete and simply connected complex space forms depending on the value of holomorphic sectional curvature c are analytically isometric to complex projective space $\mathbb{C}P^n$ if $c > 0$, to complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$ or to complex Euclidean space $\mathbb{C}^n$ if $c = 0$. This paper focuses on complex space forms with $c \neq 0$ denoted by $M_n(c)$ and called *non-flat complex space forms*. Furthermore, $c = 4$ in the case of $\mathbb{C}P^n$ and $c = -4$ in the case of $\mathbb{C}H^n$.

A submanifold M in a non-flat complex space form $M_n(c)$ of real codimension equal to 1 is called *real hypersurface*. Let N be a locally defined unit normal vector on M. The Kahler structure (J, G) of the ambient space $M_n(c)$ induces on M an *almost contact metric structure* (ϕ, ξ, η, g) defined in the following way

- $\xi = -JN$ is the *structure vector field*,
- ϕ is a skew-symmetric tensor field of type (1,1) called *structure tensor field* and defined to be the tangential component of $JX = \phi X + \eta(X)N$, for all tangent vectors X to M,
- η is a 1-form and is given by the relation $\eta(X) = g(X, \xi)$ for all tangent vectors X to M,
- g is the induced Riemannian metric on M.

A big class of real hypersurfaces in $M_n(c)$ are *Hopf hypersurfaces*, which are real hypersurfaces whose structure vector field ξ is an eigenvector of the shape operator A of M, i.e.,

$$A\xi = \alpha\xi, \tag{1}$$

where $\alpha = g(A\xi, \xi)$ and is called *Hopf principal curvature*.

Takagi classified homogeneous real hypersurfaces in complex projective space $\mathbb{C}P^n, n \geq 2$. The real hypersurfaces are divided into six types:

- type (A) which are either geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$, or tubes of radius r, with $0 < r < \frac{\pi}{2}$ over totally geodesic $\mathbb{C}P^k$,$1 \leq k \leq n-2$,
- type (B) which are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric Q^{n-1},
- type (C) which are tubes over the Serge embedding of $\mathbb{C}P^1 \times \mathbb{C}P^m$, with $2m+1 = n$ and $n \geq 5$,
- type (D) which are tubes over the Plücker embedding of the Grassmann manifold $G_{2,5}$ and $n = 9$,
- type (E) which are tubes over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$, where $SO(n)$ is a subgroup of $O(n)$ of dimension n, which consists of all the orthogonal matrices with determinant equal 1. (see [1–3]).

The above real hypersurfaces are Hopf ones with constant principal curvatures (see [4]).

In the case of the ambient space being the complex hyperbolic $\mathbb{C}H^n$, Montiel in [5] studied real hypersurfaces with two constant principal curvatures. Additionally, he proved that such real hypersurfaces are Hopf ones. Berndt in [6] classified Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n, n \geq 2$. The following list includes the Hopf hypersurfaces with constant principal curvatures.

- type (A) which are either horospheres, or geodesic hyperspheres, or tubes over totally geodesic complex hyperbolic hyperplane, or tubes over totally geodesic $\mathbb{C}H^k$, $1 \leq k \leq n-2$,
- type (B) which are tubes over totally geodesic real hyperbolic space $\mathbb{R}H^2$ (type (B)).

All of them are homogeneous ones, but in contrast to the case of complex projective space, it is proved that there are also non-Hopf hypersurfaces in $\mathbb{C}H^n$ which are homogeneous.

Let $\tilde{M}$ be a Riemannian manifold of dimension m and g its Riemannian metric. Then the *Weyl curvature tensor* $W(X,Y)Z$ of $\tilde{M}$ is given by

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z + \frac{1}{m-2}[g(SX,Z)Y - g(SY,Z)X + g(X,Z)SY - g(Y,Z)SX] \\ &- \frac{\rho}{(m-1)(m-2)}[g(X,Z)Y - g(Y,Z)X], \quad \text{for all } X, Y, Z \text{ tangent to } M, \end{aligned}$$

with R being the Riemannian curvature tensor, S being the Ricci tensor and ρ being the scalar curvature of $\tilde{M}$. If $m = 3$ then $W(X,Y)Z = 0$ and if $m \geq 4$ then $\tilde{M}$ is locally conformal flat if and only if $W(X,Y)Z = 0$. The condition of locally conformal flat holds for three dimensional Riemannian manifolds if and only if the Cotton tensor of $\tilde{M}$, which is given by

$$C(X,Y) = (\nabla_X S)Y - (\nabla_Y S)X - \frac{1}{2(m-2)}[(\nabla_X \rho)Y - (\nabla X \rho)Y],$$

vanishes identically.

The Weyl curvature tensor of real hypersurfaces M in $M_n(c)$ satisfies the relation

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z + \frac{1}{2n-3}[g(SX,Z)Y - g(SY,Z)X + g(X,Z)SY - g(Y,Z)SX] \\ &- \frac{\rho}{2(n-1)(2n-3)}[g(X,Z)Y - g(Y,Z)X], \end{aligned}$$

for all X, Y, Z tangent to M, where R is the Riemannian curvature tensor, S is the Ricci tensor, ρ is the scalar curvature of M and g is the induced Riemannian metric on M. In [7] the non-existence of real hypersurfaces in $M_n(c)$ with harmonic Weyl curvature tensor, i.e., $\delta W = 0$ with δ denoting the codifferential of the exterior differential d is proved. Moreover, in [8] the classification of real hypersurfaces in $\mathbb{C}P^n$ with ξ-parallel Weyl curvature tensor, i.e., $\nabla_\xi W = 0$ is provided. Finally, in [9] real hypersurfaces in $\mathbb{C}H^n$, $n \geq 3$ satisfying the previous geometric condition are classified.

In 1959 Tachibana defined *-Ricci tensor S^* on almost Hermitian manifold. In [10] Hamada gave the definition of *-Ricci tensor S^* on real hypersurfaces in $M_n(c)$ in the following way

$$g(S^*X, Y) = \frac{1}{2} trace(Z \to R(X, \varphi Y)\varphi Z),$$

for all X, Y tangent to M and *trace* is the sum of elements of the main diagonal of the matrix, which corresponds to the above endomorphism. He also presented *- *Einstein*, i.e., $g(S^*X, Y) = \lambda g(X, Y)$, where λ is a constant multiple of $g(X, Y)$ and provided classification of *-Einstein hypersurfaces. Ivey and Ryan in [11] extended the Hamada's work and studied the equivalence of *- Einstein condition with other geometric conditions such as the pseudo-Einstein and the pseudo-Ryan condition.

Motivated by the revious results and work we define *-Weyl curvature tensor of real hypersurfaces in the following way

$$\begin{aligned} W^*(X, Y)Z &= R(X, Y)Z + \frac{1}{2n-3}[g(S^*X, Z)Y - g(S^*Y, Z)X + g(X, Z)S^*Y - g(Y, Z)S^*X] \\ &- \frac{\rho^*}{2(n-1)(2n-3)}[g(X, Z)Y - g(Y, Z)X], \end{aligned} \tag{2}$$

for all X, Y, Z tangent to M and S^* is the *-Ricci tensor and ρ^* is the *-scalar curvature corresponding to S^* of M.

First it is examined if there are real hypersurfaces of dimension equal to or greater than three with vanishing *-Weyl curvature tensor. The following Theorem is proved

Theorem 1. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, with vanishing *-Weyl curvature tensor. Then M is an open subset of a real hypersurface of type (A) or of a Hopf hypersurface with $A\xi = 0$.*

Next it is examined if there are three-dimensional real hypersurface in $M_2(c)$ with vanishing *-Weyl curvature tensor and the following Theorem is obtained

Theorem 2. *Every real hypersurface M in $M_2(c)$ with vanishing *-Weyl curvature tensor is a Hopf hypersurface. Furthermore, M is an open subset of a real hypersurface of type (A) or of a Hopf hypersurface with $A\xi = 0$.*

The paper has the following outline: In Section 2 relations and Theorems concerning real hypersurfaces in non-flat complex space forms are provided. In Section 3 Theorems 1 and 2 are proved. Section 4 concerns discussion on the new tensor and ideas of further research and Section 5 includes the conclusions of the paper.

2. Preliminaries

The manifolds, vector fields, etc., are considered of class C^∞. We consider M to be a connected real hypersurface without boundary in $M_n(c)$ equipped with a Kahler structure (J, G) and $\overline{\nabla}$ is the

Levi-Civita connection of $M_n(c)$ and N a locally unit normal vector field on M. Then the shape operator A of M with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

and the Levi-Civita connection ∇ of the induced metric g on M satisfies

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N.$$

As mentioned in the Introduction, on M an almost contact metric structure (ϕ, ξ, η, g) is defined and the following relations are satisfied (see [12])

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for all tangent vectors X, Y to M. Relation (3) implies

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi).$$

Due to the fact that the complex structure J is parallel, i.e., $\overline{\nabla} J = 0$ we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \phi AX \tag{4}$$

for all X, Y tangent to M. Moreover, the ambient space is of holomorphic sectional curvature c and this results in the Gauss and Codazzi equations becoming respectively

$$\begin{aligned} R(X,Y)Z = \tfrac{c}{4}[&g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &-2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi], \tag{6}$$

for all tangent vectors X, Y, Z to M, where R is the Riemannian curvature tensor of M.

Let P be a point of M, then the tangent space $T_P M$ is decomposed into

$$T_P M = span\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called (*maximal*) *holomorphic distribution* (*if* $n \geq 3$).

The following Theorem concerns the shape operator of M and is proved by Maeda [13] in the case of $\mathbb{C}P^n, n \geq 2$, and by Ki and Suh [14] in the case of $\mathbb{C}H^n, n \geq 2$ (also Corollary 2.3 in [15]).

Theorem 3. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then*

(i) *α is constant.*

(ii) *If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then*

$$(\lambda - \frac{\alpha}{2})A(\phi W) = (\frac{\lambda\alpha}{2} + \frac{c}{4})(\phi W). \tag{7}$$

(iii) *If the vector field W satisfies $AW = \lambda W$ and $A(\phi W) = \nu(\phi W)$ then*

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{8}$$

We consider M a three dimensional real hypersurface in $M_2(c)$ and P a point of M such that in the neighborhood of P relation $A\xi \neq \alpha\xi$ holds. Let U be a unit vector lying in the $span\{A\xi, \xi\}$ satisfying relation $g(U, \xi) = 0$. Then, we can consider the standard non-Hopf local orthonormal frame $\{U, \phi U, \xi\}$ in the neighborhood of P (see [16] p. 445). Therefore, the shape operator A is given by

$$A\xi = \alpha\xi + \beta U, \quad AU = \gamma U + \delta(\phi U) + \beta\xi \;\text{ and }\; A(\phi U) = \delta U + \mu(\phi U). \tag{9}$$

The following Lemma holds for three dimensional non-Hopf real hypersurfaces in $M_2(c)$

Lemma 1. *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M*

$$\begin{aligned} &\nabla_U\xi = -\delta U + \gamma(\phi U), && \nabla_{\phi U}\xi = -\mu U + \delta(\phi U), && \nabla_\xi\xi = \beta(\phi U), \\ &\nabla_U U = \kappa_1(\phi U) + \delta\xi, && \nabla_{\phi U}U = \kappa_2(\phi U) + \mu\xi, && \nabla_\xi U = \kappa_3(\phi U), \\ &\nabla_U(\phi U) = -\kappa_1 U - \gamma\xi, && \nabla_{\phi U}(\phi U) = -\kappa_2 U - \delta\xi, && \nabla_\xi(\phi U) = -\kappa_3 U - \beta\xi, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Lemma 1 is proved in page 92 [17].

The Codazzi Equation (6) for $X \in \{U, \phi U\}$ and $Y = \xi$ owing to Lemma 1 results in the following relations

$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2 \tag{10}$$

$$(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu \tag{11}$$

$$(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu \tag{12}$$

and for $X = U$ and $Y = \phi U$

$$U\delta - (\phi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu. \tag{13}$$

In the case of three dimensional Hopf hypersurfaces we consider a point P of M and we define in the neighborhood of P a local orthonormal frame as follows: since M is a Hopf hypersurface the shape operator A restricted to the holomorphic distribution $\mathbb{D}$ has distinct eigenvalues. Thus, we choose a vector W as one of the eigenvectors fields. Moreover, due to the fact that M is three dimensional, the shape operator satisfies the following relations:

$$A\xi = \alpha\xi, \quad AW = \lambda W \;\text{ and }\; A(\phi W) = \nu(\phi W), \tag{14}$$

and Thereom 3 holds.

Finally, the following Theorem concerns the classification of real hypersurfaces in $M_n(c)$, $n \geq 2$, whose shape operator A satisfies a commuting condition. It is proved by Okumura in the case of $\mathbb{C}P^n$ (see [18]) and by Montiel and Romero in the case of $\mathbb{C}H^n$ (see [19]).

Theorem 4. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\phi = \phi A$, if and only if M is an open subset of a homogeneous real hypersurface of type (A).*

We mention that type (A_2) hypersurfaces do not occur in the case of three dimensional real hypersurface in $M_2(c)$.

3. Proof of Theorems 1 and 2

The *-Ricci tensor of a real hypersurface M in a non-flat complex space form is given by

$$S^*X = -[\frac{cn}{2}\phi^2 X + \phi A(\phi(AX))], \tag{15}$$

for all X tangent to M.

Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, with vanishing *-Weyl curvature tensor, i.e.,

$$W^*(X,Y)Z = 0. \tag{16}$$

Since M is a Hopf hypersurface ξ is an eigenvector of the shape operator relation (1) holds and relation (15) for $X = \xi$ yields $S^*\xi = 0$. Next, we consider W a unit vector field which belongs to the (maximal) holomorphic distribution such that relation $AW = \lambda W$ holds at some point $P \in M$ and relation (7) is satisfied. We have two cases:

Case I: $\alpha^2 + c \neq 0$.

In this case $\lambda \neq \frac{\alpha}{2}$ so relation (7) implies $A\phi W = \nu\phi W$ and relation (8) holds.

Relation (16) for $Z = \xi$ taking into account (2) implies

$$R(X,Y)\xi + \frac{1}{2n-3}[g(S^*X,\xi)Y - g(S^*Y,\xi)X + \eta(X)S^*Y - \eta(Y)S^*X]$$
$$-\frac{\rho^*}{2(n-1)(2n-3)}[\eta(X)Y - \eta(Y)X] = 0, \tag{17}$$

for all X, Y tangent to M.

The inner product of relation (17) for $X = W$ and $Y = \xi$ with W because of (3), (5), (15), $S^*\xi = 0$, $AW = \lambda W$ and $A(\phi W) = \nu(\phi W)$ yields

$$(\frac{c}{4} + \alpha\lambda) - \frac{1}{2n-3}(\frac{cn}{2} + \lambda\nu) + \frac{\rho^*}{2(n-1)(2n-3)} = 0. \tag{18}$$

Furthermore, the inner product of relation (17) for $X = \phi W$ and $Y = \xi$ with ϕW due to (3), (5) and (15), $S^*\xi = 0$, $AW = \lambda W$ and $A(\phi W) = \nu(\phi W)$ implies

$$(\frac{c}{4} + \alpha\nu) - \frac{1}{2n-3}(\frac{cn}{2} + \lambda\nu) + \frac{\rho^*}{2(n-1)(2n-3)} = 0. \tag{19}$$

Combination of relations (18) and (19) results in

$$\alpha(\lambda - \nu) = 0.$$

So, either $\alpha = 0$ and M is an open subset of a Hopf hypersurface with $A\xi = 0$ or $\lambda = \nu$ which implies that $A\phi = \phi A$ and because of Theorem 4 M is an open subset of a real hypersurface of type (A).

Case II: $\alpha^2 + c = 0$.

This case occurs only when the ambient space is the complex hyperbolic space $\mathbb{C}H^n$. Thus, $\alpha^2 - 4 = 0$ and this results in $\alpha = 2$. We consider W a unit vector field, which belongs to the (maximal) holomorphi distribution such that relation $AW = \lambda W$ holds at some point $P \in M$. Therefore, relation (7) due to $\alpha = 2$ and $c = -4$ implies

$$(\lambda - 1)A(\phi W) = (\lambda - 1)(\phi W).$$

First we suppose that $\lambda \neq 1$. Then the above relation implies $A(\phi W) = \phi W$. So, the inner product of relation (17) for $X = W$ and $Y = \xi$ with W because of (3), (5) and (15) for $X = \xi$ which implies $S^*\xi = 0$, $AW = \lambda W$ and $A(\phi W) = \phi W$ results in

$$(2\lambda - 1) - \frac{1}{2n-3}(\lambda - 2n) + \frac{\rho^*}{2(n-1)(2n-3)} = 0. \tag{20}$$

Moreover, the inner product of relation (17) for $X = \phi W$ and $Y = \xi$ with ϕW due to (3), (5), (15), $S^*\xi = 0$, $AW = \lambda W$ and $A(\phi W) = \phi W$ implies

$$1 - \frac{1}{2n-3}(\lambda - 2n) + \frac{\rho^*}{2(n-1)(2n-3)} = 0. \tag{21}$$

Combination of relations (20) and (21) yields $\lambda = 1$, which is a contradiction.

Therefore, we have $\lambda = 1$ for any vector field $W \in \mathbb{D}$ and M is an open subset of a horosphere, which is a real hypersurface of type (A) and this completes the proof of Theorem 1.

Remark 1. *Examples of Hopf hypersurfaces with $\alpha = 0$ are the following:*

- *A geodesic hypersphere of radius $r = \frac{\pi}{4}$ in $\mathbb{C}P^n$ has $\alpha = 0$.*
- *In [20,21] there are examples of Hopf hypersurfaces with $A\xi = 0$, which do not have constant principal curvatures, i.e., the eigenvalues of the shape operator corresponding to the (maximal) holomorphic distribution are not constant.*

Next we examine non-Hopf three-dimensional real hypersurfaces M in $M_2(c)$ whose *-Weyl tensor vanishes identically, i.e., relation (16) holds. We consider $\mathcal{N}$ the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P\},$$

and $\{U, \phi U, \xi\}$ be the local orthonormal frame in the neighborhood of a point P defined as in Section 2.

Relation (2) for $Z = \xi$ and due to $n = 2$ implies

$$R(X,Y)\xi + g(S^*X,\xi)Y - g(S^*Y,\xi)X + \eta(X)S^*Y - \eta(Y)S^*X - \frac{\rho^*}{2}[\eta(X)Y - \eta(Y)X], \tag{22}$$

for all X, Y tangent to M. The inner product of relation (22) for $X = U$ and $Y = \xi$ with φU and U taking into account relations (9), (5) and (15) yields respectively

$$\alpha\delta = 0 \text{ and } \alpha\gamma + \delta^2 + \frac{\rho^*}{2} = \frac{3c}{4} + \beta^2 + \gamma\mu. \tag{23}$$

Moreover, the inner product of relation (22) for $X = \varphi U$ and $Y = \xi$ with φU because of relations (9), (5) and (15) and the second of (23) results in

$$\alpha\mu = \alpha\gamma - \beta^2. \tag{24}$$

Suppose that $\delta \neq 0$ then the first of (23) gives $\alpha = 0$. Substitution of the latter in (24) results in $\beta = 0$, which is a contradiction. Thus, relation $\delta = 0$ holds.

Relation (22) for $X = U$ and $Y = \varphi U$ because of (5) implies $\mu = 0$. So, relation (24) results in $\beta^2 = \alpha\gamma$. Differentiating the latter with respect to φU taking into account relations (10)–(13) results in $c = 0$.

So $\mathcal{N}$ is empty and the following Proposition has been proved.

Proposition 1. *Every real hypersurface in $M_2(c)$ whose *-Weyl curvature tensor vanishes identically is a Hopf hypersurface.*

The above proposition with Theorem 1 for the case of $n = 2$ completes the proof of Theorem 2.

4. Discussion

In literature it is known that there are no Einstein real hypersurfaces in non-flat complex space forms, i.e., real hypersurfaces whose Ricci tensor satisfies relation $S = \alpha g$, where α is constant (see [15]). Therefore, new notions such as η-Einstein, i.e., the Ricci tensor satisfies relation $S = \alpha + \eta \otimes \xi$ or *-Ricci Einstein, i.e., the *-Ricci tensor satisfies $S^* = \rho^* g$, with ρ^* being constant, are introduced and the real hypersurfaces are studied with respect to the previous relations (see [10,11,15]). Thus, the next step is to introduce new tensors on real hypersurfaces in non-flat complex space forms related to the *-Ricci tensor, since there are results concerning notions and tensors related to the Ricci tensor. In this paper, we introduced the *-Weyl curvature tensor and studied real hypersurfaces in non-flat complex space forms in terms of it. Further work can be done in this direction. So, at this point some ideas for further research are mentioned:

1. it is worthwhile to study if there are non-Hopf real hypersurfaces of dimension greater than three in non-flat complex space forms with vanishing *-Weyl curvature tensor,
2. the *-Weyl curvature tensor could also be defined on real hypersurfaces in other symmetric Hermitian space forms such as the complex two-plane Grassmannians or the complex hyperbolic two-plane Grassmannians and it could be examined if there are real hypersurfaces with vanishing *-Weyl curvature tensor.

Overall, real hypersurfaces in non-flat complex space forms can be potentially applied to finding solutions of nonlinear dynamical differential equations. Ideas for research in this direction can be derived methods based on Lie algebra. For a first idea in this direction one could have a look in works (1) A Lie algebra approach to susceptible-infected-susceptible epidemics (see [22]), (2) Lie algebraic discussion for affinity based information diffusion in social networks (see [23]).

5. Conclusions

In this section we conclude the work which is presented in this paper.

- We introduced a new type of tensor on real hypersurfaces in non-flat complex space forms by defining the *-Weyl curvature tensor on them. The new tensor is related to the *-Ricci tensor of a real hypersurface.
- We initiated the study of real hypersurfaces in non-flat complex space forms in terms of this new tensor. The first geometric condition is that of the vanishing *-Weyl curvature tensor. The motivation for choosing this geometric condition is the existing results for Riemannian manifolds in terms of the Weyl curvature tensor. Thus, we proved two classifications Theorems. The first Theorem concerns Hopf hypersurfaces in non-flat complex space forms of dimension greater or equal to three with vanishing *-Weyl curvature tensor. The second Theorem provides a complete classification for three dimensional real hypersurfaces with vanishing *-Weyl curvature tensor.

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