

Edge Even Graceful Labeling of Cylinder Grid Graph

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Abstract: Edge even graceful labeling (e.e.g., l.) of graphs is a modular technique of edge labeling of graphs, introduced in 2017. An e.e.g., l. of simple finite undirected graph $G = (V(G), E(G))$ of order $P = |V(G)|$ and size $q = |E(G)|$ is a bijection $f : E(G) \rightarrow \{2, 4, \dots, 2q\}$, such that when each vertex $v \in V(G)$ is assigned the modular sum of the labels (images of f) of the edges incident to v , the resulting vertex labels are distinct mod $2r$, where $r = \max(p, q)$. In this work, the family of cylinder grid graphs are studied. Explicit formulas of e.e.g., l. for all of the cases of each member of this family have been proven.

Keywords: graceful labeling; edge even graceful labeling; cylinder grid graph

1. Introduction

The field of graph theory plays an important role in various areas of pure and applied sciences. One of the important areas in graph theory is graph labeling of a graph G which is an assignment of integers either to the vertices or edges or both subject to certain conditions. Graph labeling began nearly 50 years ago. Over these decades, more than 200 methods of labeling techniques were invented, and more than 2500 papers were published. In spite of this huge literature, just few general results were discovered. Nowadays, graph labeling has much attention from different brilliant researchers in graph theory, which has rigorous applications in many disciplines, e.g., communication networks, coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing, database management, and graph decomposition problems. More interesting applications of graph labeling can be found in References [1–11]. A function f is called a graceful labeling of a graph G if $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 2, \dots, q\}$, defined as $f^*(e = uv) = |f(u) - f(v)|$, is bijective. This type of graph labeling was first introduced by Rosa in 1967 [12] as a β -valuation, and later, Solomon W. Golomb [13] termed it as graceful labeling. A function f is called an odd graceful labeling of a graph G if $f : V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 3, \dots, 2q - 1\}$, defined as $f^*(e = uv) = |f(u) - f(v)|$, is bijective. This type of graph labeling first introduced by Gnanajothi in 1991 [14]. For more results on this type of labeling, see References [15,16]. A function f is called an edge graceful labeling of a graph G if $f : E(G) \rightarrow \{1, 2, \dots, q\}$ is bijective and the induced function $f^* : V(G) \rightarrow \{0, 1, 2, \dots, p - 1\}$, defined as $f^*(u) = \sum_{e=uv \in E(G)} f(e) \pmod{p}$, is bijective. This type of graph labeling was first introduced by Lo in 1985 [17]. For more results on this labeling see [18,19]. A function f is called an edge odd graceful labeling of a graph G if $f : E(G) \rightarrow \{1, 3, \dots, 2q - 1\}$ is bijective and the induced function $f^* : V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ defined as $f^*(u) = \sum_{e=uv \in E(G)} f(e) \pmod{2q}$ is injective. This type of

graph labeling was first introduced by Solairaju and Chithra in 2009 [20]. For more results on this labeling, see References [21–23]. A function f is called an edge even graceful labeling of a graph G if $f : E(G) \rightarrow \{2, 4, \dots, 2q - 2\}$ is bijective and the induced function $f^* : V(G) \rightarrow \{0, 2, 4, \dots, 2q - 2\}$, defined as $f^*(u) = \sum_{e=uv \in E(G)} f(e) \pmod{2r}$ where $r = \max\{p, q\}$, is injective. This type of graph labeling was first introduced by Elsonbaty and Daoud in 2017 [24,25]. For a summary of the results on these five types of graceful labels as well as all known labeling techniques, see Reference [26].

2. Cylinder Grid Graph

The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 , is the graph with vertex set $V(G_1) \times V(G_2)$, and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ whenever $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$. The cylinder grid graph $C_{m,n}$ is the graph formed from the Cartesian product $P_m \times C_n$ of the path graph P_m and the cycle graph C_n . That is, the cylinder grid graph consists of m copies of C_n represented by circles, and will be numbered from the innermost circle to the outer circle as $C_n^{(1)}, C_n^{(2)}, C_n^{(3)}, \dots, C_n^{(m-1)}, C_n^{(m)}$ and we call them simply circles; n copies of P_m represented by paths transverse the m circles and will be numbered clockwise as $P_m^{(1)}, P_m^{(2)}, P_m^{(3)}, \dots, P_m^{(n-1)}, P_m^{(n)}$ and we call them paths (see Figure 1).

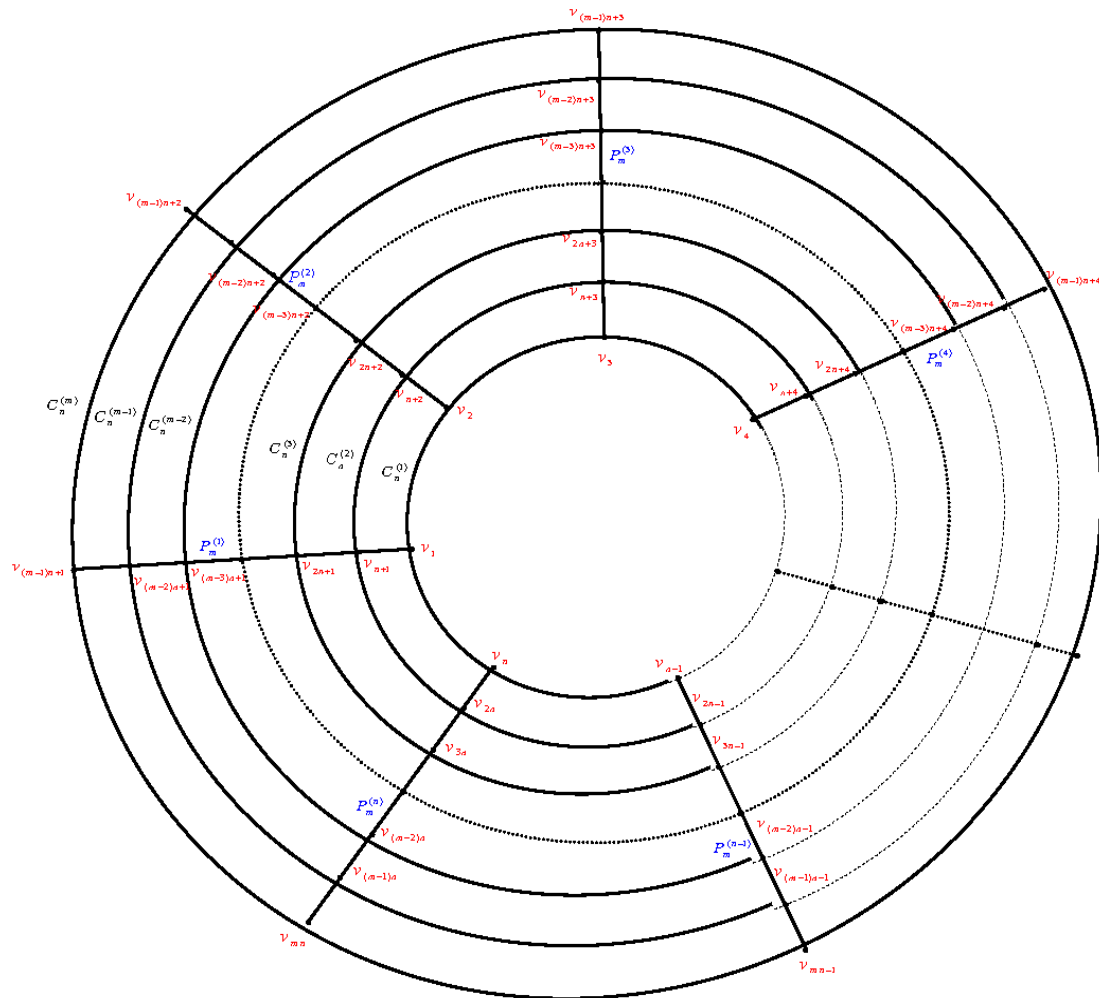


Figure 1. Cylinder grid graph $C_{m,n}$.

Theorem 1. *If m is an even positive integer greater than or equal 2 and $n \geq 2$, then the cylinder grid graph $C_{m,n}$ is an edge even graceful graph.*

Proof. Using standard notation $p = |V(C_{m,n})| = mn, q = |E(C_{m,n})| = 2mn - n$ and $r = \max(p, q) = 2mn - n$ and $f : E(C_{m,n}) \rightarrow \{2, 4, 6, \dots, 4mn - 2n - 2\}$. Let the cylinder grid graph $C_{m,n}$ be as in Figure 2. \square

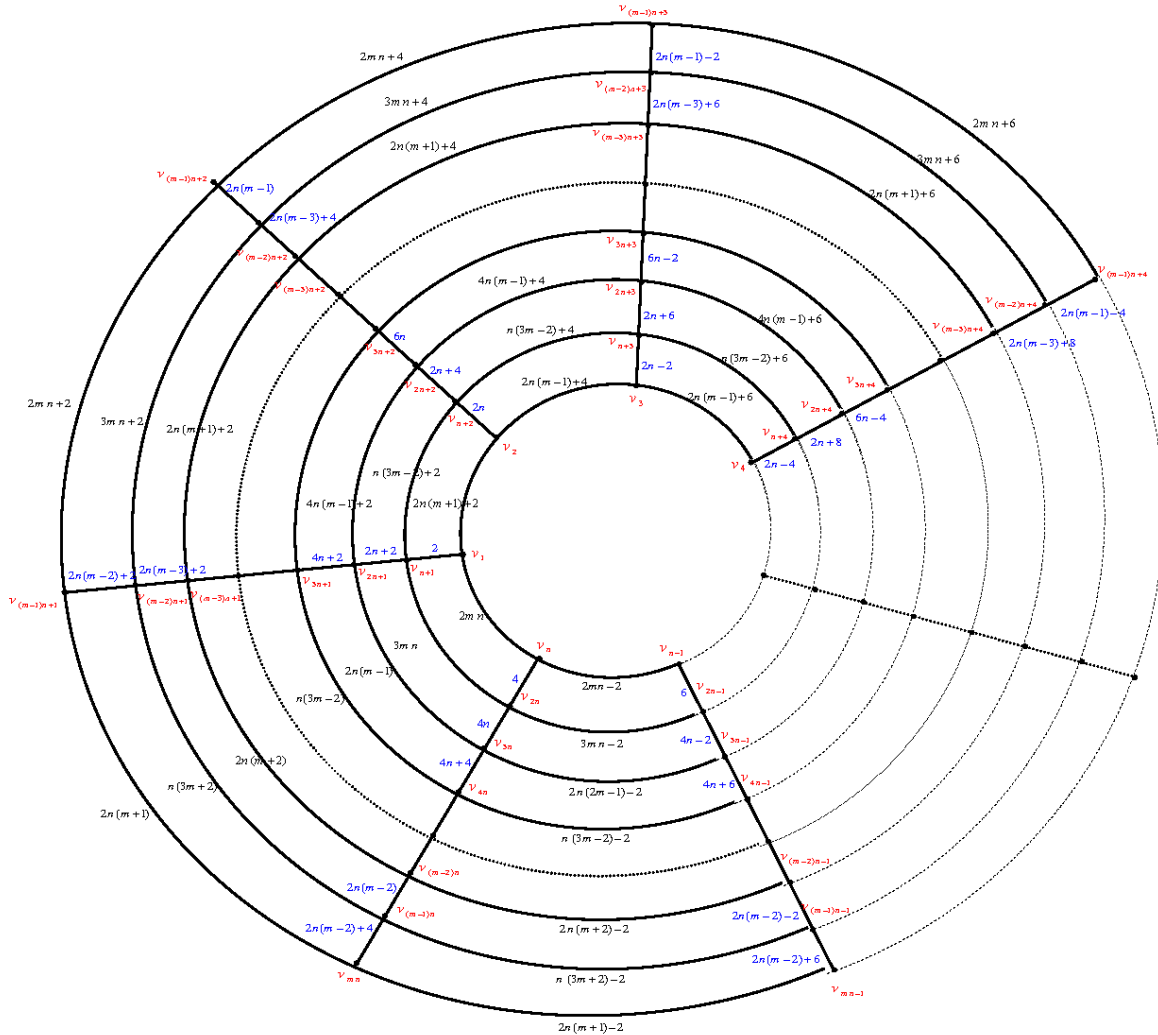


Figure 2. The cylinder grid graph $C_{m,n}$, m is even and $n \geq 2$.

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ as follows: Move anticlockwise to label the edges $v_1v_{n+1}, v_nv_{2n}, v_{n-1}v_{2n-1}, \dots, v_3v_{n+3}, v_2v_{n+2}$ by $2, 4, 6, \dots, 2n - 2, 2n$, then move clockwise to label the edges $v_{n+1}v_{2n+1}, v_{n+2}v_{2n+2}, v_{n+3}v_{2n+3}, \dots, v_{2n-1}v_{3n-1}, v_{2n}v_{3n}$ by $2n + 2, 2n + 4, 2n + 6, \dots, 4n - 2, 4n$, then move anticlockwise to label the edges $v_{2n+1}v_{3n+1}, v_{3n}v_{4n}, v_{3n-1}v_{4n-1}, \dots, v_{2n+3}v_{3n+3}, v_{2n+2}v_{3n+2}$ by $4n + 2, 4n + 4, 4n + 6, \dots, 6n - 2, 6n$ and so on. Finally, move anticlockwise to label the edges $v_{(m-2)n+1}v_{(m-1)n+1}, v_{(m-1)n}v_{mn}, v_{(m-1)n-1}v_{mn-1}, \dots, v_{(m-2)n+3}v_{(n-1)m+3}, v_{(m-2)n+2}v_{(m-1)n+2}$ by $2n(m - 1) + 2, 2n(m - 2) + 4, 2n(m - 2) + 6, 2n(m - 2) + 8, \dots, 2n(m - 1) - 2, 2n(m - 1)$.

Secondly, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ beginning with the edges of the innermost circle $C_n^{(1)}$ then the edges of outer circle $C_n^{(m)}$, then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(2)}$.

Finally, we label the edges of the circles $C_n^{(m-1)}, C_n^{(m-3)}, \dots, C_n^{(3)}$ as follows: $f(v_iv_{i+1}) = 2n(m - 1) + 2i, 1 \leq i \leq n - 1, f(v_nv_1) = 2mn; f(v_{(m-1)n+i}v_{(m-1)n+i+1}) = 2mn + 2i, 1 \leq i \leq n - 1, f(v_{mn}v_{(m-1)n+1}) =$

$2n(m + 1)$; $f(v_{(k-1)n+i}v_{(k-1)n+i+1}) = n(3m - k) + 2i, 1 \leq i \leq n - 1, f(v_{kn}v_{(k-1)n+1}) = n(3m - k + 2), 2 \leq k \leq m - 2; f(v_{(k-1)n+i}v_{(k-1)n+i+1}) = n(4m - k - 1) + 2i, 1 \leq i \leq n - 1, f(v_{kn}v_{(k-1)n+1}) = n(4m - k + 1), 3 \leq k \leq m - 1, k$ is odd.

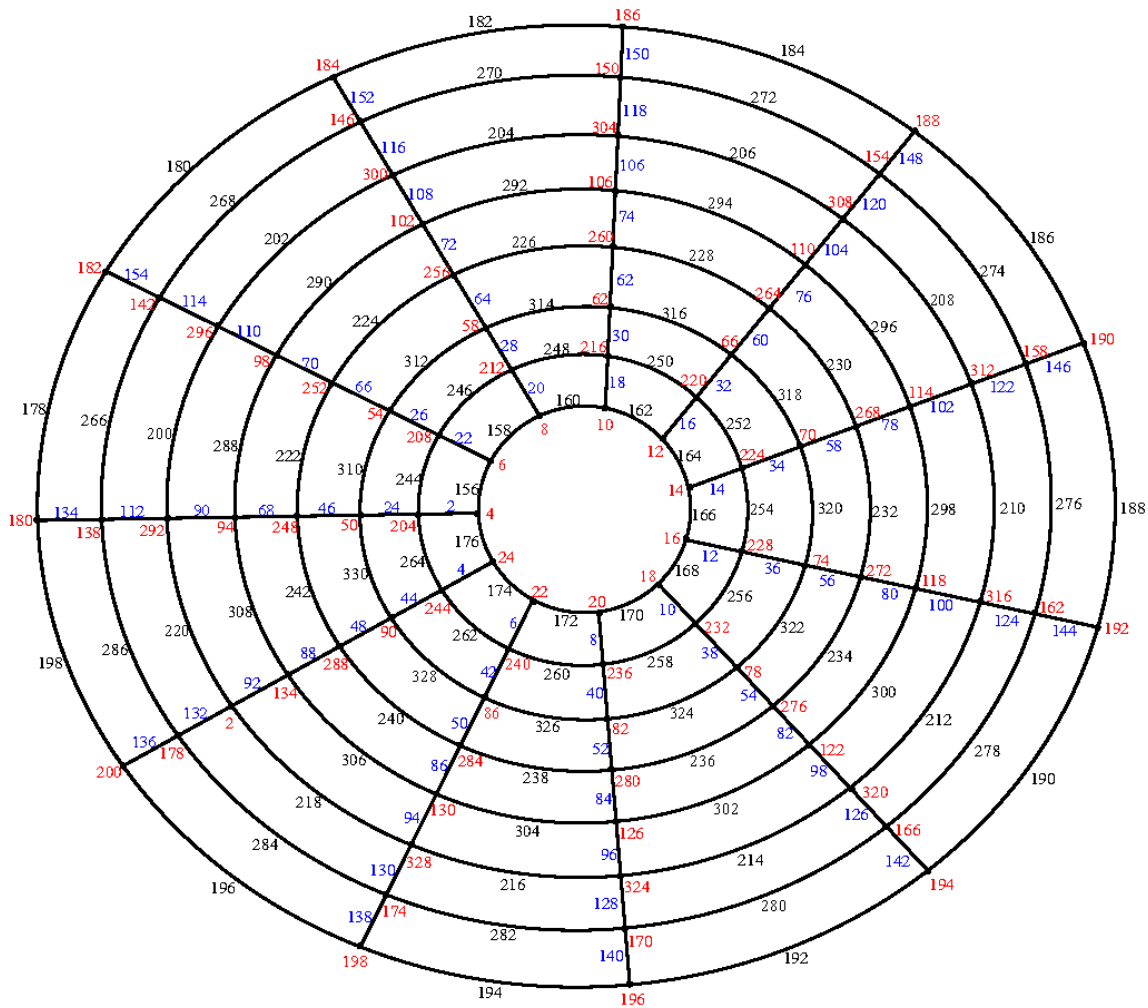
Thus, the labels of corresponding vertices mod $(4mn - 2n)$ will be: $f^*(v_i) \equiv 2i + 2; f^*(v_{n+i}) \equiv 2mn + 2n + 4i + 2; f^*(v_{2n+i}) \equiv 4n + 4i + 2; f^*(v_{3n+i}) \equiv 2mn + 6n + 4i + 2; \dots; f^*(v_{(m-3)n+i}) \equiv 4mn - 6n + 4i + 2; f^*(v_{(m-2)n+i}) \equiv 2mn - 4n + 4i + 2; f^*(v_{(m-1)n+i}) \equiv 2mn + 2i + 2, 1 \leq i \leq n$.

Illustration: An e.e.g., l , of the cylinder grid graphs $C_{8,11}$ and $C_{8,12}$ are shown in Figure 3.

Theorem 2. If $m = 3$ and n is an odd positive integer greater than 3, then the cylinder grid graph $C_{3,n}$, is an edge even graceful graph.

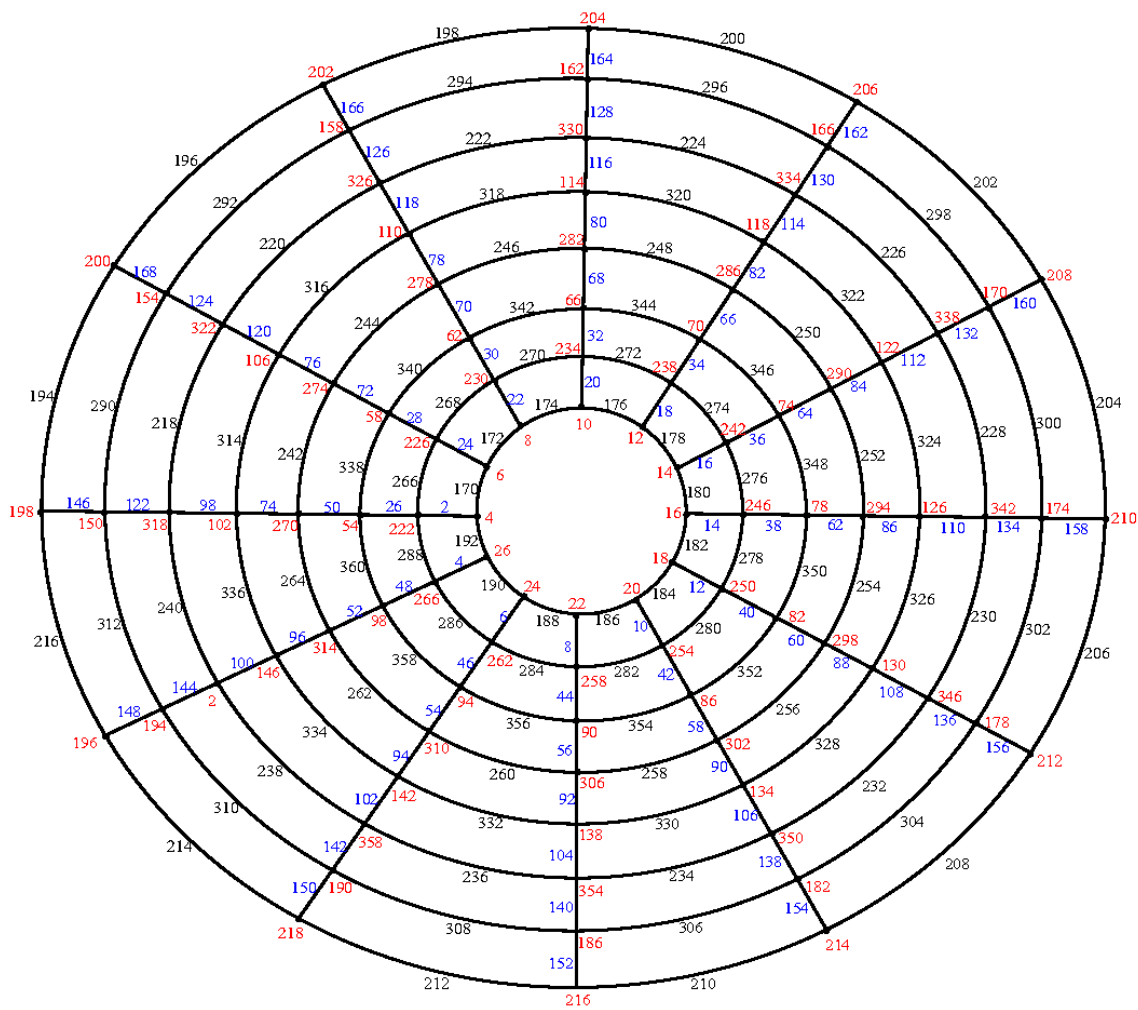
Proof. Using standard notation $p = |V(C_{3,n})| = 3n, q = |E(C_{3,n})| = 5n, r = \max(p, q) = 5n$, and $f : E(C_{3,n}) \rightarrow \{2, 4, 6, \dots, 10n - 2\}$. There are three cases:

Case (1): If $n \equiv 1 \pmod{6}$, let the cylinder grid graph $C_{3,n}$ be as in Figure 4.



(a) $C_{8,11}$

Figure 3. Cont.



(b) $C_{8,12}$

Figure 3. An edge even graceful labeling (e.e.g., l.) of the cylinder grid graphs $C_{8,11}$ and $C_{8,12}$.

First, we label the edges of the paths $P_3^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_3^{(1)}$ as follows: Move clockwise to label the edges $f(v_1v_{n+1}) = 2, f(v_2v_{n+2}) = 6, f(v_iv_{n+i}) = 2i + 2, 3 \leq i \leq n$. Then, move anticlockwise to label the edges $f(v_{n+1}v_{2n+1}) = 2n + 4, f(v_{2n}v_{3n}) = 2n + 6, f(v_{2n-1}v_{3n-1}) = 2n + 8, f(v_{2n-2}v_{3n-2}) = 2n + 10, \dots, f(v_{n+3}v_{2n+3}) = 4n, f(v_{n+2}v_{2n+2}) = 4n + 2$.

Secondly, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq 3$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(3)}$, and then the edges of the circle $C_n^{(2)}$. Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 4n + 4, f(v_2v_3) = 4n + 6, \dots, f(v_{\frac{n-1}{3}v_{\frac{n+2}{3}}}) = \frac{14n+10}{3}, f(v_{\frac{n+2}{3}v_{\frac{n+5}{3}}}) = \frac{14n+4}{3}, f(v_{\frac{n+5}{3}v_{\frac{n+8}{3}}}) = \frac{14n+16}{3}, f(v_{\frac{n+8}{3}v_{\frac{n+11}{3}}}) = \frac{14n+22}{3}, f(v_{\frac{n+11}{3}v_{\frac{n+14}{3}}}) = \frac{14n+34}{3}, f(v_{\frac{n+14}{3}v_{\frac{n+17}{3}}}) = \frac{14n+28}{3}, f(v_{\frac{n+17}{3}v_{\frac{n+20}{3}}}) = \frac{14n+40}{3}, f(v_{\frac{n+20}{3}v_{\frac{n+23}{3}}}) = \frac{14n+46}{3}, f(v_{\frac{n+23}{3}v_{\frac{n+26}{3}}}) = \frac{14n+58}{3}, f(v_{\frac{n+26}{3}v_{\frac{n+29}{3}}}) = \frac{14n+52}{3}, f(v_{\frac{n+29}{3}v_{\frac{n+32}{3}}}) = \frac{14n+64}{3}, f(v_{\frac{n+32}{3}v_{\frac{n+35}{3}}}) = \frac{14n+70}{3}, f(v_{\frac{n+35}{3}v_{\frac{n+38}{3}}}) = \frac{14n+82}{3}, f(v_{\frac{n+38}{3}v_{\frac{n+41}{3}}}) = \frac{14n+76}{3}, f(v_{\frac{n+41}{3}v_{\frac{n+44}{3}}}) = \frac{14n+88}{3}, f(v_{\frac{n+44}{3}v_{\frac{n+47}{3}}}) = \frac{14n+94}{3}, \dots, f(v_{n-13}v_{n-12}) = 6n - 22, f(v_{n-12}v_{n-11}) = 6n - 24, f(v_{n-11}v_{n-10}) = 6n - 20, f(v_{n-10}v_{n-9}) = 6n - 18, f(v_{n-9}v_{n-8}) = 6n - 14, f(v_{n-8}v_{n-7}) = 6n - 16, f(v_{n-7}v_{n-6}) = 6n - 12, f(v_{n-6}v_{n-5}) = 6n - 10, f(v_{n-5}v_{n-4}) = 6n - 6, f(v_{n-4}v_{n-3}) = 6n - 8, f(v_{n-3}v_{n-2}) = 6n - 4, f(v_{n-2}v_{n-1}) = 6n - 2, f(v_{n-1}v_n) = 6n + 2, f(v_nv_1) = 6n. $\square$$

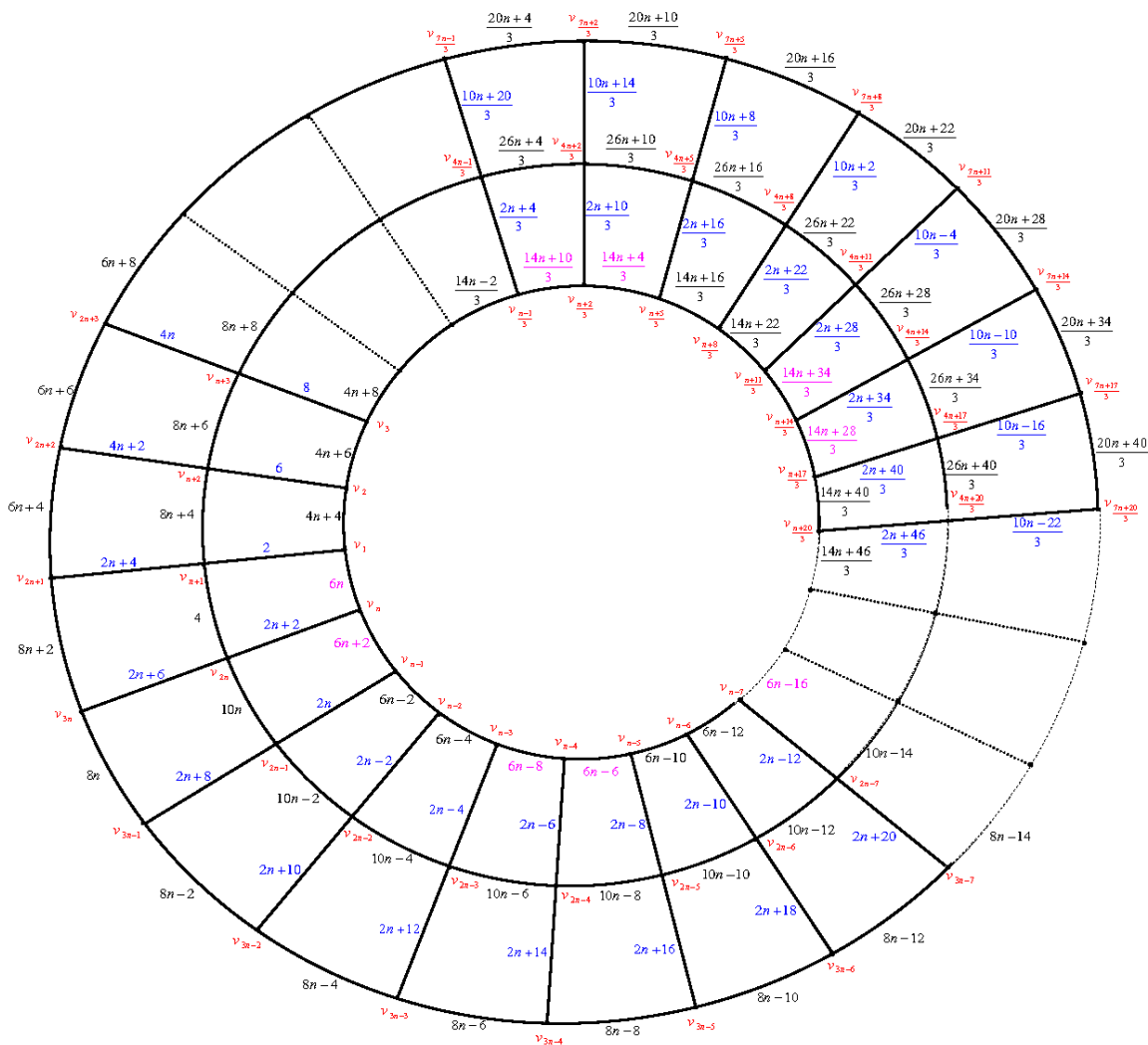


Figure 4. The cylinder grid graph $C_{3,n}, n \equiv 1 \pmod{6}$.

Label the edges of the circle $C_n^{(2)}$ as follows: $f(v_{n+i}v_{n+i+1}) = 8n + 2i + 2, 1 \leq i \leq n - 1, f(v_{2n}v_{n+1}) = 4$.
 Label the edges of the circle $C_n^{(3)}$ as follows: $f(v_{2n+i}v_{2n+i+1}) = 6n + 2i + 2, 1 \leq i \leq n$.
 The labels of corresponding of vertices mod $10n$ are as follows:
 The labels of vertices of the circle $C_n^{(1)}$ are as follows: $f^*(v_1) \equiv 6, f^*(v_2) \equiv 8n + 16, f^*(v_3) \equiv 8n + 22, \dots, f^*(v_{\frac{n-1}{3}}) \equiv 4, f^*(v_{\frac{n+2}{3}}) \equiv 8, f^*(v_{\frac{n+5}{3}}) \equiv 12, f^*(v_{\frac{n+8}{3}}) \equiv 20, f^*(v_{\frac{n+11}{3}}) \equiv 28, f^*(v_{\frac{n+14}{3}}) \equiv 32, f^*(v_{\frac{n+17}{3}}) \equiv 36, f^*(v_{\frac{n+20}{3}}) \equiv 44, f^*(v_{\frac{n+23}{3}}) \equiv 52, f^*(v_{\frac{n+26}{3}}) \equiv 56, f^*(v_{\frac{n+29}{3}}) \equiv 60, f^*(v_{\frac{n+32}{3}}) \equiv 68, f^*(v_{\frac{n+35}{3}}) \equiv 76, f^*(v_{\frac{n+38}{3}}) \equiv 80, f^*(v_{\frac{n+41}{3}}) \equiv 84, f^*(v_{\frac{n+44}{3}}) \equiv 92, f^*(v_{\frac{n+47}{3}}) \equiv 100, \dots, f^*(v_{n-12}) \equiv 4n - 68, f^*(v_{n-11}) \equiv 4n - 64, f^*(v_{n-10}) \equiv 4n - 56, f^*(v_{n-9}) \equiv 4n - 48, f^*(v_{n-8}) \equiv 4n - 44, f^*(v_{n-7}) \equiv 4n - 40, f^*(v_{n-6}) \equiv 4n - 32, f^*(v_{n-5}) \equiv 4n - 24, f^*(v_{n-4}) \equiv 4n - 20, f^*(v_{n-3}) \equiv 4n - 16, f^*(v_{n-2}) \equiv 4n - 8, f^*(v_{n-1}) \equiv 4n, f^*(v_n) \equiv 4n + 4$.
 The labels of vertices of the circle $C_n^{(2)}$ are $f^*(v_{i+1}) = 4i + 10, 1 \leq i \leq n - 1, f^*(v_{2n}) = 4n + 12$.
 The labels of vertices of the circle $C_n^{(3)}$ are $f^*(v_{2i+1}) = 6n + 2i + 8, 1 \leq i \leq n$.
 Case (2): If $n \equiv 3 \pmod{6}$, let the cylinder grid graph $C_{3,n}$ be as in Figure 5.
 First, we label the edges of the paths $P_3^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_3^{(1)}$ as the same in case (1).
 Secondly, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq 3$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(3)}$, and then the edges of the circle $C_n^{(2)}$.

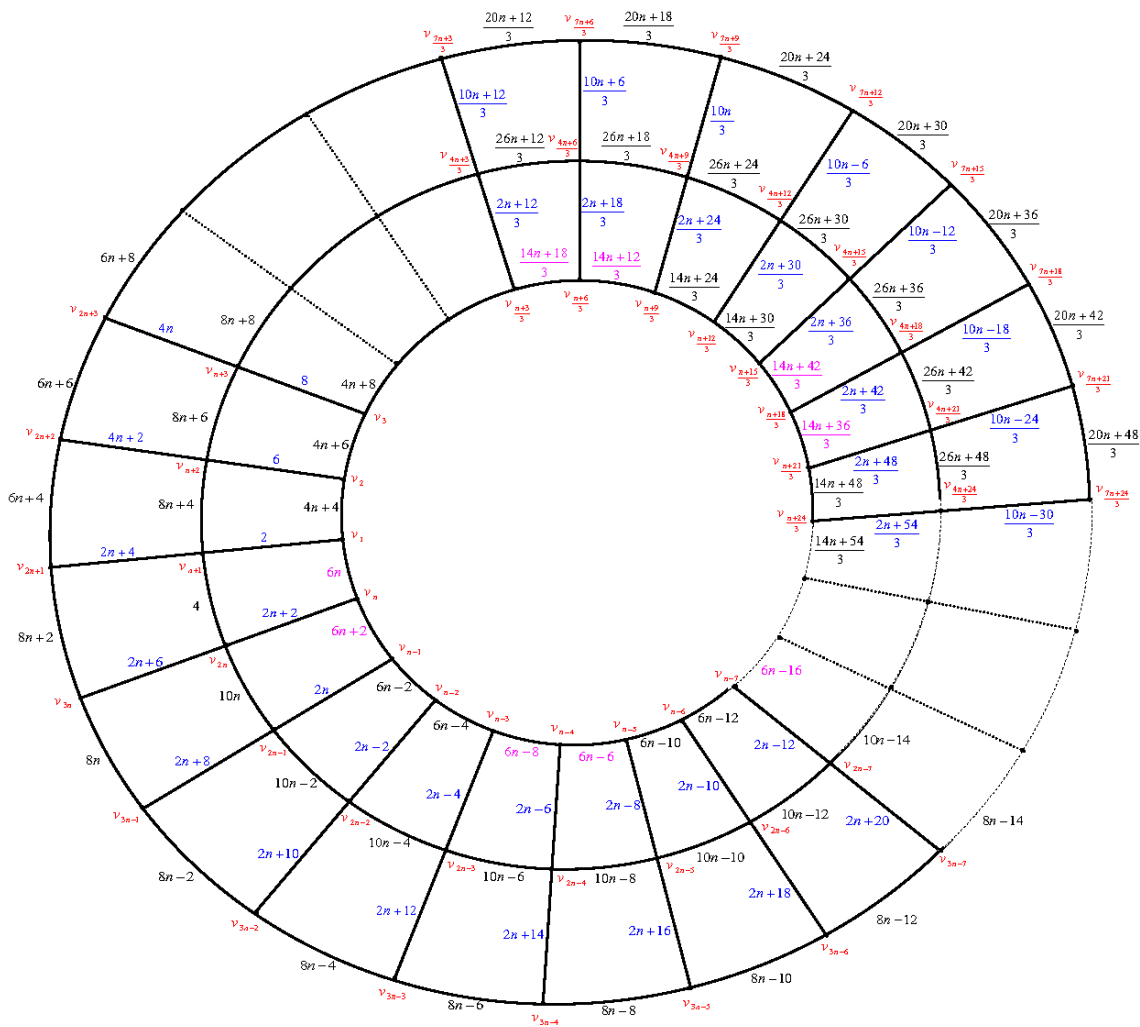


Figure 5. The cylinder grid graph $C_{3,n}, n \equiv 3 \pmod 6$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 4n + 4, f(v_2v_3) = 4n + 6, \dots, f(v_{\frac{n+3}{3}}v_{\frac{n+6}{3}}) = \frac{14n+18}{3}, f(v_{\frac{n+6}{3}}v_{\frac{n+9}{3}}) = \frac{14n+12}{3}, f(v_{\frac{n+9}{3}}v_{\frac{n+12}{3}}) = \frac{14n+24}{3}, f(v_{\frac{n+12}{3}}v_{\frac{n+15}{3}}) = \frac{14n+30}{3}, f(v_{\frac{n+15}{3}}v_{\frac{n+18}{3}}) = \frac{14n+42}{3}, f(v_{\frac{n+18}{3}}v_{\frac{n+21}{3}}) = \frac{14n+36}{3}, f(v_{\frac{n+21}{3}}v_{\frac{n+24}{3}}) = \frac{14n+48}{3}, f(v_{\frac{n+24}{3}}v_{\frac{n+27}{3}}) = \frac{14n+54}{3}, f(v_{\frac{n+27}{3}}v_{\frac{n+30}{3}}) = \frac{14n+66}{3}, f(v_{\frac{n+30}{3}}v_{\frac{n+33}{3}}) = \frac{14n+60}{3}, f(v_{\frac{n+33}{3}}v_{\frac{n+36}{3}}) = \frac{14n+72}{3}, f(v_{\frac{n+36}{3}}v_{\frac{n+39}{3}}) = \frac{14n+78}{3}, f(v_{\frac{n+39}{3}}v_{\frac{n+42}{3}}) = \frac{14n+90}{3}, f(v_{\frac{n+42}{3}}v_{\frac{n+45}{3}}) = \frac{14n+84}{3}, f(v_{\frac{n+45}{3}}v_{\frac{n+48}{3}}) = \frac{14n+96}{3}, f(v_{\frac{n+48}{3}}v_{\frac{n+51}{3}}) = \frac{14n+102}{3}, \dots, f(v_{n-13}v_{n-12}) = 6n - 22, f(v_{n-12}v_{n-11}) = 6n - 24, f(v_{n-11}v_{n-10})f(v_{n-9}v_{n-8}) = 6n - 14, f(v_{n-8}v_{n-7}) = 6n - 16, f(v_{n-7}v_{n-6}) = 6n - 12, f(v_{n-6}v_{n-5}) = 6n - 10, f(v_{n-5}v_{n-4}) = 6n - 6, f(v_{n-4}v_{n-3}) = 6n - 8, f(v_{n-3}v_{n-2}) = 6n - 4, f(v_{n-2}v_{n-1}) = 6n - 2, f(v_{n-1}v_n) = 6n + 2, f(v_nv_1) = 6n.$

The labels of corresponding vertices mod $10n$ are as follows: The label of vertices of the circle $C_n^{(1)}$ are $f^*(v_1) \equiv 6, f^*(v_2) \equiv 8n + 16, f^*(v_3) \equiv 8n + 22, \dots, f^*(v_{\frac{n}{3}-1}) \equiv 10n - 2, f^*(v_{\frac{n}{3}}) \equiv 4, f^*(v_{\frac{n}{3}+1}) \equiv 12, f^*(v_{\frac{n}{3}+2}) \equiv 16, f^*(v_{\frac{n}{3}+3}) \equiv 20, f^*(v_{\frac{n}{3}+4}) \equiv 28, f^*(v_{\frac{n}{3}+5}) \equiv 36, f^*(v_{\frac{n}{3}+6}) \equiv 40, f^*(v_{\frac{n}{3}+7}) \equiv 44, f^*(v_{\frac{n}{3}+8}) \equiv 52, f^*(v_{\frac{n}{3}+9}) \equiv 60, f^*(v_{\frac{n}{3}+10}) \equiv 64, f^*(v_{\frac{n}{3}+11}) \equiv 68, f^*(v_{\frac{n}{3}+12}) \equiv 76, f^*(v_{\frac{n}{3}+13}) \equiv 84, f^*(v_{\frac{n}{3}+14}) \equiv 88, f^*(v_{\frac{n}{3}+15}) \equiv 92, f^*(v_{\frac{n}{3}+16}) \equiv 100, \dots, f^*(v_{n-12}) \equiv 4n - 68, f^*(v_{n-11}) \equiv 4n - 64, f^*(v_{n-10}) \equiv 4n - 56, f^*(v_{n-9}) \equiv 4n - 48, f^*(v_{n-8}) \equiv 4n - 44, f^*(v_{n-7}) \equiv 4n - 40, f^*(v_{n-6}) \equiv 4n - 32, f^*(v_{n-5}) \equiv 4n - 24, f^*(v_{n-4}) \equiv 4n - 20, f^*(v_{n-3}) \equiv 4n - 16, f^*(v_{n-2}) \equiv 4n - 8, f^*(v_{n-1}) \equiv 4n, f^*(v_n) \equiv 4n + 4.$

The labels of vertices of the circles $C_n^{(2)}$ and $C_n^{(3)}$ are the same as in case (1).

Case (3): If $n \equiv 5 \pmod 6$, let the cylinder grid graph $C_{3,n}$ be as in Figure 6.

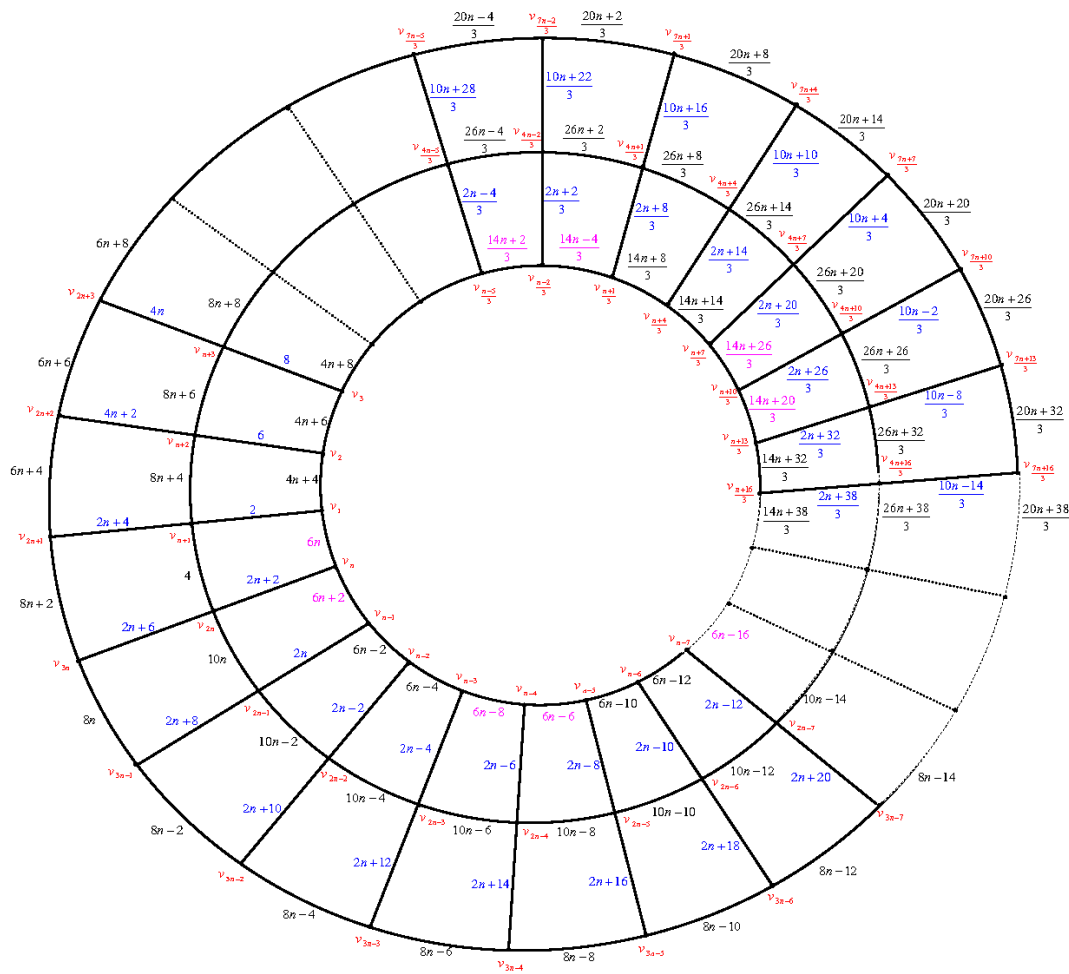


Figure 6. The cylinder grid graph $C_{3,n}, n \equiv 5 \pmod{6}$.

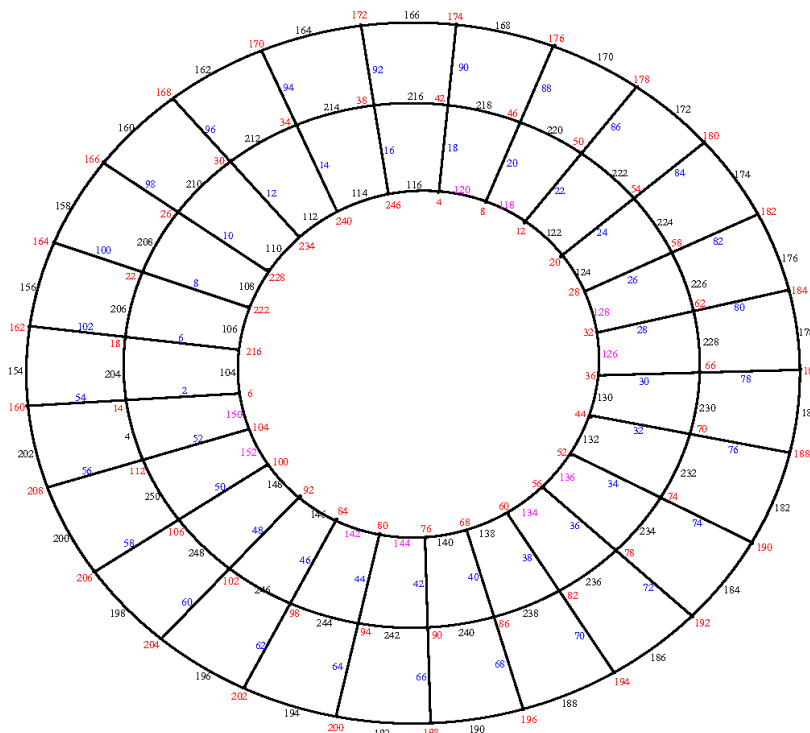
First, we label the edges of the paths $P_3^{(k)}, 1 \leq k \leq 2$ beginning with the edges of the path $P_3^{(1)}$ as the same in case (1). Second, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq 3$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(3)}$, and then the edges of the circle $C_n^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 4n + 4, f(v_2v_3) = 4n + 6, \dots, f(v_{\frac{n-5}{3}}v_{\frac{n-2}{3}}) = \frac{14n+2}{3}, f(v_{\frac{n-2}{3}}v_{\frac{n+1}{3}}) = \frac{14n-4}{3}, f(v_{\frac{n+1}{3}}v_{\frac{n+4}{3}}) = \frac{14n+8}{3}, f(v_{\frac{n+4}{3}}v_{\frac{n+7}{3}}) = \frac{14n+14}{3}, f(v_{\frac{n+7}{3}}v_{\frac{n+10}{3}}) = \frac{14n+26}{3}, f(v_{\frac{n+10}{3}}v_{\frac{n+13}{3}}) = \frac{14n+30}{3}, f(v_{\frac{n+13}{3}}v_{\frac{n+16}{3}}) = \frac{14n+32}{3}, f(v_{\frac{n+16}{3}}v_{\frac{n+19}{3}}) = \frac{14n+38}{3}, f(v_{\frac{n+19}{3}}v_{\frac{n+22}{3}}) = \frac{14n+50}{3}, f(v_{\frac{n+22}{3}}v_{\frac{n+25}{3}}) = \frac{14n+44}{3}, f(v_{\frac{n+25}{3}}v_{\frac{n+28}{3}}) = \frac{14n+56}{3}, f(v_{\frac{n+28}{3}}v_{\frac{n+31}{3}}) = \frac{14n+62}{3}, f(v_{\frac{n+31}{3}}v_{\frac{n+34}{3}}) = \frac{14n+74}{3}, f(v_{\frac{n+34}{3}}v_{\frac{n+37}{3}}) = \frac{14n+68}{3}, f(v_{\frac{n+37}{3}}v_{\frac{n+40}{3}}) = \frac{14n+80}{3}, f(v_{\frac{n+40}{3}}v_{\frac{n+43}{3}}) = \frac{14n+86}{3}, \dots, f(v_{n-13}v_{n-12}) = 6n - 22, f(v_{n-12}v_{n-11}) = 6n - 24, f(v_{n-11}v_{n-10}) = 6n - 20, f(v_{n-10}v_{n-9}) = 6n - 18, f(v_{n-9}v_{n-8}) = 6n - 14, f(v_{n-8}v_{n-7}) = 6n - 16, f(v_{n-7}v_{n-6}) = 6n - 12, f(v_{n-6}v_{n-5}) = 6n - 10, f(v_{n-5}v_{n-4}) = 6n - 6, f(v_{n-4}v_{n-3}) = 6n - 8, f(v_{n-3}v_{n-2}) = 6n - 4, f(v_{n-2}v_{n-1}) = 6n - 2, f(v_{n-1}v_n) = 6n + 2, f(v_nv_1) = 6n.$

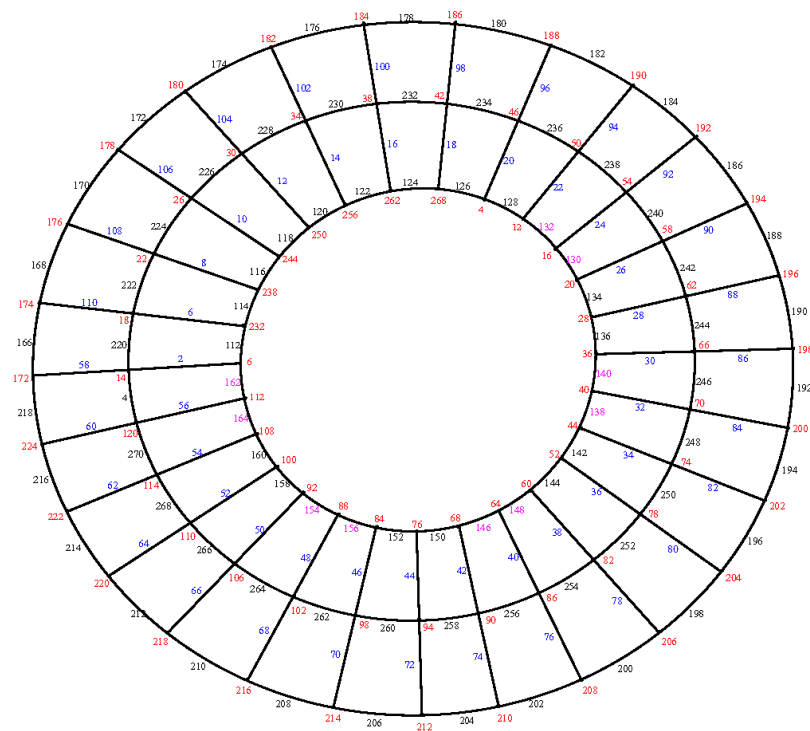
The labels of corresponding vertices mod10n are as follows: The labels of vertices of the circle $C_n^{(1)}$: $f^*(v_1) \equiv 6, f^*(v_2) \equiv 8n + 16, f^*(v_3) \equiv 8n + 22, \dots, f^*(v_{\frac{n-5}{3}}) \equiv 10n - 4, f^*(v_{\frac{n-2}{3}}) \equiv 0, f^*(v_{\frac{n+1}{3}}) \equiv 4, f^*(v_{\frac{n+4}{3}}) \equiv 12, f^*(v_{\frac{n+7}{3}}) \equiv 20, f^*(v_{\frac{n+10}{3}}) \equiv 24, f^*(v_{\frac{n+13}{3}}) \equiv 28, f^*(v_{\frac{n+16}{3}}) \equiv 36, f^*(v_{\frac{n+19}{3}}) \equiv 44, f^*(v_{\frac{n+22}{3}}) \equiv 44, f^*(v_{\frac{n+25}{3}}) \equiv 48, f^*(v_{\frac{n+28}{3}}) \equiv 52, f^*(v_{\frac{n+31}{3}}) \equiv 60, f^*(v_{\frac{n+34}{3}}) \equiv 72, f^*(v_{\frac{n+37}{3}}) \equiv 76, f^*(v_{\frac{n+40}{3}}) \equiv 84, f^*(v_{\frac{n+43}{3}}) \equiv 92, f^*(v_{\frac{n+46}{3}}) \equiv 96, f^*(v_{\frac{n+49}{3}}) \equiv 100, \dots, f^*(v_{n-12}) \equiv 4n - 68, f^*(v_{n-11}) \equiv 4n - 64, f^*(v_{n-10}) \equiv 4n - 56, f^*(v_{n-9}) \equiv 4n - 48, f^*(v_{n-8}) \equiv 4n - 44, f^*(v_{n-7}) \equiv 4n - 40, f^*(v_{n-6}) \equiv 4n - 32, f^*(v_{n-5}) \equiv 4n - 24, f^*(v_{n-4}) \equiv 4n - 20, f^*(v_{n-3}) \equiv 4n - 16, f^*(v_{n-2}) \equiv 4n - 8, f^*(v_{n-1}) \equiv 4n, f^*(v_n) \equiv 4n + 4.$

The labels of vertices of the circles $C_n^{(2)}$ and $C_n^{(3)}$ are the same as in case (1).

Illustration: An e.e.g., l. of the cylinder grid graphs $C_{3,25}$, $C_{3,27}$ and $C_{3,29}$ are shown in Figure 7.



(a) $C_{3,25}$



(b) $C_{3,27}$

Figure 7. Cont.

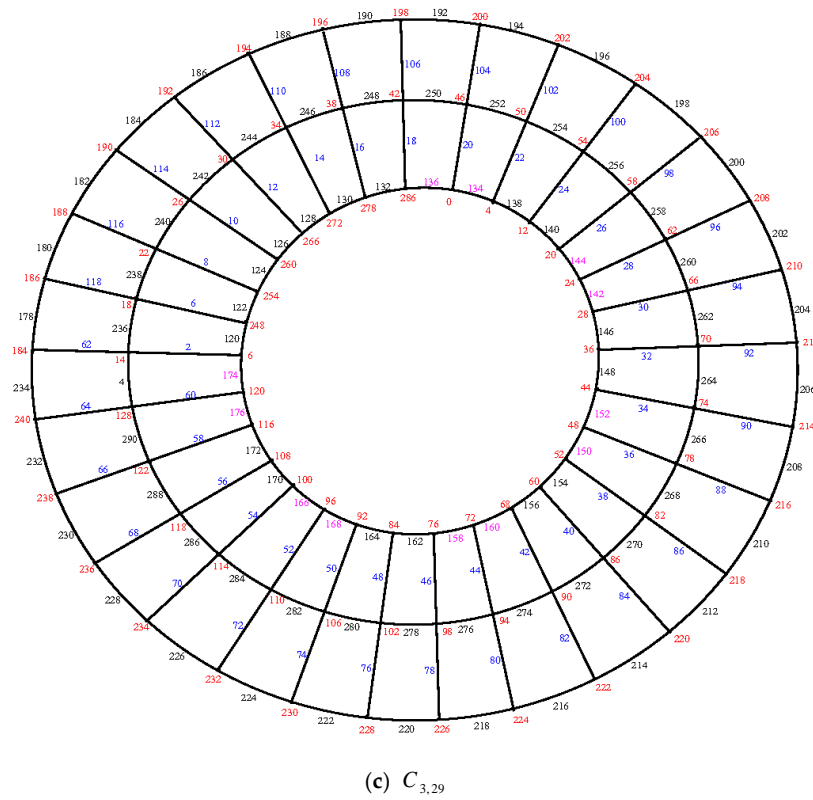


Figure 7. An e.e.g., l. of the cylinder grid graphs $C_{3,25}$, $C_{3,27}$ and $C_{3,29}$.

Remark 1. Note that $C_{3,5}$ is an edge even graceful graph but it does not follow the previous rule (see Figure 8).

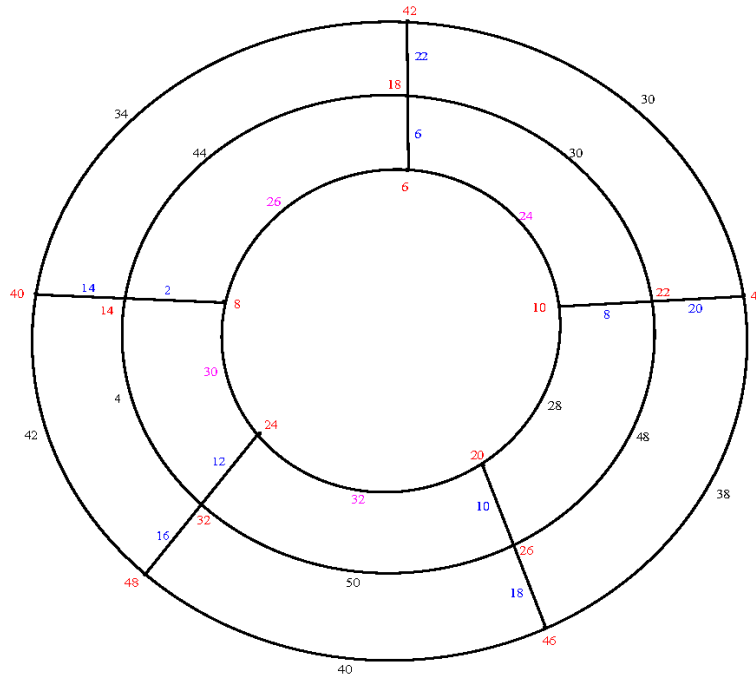


Figure 8. An e.e.g., l. of the cylinder grid graph $C_{3,5}$.

Theorem 3. *If m is an odd positive integer greater than 3 and n is an even positive integer, $n \geq 2$, then the cylinder grid graph $C_{m,n}$, is an edge even graceful graph.*

Proof. Using standard notation $p = |V(C_{m,n})| = mn$, $q = |E(C_{m,n})| = 2mn - n$ and $r = \max(p, q) = 2mn - n$ and $f : E(C_{m,n}) \rightarrow \{2, 4, 6, \dots, 4mn - 2n - 2\}$. \square

Let the cylinder grid graph $C_{m,n}$ be as in Figure 9. There are six cases:

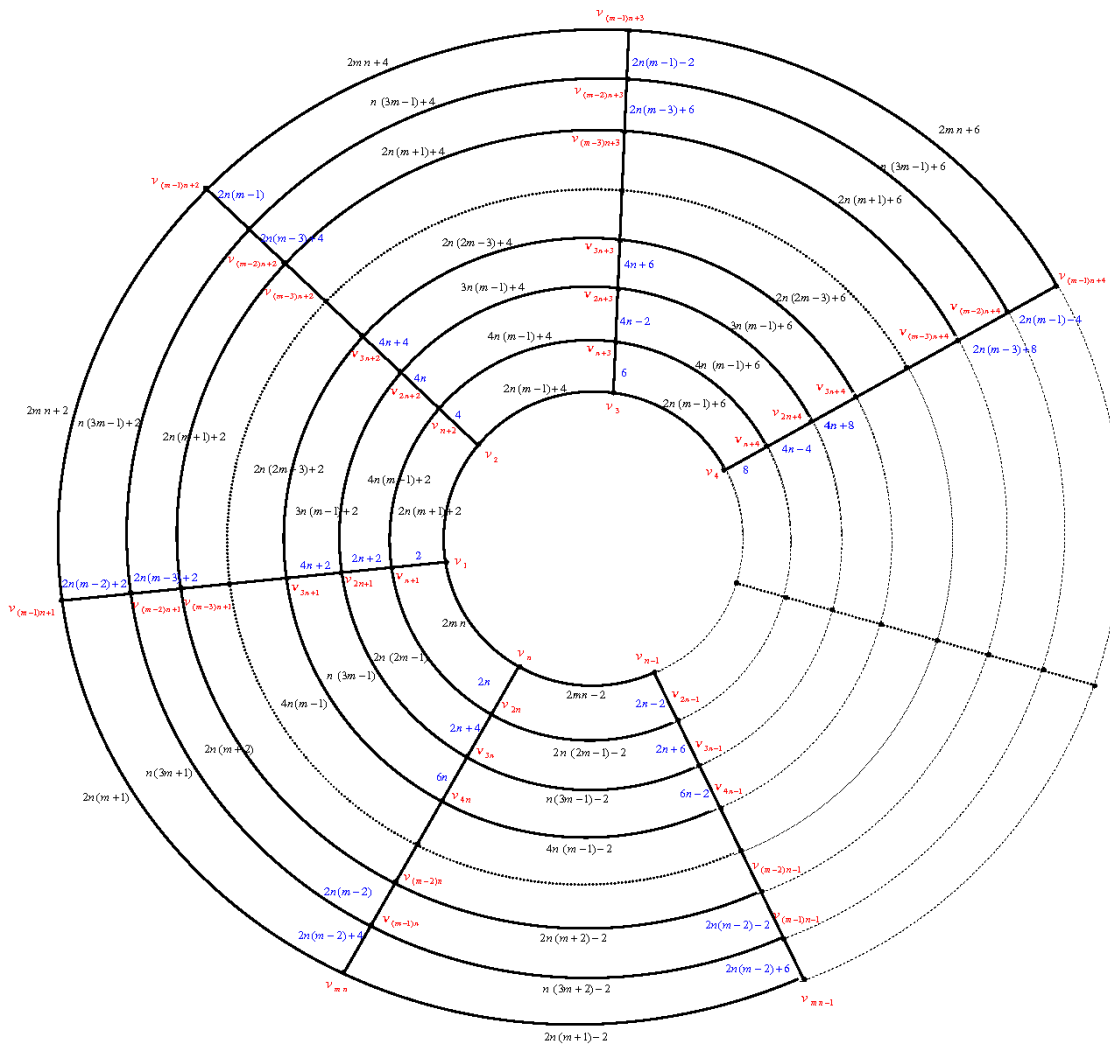


Figure 9. The cylinder grid graph $C_{m,n}$, m is odd greater than 3 and $n \geq 2$.

Case (1): $n \equiv 0 \pmod{12}$. First, we label the edges of the paths $P_m^{(k)}$, $1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ as follows: Move clockwise to label the edges $v_1v_{n+1}, v_2v_{n+2}, v_3v_{n+3}, \dots, v_{n-1}v_{2n-1}, v_nv_{2n}$ by $2, 4, 6, \dots, 2n - 2, 2n$, then move anticlockwise to label the edges $v_{n+1}v_{n+2}, v_{n+2}v_{n+3}, v_{n+3}v_{n+4}, \dots, v_{2n-1}v_{2n-2}, v_{2n}v_{2n-1}$ by $2n + 2, 2n + 4, 2n + 6, \dots, 4n - 2, 4n$, then move clockwise to label the edges $v_{2n+1}v_{3n+1}, v_{2n+2}v_{3n+2}, v_{2n+3}v_{3n+3}, \dots, v_{3n-1}v_{4n-1}, v_{3n}v_{4n}$ by $4n + 2, 4n + 4, 4n + 6, \dots, 6n - 2, 6n$ and so on.

Finally, move anticlockwise to label the edges $v_{(m-2)n+1}v_{(m-1)n+1}, v_{(m-1)n}v_{mn}, v_{(m-1)n-1}v_{mn-1}, \dots, v_{(m-2)n+3}v_{(m-1)n+3}, v_{(m-2)n+2}v_{(m-1)n+2}$ by $2n(m - 2) + 2, 2n(m - 2) + 4, 2n(m - 2) + 6, \dots, 2n(m - 1) - 2, 2n(m - 1)$.

Secondly, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ beginning with the edges of the inner most circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, then the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$. Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m - 1) + 2, f(v_2v_3) = 2n(m - 1) + 6, f(v_3v_4) = 2n(m - 1) + 4, f(v_4v_5) = 2n(m - 1) + 8, f(v_5v_6) = 2n(m - 1) + 10, f(v_6v_7) = 2n(m - 1) + 14, f(v_7v_8) = 2n(m - 1) + 12, f(v_8v_9) = 2n(m - 1) + 16, f(v_9v_{10}) = 2n(m - 1) + 18, f(v_{10}v_{11}) = 2n(m - 1) + 22, f(v_{11}v_{12}) = 2n(m - 1) + 20, f(v_{12}v_{13}) = 2n(m - 1) + 24, \dots, f(v_{n-7}v_{n-6}) = 2mn - 14, f(v_{n-6}v_{n-5}) = 2mn - 10, f(v_{n-5}v_{n-4}) = 2mn - 12, f(v_{n-4}v_{n-3}) = 2mn - 8, f(v_{n-3}v_{n-2}) = 2mn - 2, f(v_{n-2}v_{n-1}) = 2mn - 6, f(v_{n-1}v_n) = 2mn, f(v_nv_1) = 2mn - 4.$

Label the edges of the circle $C_n^{(m)}$ as follows: $f(v_{(m-1)n+i}v_{(m-1)n+i+1}) = 2mn + 2i, f(v_{mn}v_{(m-1)n+1}) = 2n(m + 1), 1 \leq i \leq n - 1.$

Label the edges of the circle $C_n^{(m-2)}$ as follows: $f(v_{(m-3)n+i}v_{(m-3)n+i+1}) = 2n(m + 1) + 2i, f(v_{(m-2)n}v_{(m-3)n+1}) = 2n(m + 2), 1 \leq i \leq n - 1.$

Label the edges of the circle $C_n^{(m-4)}$ as follows: $f(v_{(m-5)n+i}v_{(m-5)n+i+1}) = 2n(m + 2) + 2i, f(v_{(m-4)n}v_{(m-5)n+1}) = 2n(m + 3), 1 \leq i \leq n - 1,$ and so on.

Label the edges of the circle $C_n^{(3)}$ as follows: $f(v_{2n+i}v_{2n+i+1}) = 3n(m - 1) + 2i, f(v_{3n}v_{2n+1}) = n(3m - 1), 1 \leq i \leq n - 1,$

Label the edges of the circle $C_n^{(m-1)}$ as follows: $f(v_{(m-2)n+i}v_{(m-2)n+i+1}) = n(3m - 1) + 2i, f(v_{(m-1)n}v_{(m-1)n+1}) = n(3m + 1) - 1, 1 \leq i \leq n - 1,$

Label the edges of the circle $C_n^{(m-3)}$ as follows: $f(v_{(m-4)n+i}v_{(m-4)n+i+1}) = n(3m + 2) + 2i, f(v_{(m-3)n}v_{(m-4)n+1}) = 3n(m + 1), 1 \leq i \leq n - 1, \dots,$ and so on.

Label the edges of the circle $C_n^{(4)}$ as follows: $f(v_{3n+i}v_{3n+i+1}) = 2n(2m - 3) + 2i, f(v_{4n}v_{3n+1}) = 4n(m - 1), 1 \leq i \leq n - 1,$

Label the edges of $C_n^{(2)}$ as follows: $f(v_{n+i}v_{n+i+1}) = 4n(m - 1) + 2i, f(v_{2n}v_{2n+1}) = 2n(m - 1), 1 \leq i \leq n - 1,$

Thus, the labels of corresponding vertices mod $(4mn - 2n)$ will be:

The label the vertices of $C_n^{(1)}$ are: $f^*(v_1) \equiv 0; f^*(v_2) \equiv 4mn - 4n + 12; f^*(v_3) \equiv 4mn - 4n + 16; f^*(v_4) \equiv 4mn - 4n + 20; f^*(v_5) \equiv 4mn - 4n + 28; f^*(v_6) \equiv 4mn - 4n + 36; f^*(v_7) \equiv 4mn - 4n + 40; f^*(v_8) \equiv 4mn - 4n + 44; f^*(v_9) \equiv 4mn - 4n + 52; f^*(v_{10}) \equiv 4mn - 4n + 60; f^*(v_{11}) \equiv 4mn - 4n + 64; f^*(v_{12}) \equiv 4mn - 4n + 68; \dots; f^*(v_{n-6}) \equiv 4n - 36; f^*(v_{n-5}) \equiv 4n - 32; f^*(v_{n-4}) \equiv 4n - 28; f^*(v_{n-3}) \equiv 4n - 16; f^*(v_{n-2}) \equiv 4n - 12; f^*(v_{n-1}) \equiv 4n - 8; f^*(v_n) \equiv 4n - 4.$

The label the vertices of $C_n^{(2)}, C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$ respectively are: $f^*(v_{n+i}) \equiv 4i + 2; f^*(v_{2n+i}) \equiv 2mn + 4n + 4i + 2; f^*(v_{3n+i}) \equiv 4n + 4i + 2; \dots; f^*(v_{(m-3)n+i}) \equiv 4mn - 6n + 4i + 2; f^*(v_{(m-2)n+i}) \equiv 2mn - 6n + 4i + 2; f^*(v_{(m-1)n+i}) \equiv 2mn + 2i + 2, 1 \leq i \leq n.$

Case (2): $n \equiv 2 \pmod{12}, n \neq 2.$

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ begin with the edges of the path $P_m^{(1)}$ as the same in case (1).

Secondly, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ begin with the edges of the inner most circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m - 1) + 2, f(v_2v_3) = 2n(m - 1) + 6, f(v_3v_4) = 2n(m - 1) + 4, f(v_4v_5) = 2n(m - 1) + 8, f(v_5v_6) = 2n(m - 1) + 10, f(v_6v_7) = 2n(m - 1) + 14, f(v_7v_8) = 2n(m - 1) + 12, f(v_8v_9) = 2n(m - 1) + 16, f(v_9v_{10}) = 2n(m - 1) + 18, f(v_{10}v_{11}) = 2n(m - 1) + 22, f(v_{11}v_{12}) = 2n(m - 1) + 20, f(v_{12}v_{13}) = 2n(m - 1) + 24, \dots, f(v_{n-9}v_{n-8}) = 2mn - 18, f(v_{n-8}v_{n-7}) = 2mn - 14, f(v_{n-7}v_{n-6}) = 2mn - 16, f(v_{n-6}v_{n-5}) = 2mn - 12, f(v_{n-5}v_{n-4}) = 2mn - 10, f(v_{n-4}v_{n-3}) = 2mn - 6, f(v_{n-3}v_{n-2}) = 2mn - 8, f(v_{n-2}v_{n-1}) = 2mn - 4, f(v_{n-1}v_n) = 2mn - 2, f(v_nv_1) = 2mn.$

Label the edges of the circle $C_n^{(2)}$ as follows: $f(v_{n+1}v_{n+2}) = 4n(m-1) + 4, f(v_{n+2}v_{n+3}) = 4n(m-1) + 2, f(v_{n+3}v_{n+4}) = 4n(m-1) + 8, f(v_{n+4}v_{n+5}) = 4n(m-1) + 6, f(v_{n+i}v_{n+i+1}) = 4n(m-1) + 2i, 6 \leq i \leq n-2, f(v_{2n-1}v_{2n}) = 2n(2m-1), f(v_{2n}v_{n+1}) = 2n(2m-1) - 2$. Label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}$ as in case (1).

Thus, the labels of corresponding vertices mod $(4mn - 2n)$ will be:

The label the vertices of $C_n^{(1)}$ are: $f^*(v_1) \equiv 4, f^*(v_2) \equiv 4mn - 4n + 12, f^*(v_3) \equiv 4mn - 4n + 16, f^*(v_4) \equiv 4mn - 4n + 20, f^*(v_5) \equiv 4mn - 4n + 28, f^*(v_6) \equiv 4mn - 4n + 36, f^*(v_7) \equiv 4mn - 4n + 40, f^*(v_8) \equiv 4mn - 4n + 44, f^*(v_9) \equiv 4mn - 4n + 52, f^*(v_{10}) \equiv 4mn - 4n + 60, f^*(v_{11}) \equiv 4mn - 4n + 64, f^*(v_{12}) \equiv 4mn - 4n + 68, f^*(v_{13}) \equiv 4mn - 4n + 76, \dots, f^*(v_{n-8}) \equiv 4n - 48, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 40, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 24, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 16, f^*(v_{n-1}) \equiv 4n - 8, f^*(v_n) \equiv 4n - 2$.

The label the vertices of the circle $C_n^{(2)}$ are: $f^*(v_{n+1}) \equiv 6, f^*(v_{n+2}) \equiv 10, f^*(v_{n+3}) \equiv 14, f^*(v_{n+4}) \equiv 18, f^*(v_{n+5}) \equiv 20, f^*(v_{n+i}) \equiv 4i + 2, 6 \leq i \leq n-2, f^*(v_{2n-1}) \equiv 4n, f^*(v_{2n}) \equiv 4n + 2$.

The label the vertices of $C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$ respectively are as the same as in case (1).

Remark 2. In case $n = 2$. Let the edges of the cylinder grid graph $C_{m,2}$ are labeled as shown in Figure 10. The corresponding labels of vertices mod $(8m - 4)$ are as follows: $f^*(v_1) \equiv 8, f^*(v_{2i+1}) \equiv 4m + 8i + 4, 1 \leq i \leq \frac{m-3}{2}, f^*(v_{2i}) \equiv 8i + 6, 1 \leq i \leq \frac{m-1}{2}; f^*(v'_1) \equiv 12, f^*(v'_2) \equiv 20, f^*(v'_{2i+1}) \equiv 4m + 8i + 18, 1 \leq i \leq \frac{m-3}{2}, f^*(v'_{2i}) \equiv 8i + 10, 2 \leq i \leq \frac{m-1}{2}$.

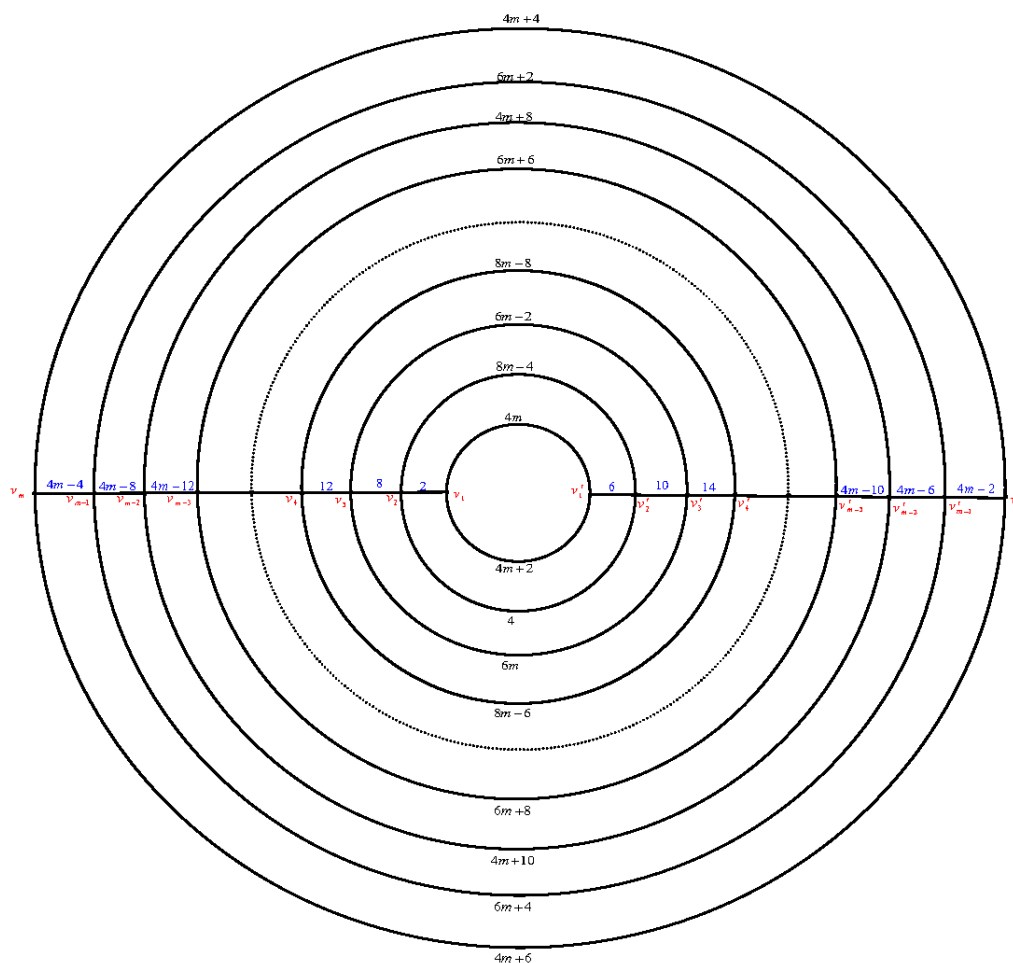


Figure 10. The cylinder grid graph $C_{m,2}$.

Case (3): $n \equiv 4 \pmod{12}$.

First we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ begin with the edges of the path $P_m^{(1)}$ as the same in case (1).

Second we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ begin with the edges of the inner most circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m-1) + 2, f(v_2v_3) = 2n(m-1) + 6, f(v_3v_4) = 2n(m-1) + 4, f(v_4v_5) = 2n(m-1) + 8, f(v_5v_6) = 2n(m-1) + 10, f(v_6v_7) = 2n(m-1) + 14, f(v_7v_8) = 2n(m-1) + 12, f(v_8v_9) = 2n(m-1) + 16, f(v_9v_{10}) = 2n(m-1) + 18, f(v_{10}v_{11}) = 2n(m-1) + 22, f(v_{11}v_{12}) = 2n(m-1) + 20, f(v_{12}v_{13}) = 2n(m-1) + 24, \dots, f(v_{n-8}v_{n-7}) = 2mn - 16, f(v_{n-7}v_{n-6}) = 2mn - 14, f(v_{n-6}v_{n-5}) = 2mn - 10, f(v_{n-5}v_{n-4}) = 2mn - 12, f(v_{n-4}v_{n-3}) = 2mn - 8, f(v_{n-3}v_{n-2}) = 2mn - 2, f(v_{n-2}v_{n-1}) = 2mn - 6, f(v_{n-1}v_n) = 2mn, \text{ and } f(v_nv_1) = 2mn - 4.$

Label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}, C_n^{(2)}$ as in case (1).

Thus we have the labels of corresponding vertices of the circle $C_n^{(1)} \pmod{4mn - 2n}$ will be: $f^*(v_1) \equiv 0, f^*(v_2) \equiv 4mn - 4n + 12, f^*(v_3) \equiv 4mn - 4n + 16, f^*(v_4) \equiv 4mn - 4n + 20, f^*(v_5) \equiv 4mn - 4n + 28, f^*(v_6) \equiv 4mn - 4n + 36, f^*(v_7) \equiv 4mn - 4n + 40, f^*(v_8) \equiv 4mn - 4n + 44, f^*(v_9) \equiv 4mn - 4n + 52, f^*(v_{10}) \equiv 4mn - 4n + 60, f^*(v_{11}) \equiv 4mn - 4n + 64, f^*(v_{12}) \equiv 4mn - 4n + 68, f^*(v_{13}) \equiv 4mn - 4n + 76, \dots, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 36, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 28, f^*(v_{n-3}) \equiv 4n - 16, f^*(v_{n-2}) \equiv 4n - 12, f^*(v_{n-1}) \equiv 4n - 8, f^*(v_n) \equiv 4n - 4.$

The label the vertices of $C_n^{(2)}, C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$ respectively are as same in case (1)

Remark 3. In case $n = 4$. Let the the edges of the cylinder grid graph $C_{m,4}$ are labeled as shown in Figure 11. The corresponding labels of vertices $\pmod{16m - 8}$ are as follows: $f^*(v_1) \equiv 6, f^*(v_2) \equiv 8, f^*(v_3) \equiv 16, f^*(v_4) \equiv 20; f^*(v_{4i+1}) \equiv 4i + 10, 1 \leq i \leq 3, f^*(v_8) \equiv 28; f^*(v_{8k+i}) \equiv 8m + 4i + 16k - 10, 1 \leq i \leq 4, 1 \leq k \leq \frac{m-5}{2}; f^*(v_{4m-11}) \equiv 0, f^*(v_{4m-10}) \equiv 2, f^*(v_{4m-9}) \equiv 4, f^*(v_{4m-8}) \equiv 10, f^*(v_{8k+4+i}) \equiv 4i + 16k + 10, 1 \leq i \leq 4, 1 \leq k \leq \frac{m-3}{2}.$

Case (4): $n \equiv 6 \pmod{12}$.

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ begin with the edges of the path $P_m^{(1)}$ as the same in case (1).

Secondly, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ begin with the edges of the inner most circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m-1) + 2, f(v_2v_3) = 2n(m-1) + 6, f(v_3v_4) = 2n(m-1) + 4, f(v_4v_5) = 2n(m-1) + 8, f(v_5v_6) = 2n(m-1) + 10, f(v_6v_7) = 2n(m-1) + 14, f(v_7v_8) = 2n(m-1) + 12, f(v_8v_9) = 2n(m-1) + 16, f(v_9v_{10}) = 2n(m-1) + 18, f(v_{10}v_{11}) = 2n(m-1) + 22, f(v_{11}v_{12}) = 2n(m-1) + 20, f(v_{12}v_{13}) = 2n(m-1) + 24, \dots, f(v_{n-9}v_{n-8}) = 2mn - 18, f(v_{n-8}v_{n-7}) = 2mn - 14, f(v_{n-7}v_{n-6}) = 2mn - 16, f(v_{n-6}v_{n-5}) = 2mn - 12, f(v_{n-5}v_{n-4}) = 2mn - 10, f(v_{n-4}v_{n-3}) = 2mn - 6, f(v_{n-3}v_{n-2}) = 2mn - 8, f(v_{n-2}v_{n-1}) = 2mn - 4, f(v_{n-1}v_n) = 2mn + 2, f(v_nv_1) = 2mn - 2.$

Label the edges of $C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}, C_n^{(2)}$ as in case (1).

Label the edges of the circle $C_n^{(m-2)}$ as follows: $f(v_{(m-3)n+1}v_{(m-3)n+2}) = 2n(m+2), f(v_{(m-3)n+i}v_{(m-3)n+i+1}) = 2n(m+1) + 2i, 2 \leq i \leq n-1, f(v_{(m-2)n}v_{(m-3)n+1}) = 2n(m+2) + 2.$

Label the edges of the circle $C_n^{(m)}$ as follows: $f(v_{(m-1)n+1}v_{(m-1)n+2}) = 2mn, f(v_{(m-1)n+i}v_{(m-1)n+i+1}) = 2mn + 2i, 2 \leq i \leq n-1, f(v_{(m-2)n}v_{(m-3)n+1}) = 2n(m+2).$

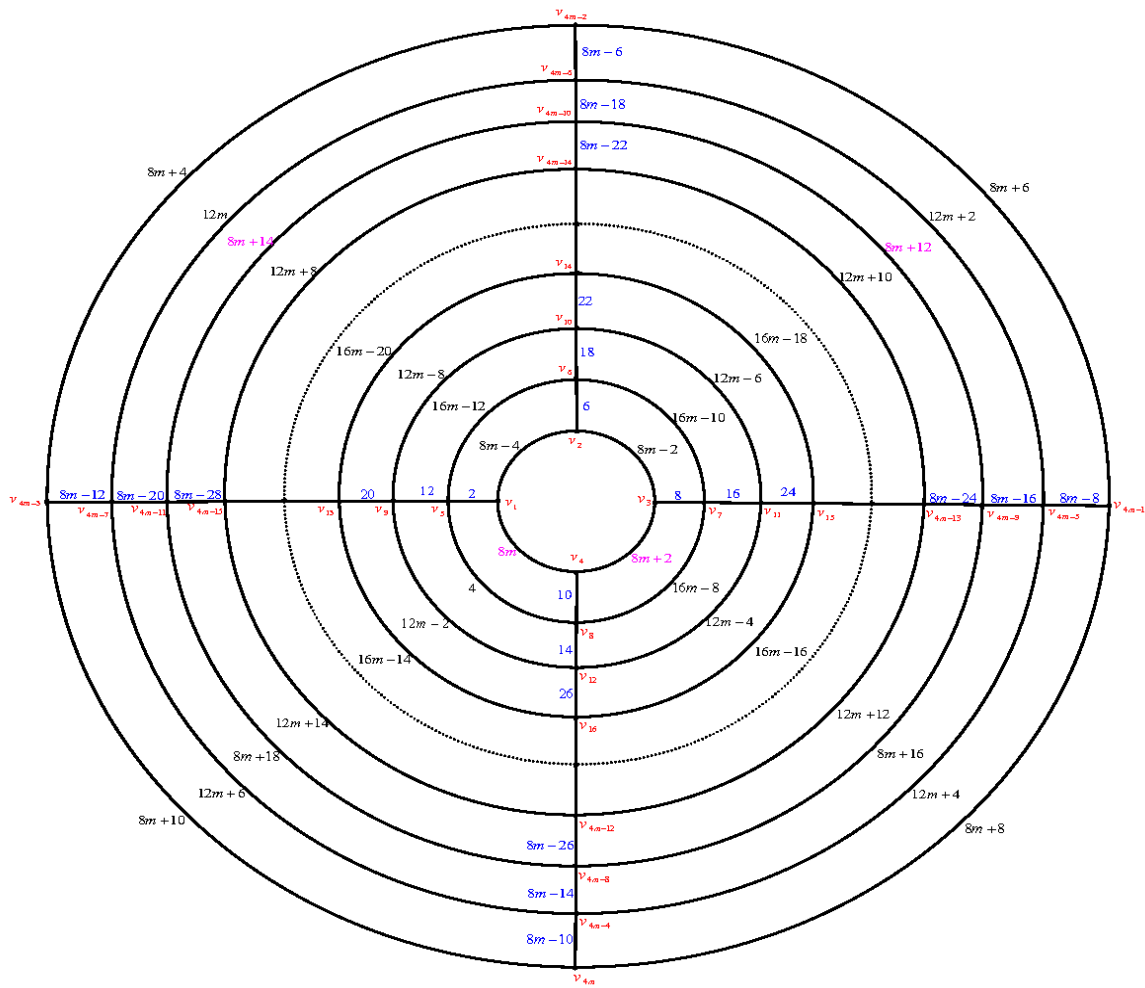


Figure 11. An e.g., l. of the cylinder grid graph $C_{m,4}$.

Thus we have the labels of corresponding vertices $\text{mod}(4mn - 2n)$ will be:

The labels the vertices of the circle $C_n^{(1)}$ are: $f^*(v_1) \equiv 2, f^*(v_2) \equiv 4mn - 4n + 12, f^*(v_3) \equiv 4mn - 4n + 16, f^*(v_4) \equiv 4mn - 4n + 20, f^*(v_5) \equiv 4mn - 4n + 28, f^*(v_6) \equiv 4mn - 4n + 36, f^*(v_7) \equiv 4mn - 4n + 40, f^*(v_8) \equiv 4mn - 4n + 44, f^*(v_9) \equiv 4mn - 4n + 52, f^*(v_{10}) \equiv 4mn - 4n + 60, f^*(v_{11}) \equiv 4mn - 4n + 64, f^*(v_{12}) \equiv 4mn - 4n + 68, \dots, f^*(v_{n-8}) \equiv 4n - 48, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 40, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 24, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 16, f^*(v_{n-1}) \equiv 4n - 4; f^*(v_n) \equiv 4n$.

The labels the vertices of the circle $C_n^{(m-2)}$ are: $f^*(v_{(m-3)n+1}) \equiv 4mn - 6n + 6; f^*(v_{(m-3)n+2}) \equiv 4mn - 6n + 8; f^*(v_{(m-3)n+i}) \equiv 4mn - 6n + 4i + 2, 3 \leq i \leq n - 1, f^*(v_{(m-2)n}) \equiv 4mn - 4n + 4$.

The labels the vertices of $C_n^{(2)}, C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-1)}, C_n^{(m-3)}$ respectively are the same as in case (1).

The labels the vertices of $C_n^{(m)}$ are: $f^*(v_{(m-1)n+1}) \equiv 2mn + 2, f^*(v_{(m-1)n+2}) \equiv 2mn + 4, f^*(v_{(m-1)n+i}) \equiv 2mn + 2i + 2, 3 \leq i \leq n$. Case (5): $n \equiv 8 \text{mod} 12$.

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ begin with the edges of the path $P_m^{(1)}$ as the same in case (1).

Secondly, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ begin with the edges of the inner most circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m - 1) + 2, f(v_2v_3) = 2n(m - 1) + 6, f(v_3v_4) = 2n(m - 1) + 4, f(v_4v_5) = 2n(m - 1) + 8, f(v_5v_6) = 2n(m - 1) + 10, f(v_6v_7) = 2n(m - 1) + 14, f(v_7v_8) = 2n(m - 1) + 12, f(v_8v_9) = 2n(m - 1) + 16, f(v_9v_{10}) = 2n(m - 1) + 18, f(v_{10}v_{11}) =$

$2n(m - 1) + 22, f(v_{11}v_{12}) = 2n(m - 1) + 20, f(v_{12}v_{13}) = 2n(m - 1) + 24, f(v_{13}v_{14}) = 2n(m - 1) + 26, \dots, f(v_{n-9}v_{n-8}) = 2mn - 20, f(v_{n-8}v_{n-7}) = 2mn - 16, f(v_{n-7}v_{n-6}) = 2mn - 14, f(v_{n-6}v_{n-5}) = 2mn - 10, f(v_{n-5}v_{n-4}) = 2mn - 12, f(v_{n-4}v_{n-3}) = 2mn - 8, f(v_{n-3}v_{n-2}) = 2mn - 6, f(v_{n-2}v_{n-1}) = 2mn - 2, f(v_{n-1}v_n) = 2mn - 4, f(v_nv_1) = 2mn + 4.$

Label the edges of the circle $C_n^{(m)}$ as follows $f(v_{(m-1)n+1}v_{(m-1)n+2}) = 2mn + 2, f(v_{(m-1)n+2}v_{(m-1)n+3}) = 2mn + 6, f(v_{(m-1)n+i}v_{(m-1)n+i+1}) = 2mn + 2i + 2, 3 \leq i \leq n - 1, f(v_{mn}v_{(m-1)n+1}) = 2mn.$

Label the edges of $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}, C_n^{(2)}$ as the same in case (1).

Thus we labels of corresponding vertices of the circle $C_n^{(1)} \pmod{4mn - 2n}$ will be: $f^*(v_1) \equiv 8, f^*(v_2) \equiv 4mn - 4n + 12, f^*(v_3) \equiv 4mn - 4n + 16, f^*(v_4) \equiv 4mn - 4n + 20, f^*(v_5) \equiv 4mn - 4n + 28, f^*(v_6) \equiv 4mn - 4n + 36, f^*(v_7) \equiv 4mn - 4n + 40, f^*(v_8) \equiv 4mn - 4n + 44, f^*(v_9) \equiv 4mn - 4n + 52, f^*(v_{10}) \equiv 4mn - 4n + 60, f^*(v_{11}) \equiv 4mn - 4n + 64, f^*(v_{12}) \equiv 4mn - 4n + 68, f^*(v_{13}) \equiv 4mn - 4n + 76, \dots, f^*(v_{n-8}) \equiv 4n - 52, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 36, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 28, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 12, f^*(v_{n-1}) \equiv 4n - 8, f^*(v_n) \equiv 4n.$

The labels the vertices of the circle $C_n^{(m)}$ are: $f^*(v_{(m-1)n+1}) \equiv 2mn - 4n + 14, f^*(v_{(m-1)n+2}) \equiv 2mn + 8, f^*(v_{(m-1)n+i}) \equiv 2mn + 2i + 6, 3 \leq i \leq n - 1, f^*(v_{mn}) \equiv 2mn + 4.$ The labels the vertices of $C_n^{(2)}, C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}$ respectively are as the same in case (1).

Case (6): $n \equiv 10 \pmod{12}$. First we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ begin with the edges of the path $P_m^{(1)}$ as the same as in case (1).

Second we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ begin with the edges of the inner most circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m - 1) + 2, f(v_2v_3) = 2n(m - 1) + 6, f(v_3v_4) = 2n(m - 1) + 4, f(v_4v_5) = 2n(m - 1) + 8, f(v_5v_6) = 2n(m - 1) + 10, f(v_6v_7) = 2n(m - 1) + 14, f(v_7v_8) = 2n(m - 1) + 12, f(v_8v_9) = 2n(m - 1) + 16, f(v_9v_{10}) = 2n(m - 1) + 18, f(v_{10}v_{11}) = 2n(m - 1) + 22, f(v_{11}v_{12}) = 2n(m - 1) + 20, f(v_{12}v_{13}) = 2n(m - 1) + 24, f(v_{13}v_{14}) = 2n(m - 1) + 26, \dots, f(v_{n-9}v_{n-8}) = 2mn - 18, f(v_{n-8}v_{n-7}) = 2mn - 14, f(v_{n-7}v_{n-6}) = 2mn - 16, f(v_{n-6}v_{n-5}) = 2mn - 12, f(v_{n-5}v_{n-4}) = 2mn - 10, f(v_{n-4}v_{n-3}) = 2mn - 6, f(v_{n-3}v_{n-2}) = 2mn - 8, f(v_{n-2}v_{n-1}) = 2mn - 4, f(v_{n-1}v_n) = 2mn - 2, f(v_nv_1) = 2mn.$

Label the edges of the circle $C_n^{(2)}$ as follows: $f(v_{n+1}v_{n+2}) = 4n(m - 1) + 4, f(v_{n+2}v_{n+3}) = 4n(m - 1) + 2, f(v_{n+i}v_{n+i+1}) = 4n(m - 1) + 2i, 3 \leq i \leq n - 2, f(v_{2n-1}v_{2n}) = 2n(2m - 1), f(v_{2n}v_{2n+1}) = 2n(2m - 1) - 2.$

Label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}$ as in case (1).

Thus we have the labels of corresponding vertices $\pmod{4mn - 2n}$ will be:

The labels the vertices of the circle $C_n^{(1)}$ are as follows: $f^*(v_1) \equiv 4, f^*(v_2) \equiv 4mn - 4n + 12, f^*(v_3) \equiv 4mn - 4n + 16, f^*(v_4) \equiv 4mn - 4n + 20, f^*(v_5) \equiv 4mn - 4n + 28, f^*(v_6) \equiv 4mn - 4n + 36, f^*(v_7) \equiv 4mn - 4n + 40, f^*(v_8) \equiv 4mn - 4n + 44, f^*(v_9) \equiv 4mn - 4n + 52, f^*(v_{10}) \equiv 4mn - 4n + 60, f^*(v_{11}) \equiv 4mn - 4n + 64, f^*(v_{12}) \equiv 4mn - 4n + 68, f^*(v_{13}) \equiv 4mn - 4n + 76, f^*(v_{14}) \equiv 4mn - 4n + 84, \dots, f^*(v_{n-8}) \equiv 4n - 48, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 40, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 24, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 16, f^*(v_{n-1}) \equiv 4n - 8, f^*(v_n) \equiv 4n - 2.$

The labels the vertices of the circle $C_n^{(2)}$ are as follows: $f^*(v_{n+1}) \equiv 6, f^*(v_{n+2}) \equiv 10, f^*(v_{n+3}) \equiv 12, f^*(v_{n+i}) \equiv 4i + 2, 4 \leq i \leq n - 2, f^*(v_{2n-1}) \equiv 4n, f^*(v_{2n}) \equiv 4n + 2.$ Label the vertices of $C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$ respectively are as the same as in case (1).

Illustration: The edge even graceful labeling of the cylinder grid graphs $C_{9,2}, C_{9,4}, C_{7,10}, C_{7,12}, C_{7,14}, C_{7,16}, C_{7,18}$ and $C_{7,20}$ are shown in Figure 12.

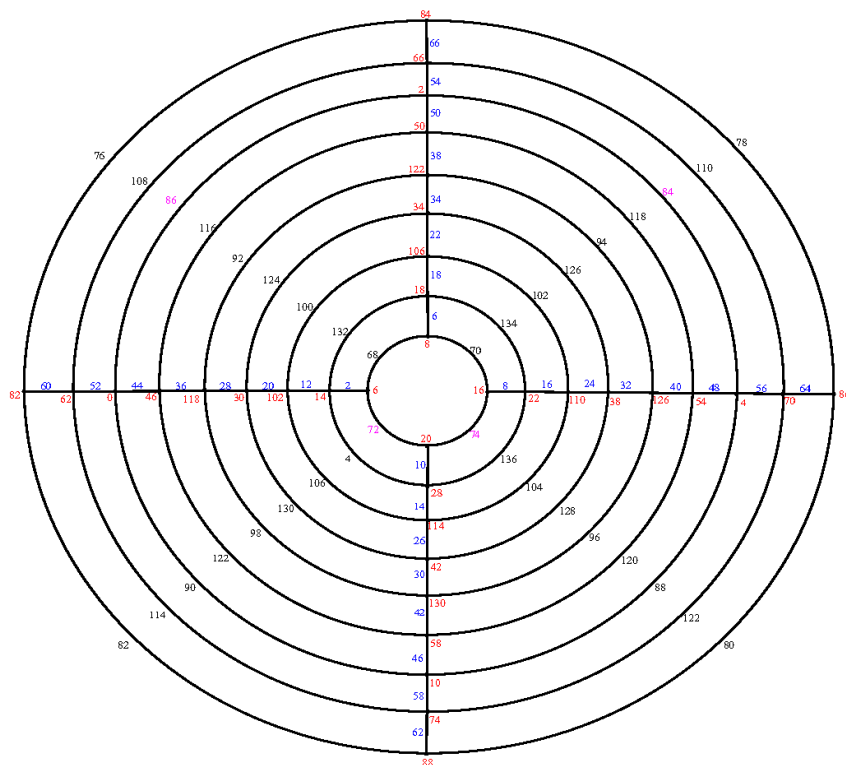
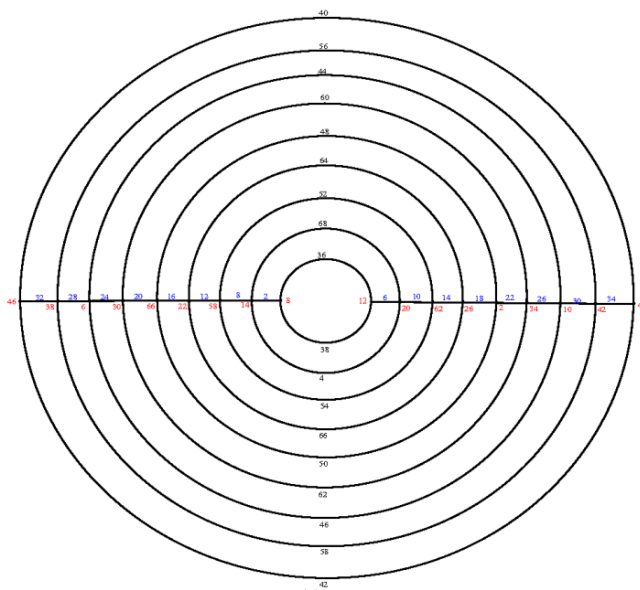
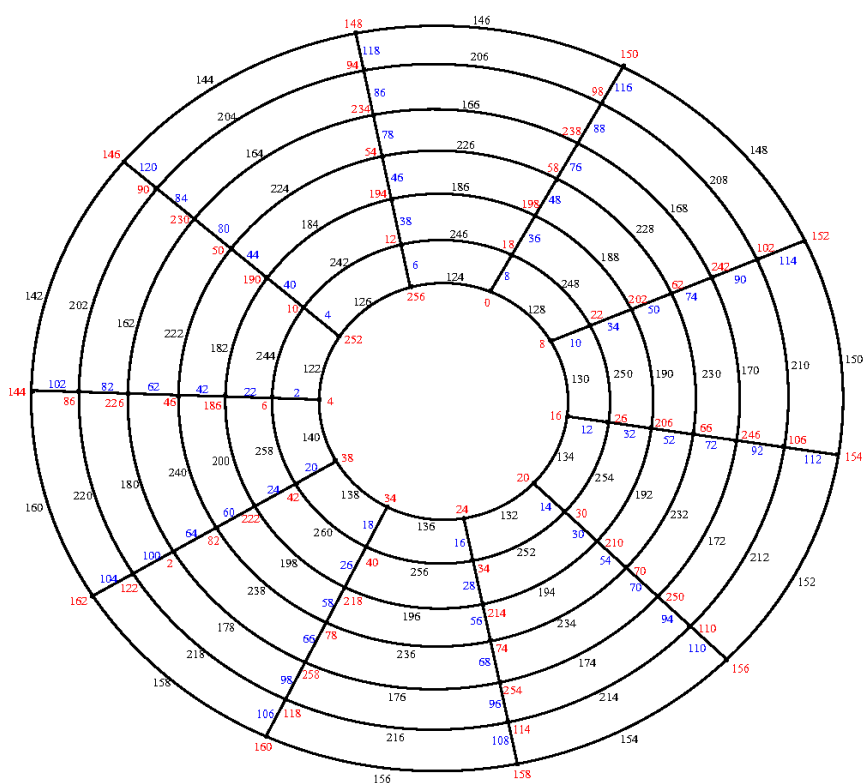
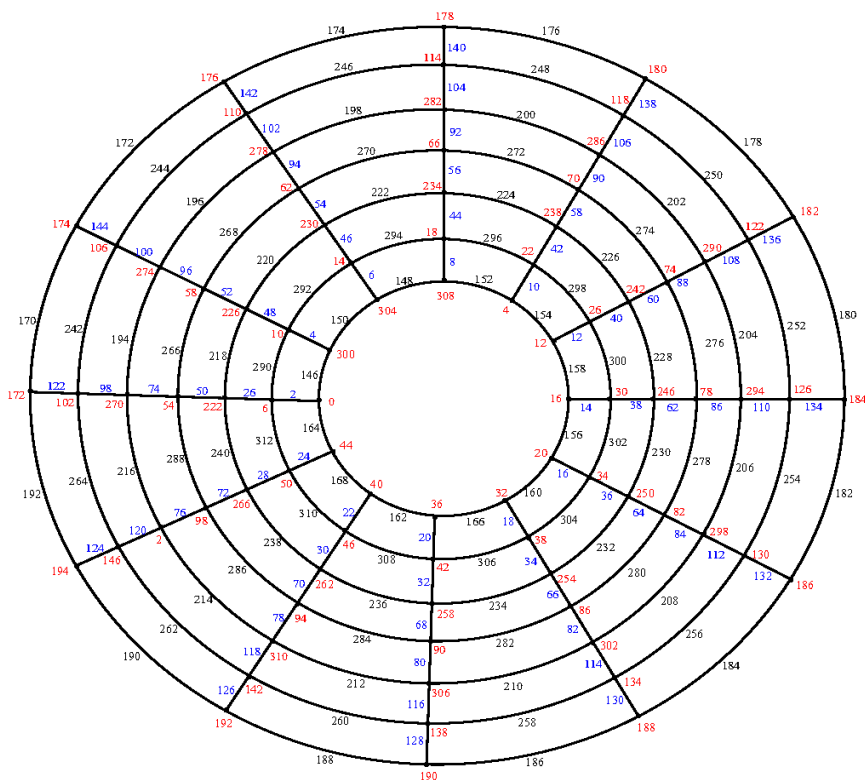


Figure 12. Cont.

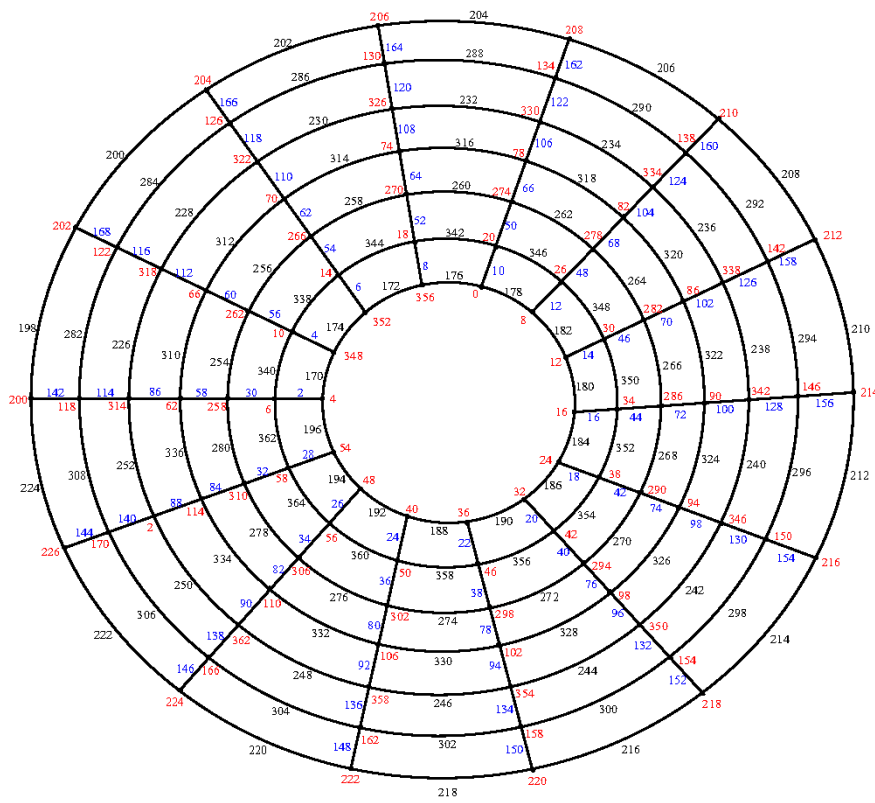


(c) $C_{7,10}$

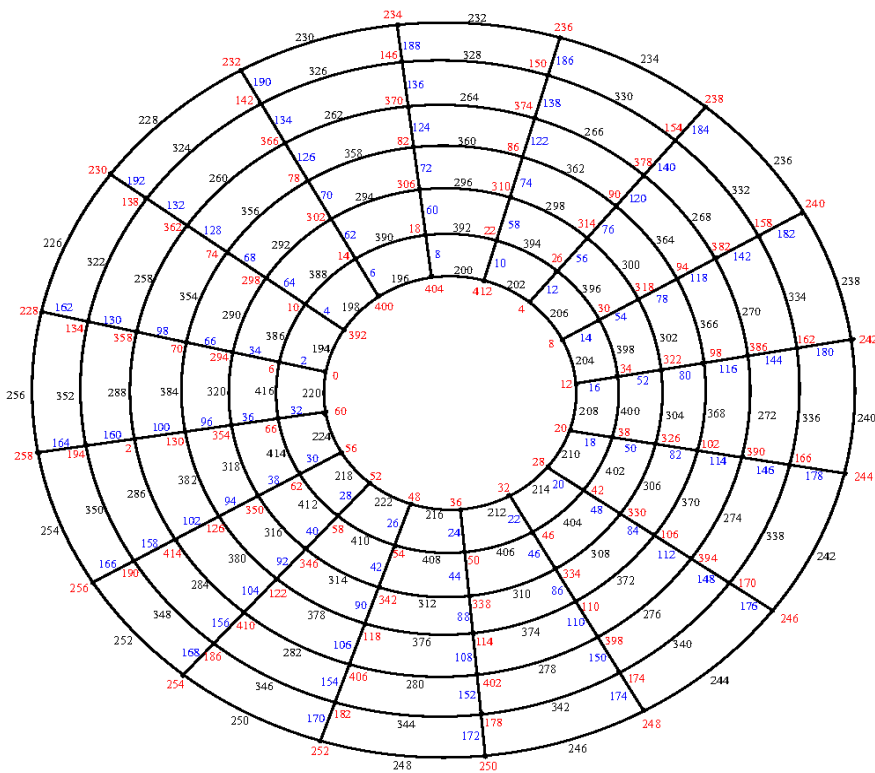


(d) $C_{7,12}$

Figure 12. Cont.

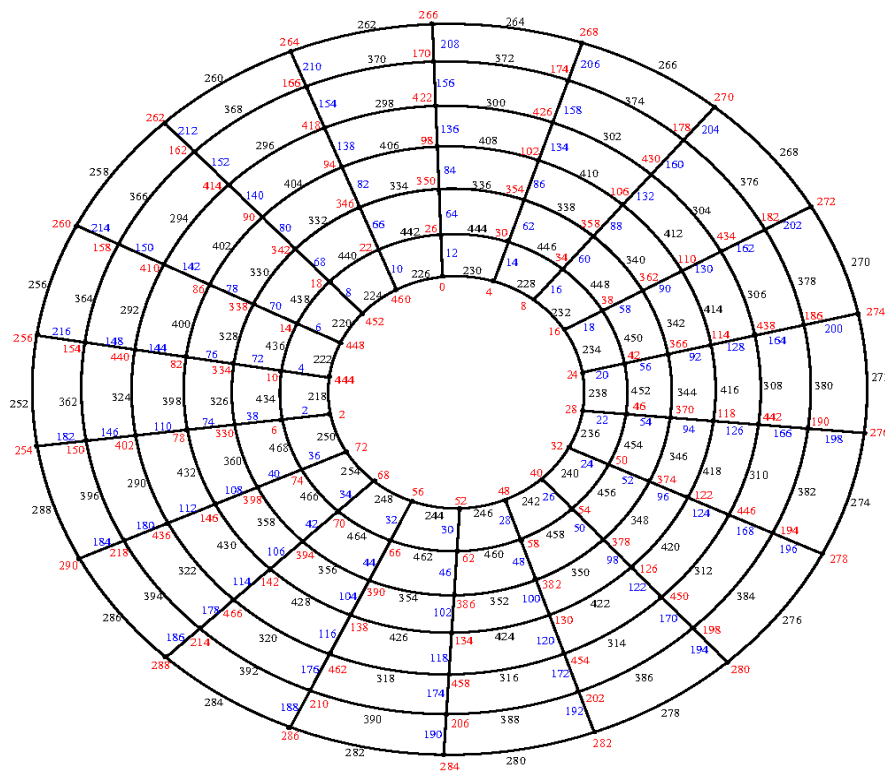


(e) $C_{7,14}$

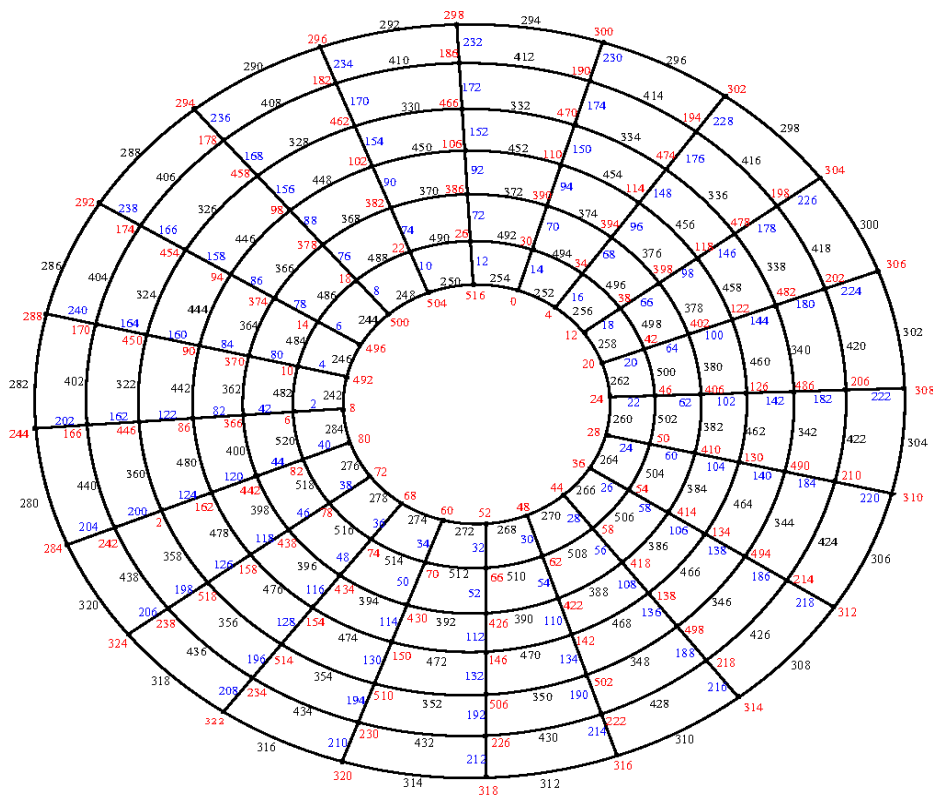


(f) $C_{7,16}$

Figure 12. Cont.



(g) $C_{7,18}$



(h) $C_{7,20}$

Figure 12. An e.g., l. of the cylinder grid graphs $C_{9,2}$, $C_{9,4}$, $C_{7,10}$, $C_{7,12}$, $C_{7,14}$, $C_{7,16}$, $C_{7,18}$, and $C_{7,20}$.

Theorem 4. *If m is an odd positive integer greater than 3 and n is an odd positive integer, $n \geq 3$, then the cylinder grid graph $C_{m,n}$, is an edge even graceful graph.*

Proof. Using standard notation $p = |V(C_{m,n})| = mn, q = |E(C_{m,n})| = 2mn - n, r = \max(p, q) = 2mn - n$, and $f : E(C_{m,n}) \rightarrow \{2, 4, 6, \dots, 4mn - 2n - 2\}$. \square

Let the cylinder grid graph $C_{m,n}$ be as in Figure 9. There are six cases:

Case (1): $n \equiv 1 \pmod{12}$.

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ as follows: Move clockwise to label the edges $v_1v_{n+1}, v_2v_{n+2}, v_3v_{n+3}, \dots, v_{n-1}v_{2n-1}, v_nv_{2n}$ by $2, 4, 6, \dots, 2n - 2, 2n$, then move anticlockwise to label the edges $v_{n+1}v_{n+2}, v_{2n}v_{3n}, v_{2n-1}v_{3n-1}, \dots, v_{n+3}v_{2n+3}, v_{n+2}v_{2n+2}$ by $2n + 2, 2n + 4, 2n + 6, \dots, 4n - 2, 4n$, then move clockwise to label the edges $v_{2n+1}v_{3n+1}, v_{2n+2}v_{3n+2}, v_{2n+3}v_{3n+3}, \dots, v_{3n-1}v_{4n-1}, v_{3n}v_{4n}$ by $4n + 2, 4n + 4, 4n + 6, \dots, 6n - 2, 6n$, and so on.

Finally, move anticlockwise to label the edges $v_{(m-2)n+1}v_{(m-1)n+1}, v_{(m-1)n}v_{mn}, v_{(m-1)n-1}v_{mn-1}, \dots, v_{(m-2)n+3}v_{(m-1)n+3}, v_{(m-2)n+2}v_{m(n-1)+2}$ by $2n(m - 2) + 2, 2n(m - 2) + 4, 2n(m - 2) + 6, \dots, 2n(m - 1) - 2, 2n(m - 1)$.

Second, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, and then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m - 1) + 2, f(v_2v_3) = 2n(m - 1) + 4, f(v_3v_4) = 2n(m - 1) + 8, f(v_4v_5) = 2n(m - 1) + 6, f(v_5v_6) = 2n(m - 1) + 10, f(v_6v_7) = 2n(m - 1) + 12, f(v_7v_8) = 2n(m - 1) + 16, f(v_8v_9) = 2n(m - 1) + 14, f(v_9v_{10}) = 2n(m - 1) + 18, f(v_{10}v_{11}) = 2n(m - 1) + 20, f(v_{11}v_{12}) = 2n(m - 1) + 24, f(v_{12}v_{13}) = 2n(m - 1) + 22, f(v_{13}v_{14}) = 2n(m - 1) + 26, f(v_{14}v_{15}) = 2n(m - 1) + 28, \dots, f(v_{n-7}v_{n-6}) = 2mn - 14, f(v_{n-6}v_{n-5}) = 2mn - 10, f(v_{n-5}v_{n-4}) = 2mn - 12, f(v_{n-4}v_{n-3}) = 2mn - 8, f(v_{n-3}v_{n-2}) = 2mn - 6, f(v_{n-2}v_{n-1}) = 2mn - 2, f(v_{n-1}v_n) = 2mn - 4, f(v_nv_1) = 2mn$.

Then, label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}, C_n^{(2)}$ as follows:

Label the edges of the circle $C_n^{(m)}$ as follows: $f(v_{(m-1)n+i}v_{(m-1)n+i+1}) = 2mn + 2i, f(v_{mn}v_{(m-1)n+1}) = 2n(m + 1), 1 \leq i \leq n - 1$.

Label the edges of the circle $C_n^{(m-2)}$ as follows: $f(v_{(m-3)n+i}v_{(m-3)n+i+1}) = 2n(m + 1) + 2i, f(v_{(m-2)n}v_{(m-3)n+1}) = 2n(m + 2), 1 \leq i \leq n - 1$.

Label the edges of the circle $C_n^{(m-4)}$ as follows: $f(v_{(m-5)n+i}v_{(m-5)n+i+1}) = 2n(m + 2) + 2i, f(v_{(m-4)n}v_{(m-5)n+1}) = 2n(m + 3), 1 \leq i \leq n - 1$, and so on.

Label the edges of the circle $C_n^{(3)}$ as follows: $f(v_{2n+i}v_{2n+i+1}) = 3n(m - 1) + 2i, f(v_{3n}v_{2n+1}) = n(3m - 1), 1 \leq i \leq n - 1$,

Label the edges of the circle $C_n^{(m-1)}$ as follows: $f(v_{(m-2)n+i}v_{(m-2)n+i+1}) = n(3m - 1) + 2i, f(v_{(m-1)n}v_{(m-1)n+1}) = n(3m + 1), 1 \leq i \leq n - 1$,

Label the edges of the circle $C_n^{(m-3)}$ as follows: $f(v_{(m-4)n+i}v_{(m-4)n+i+1}) = n(3m + 2) + 2i, f(v_{(m-3)n}v_{(m-4)n+1}) = 3n(m + 1), 1 \leq i \leq n - 1, \dots$, and so on.

Label the edges of the circle $C_n^{(4)}$ as follows: $f(v_{3n+i}v_{3n+i+1}) = 2n(2m - 3) + 2i, f(v_{4n}v_{3n+1}) = 4n(m - 1), 1 \leq i \leq n - 1$,

Label the edges of $C_n^{(2)}$ as follows: $f(v_{n+i}v_{n+i+1}) = 4n(m - 1) + 2i, f(v_{2n}v_{2n+1}) = 2n(m - 1), 1 \leq i \leq n - 1$,

Thus, the labels of corresponding vertices of the circle $C_n^{(1)} \pmod{4mn - 2n}$ will be: $f^*(v_1) \equiv 4, f^*(v_2) \equiv 4mn - 4n + 10, f^*(v_3) \equiv 4mn - 4n + 18, f^*(v_4) \equiv 4mn - 4n + 22, f^*(v_5) \equiv 4mn - 4n + 26, f^*(v_6) \equiv 4mn - 4n + 34, f^*(v_7) \equiv 4mn - 4n + 42, f^*(v_8) \equiv 4mn - 4n + 46, f^*(v_9) \equiv 4mn - 4n + 50, f^*(v_{10}) \equiv 4mn -$

$4n + 58, f^*(v_{11}) \equiv 4mn - 4n + 66, f^*(v_{12}) \equiv 4mn - 4n + 70, f^*(v_{13}) \equiv 4mn - 4n + 74, f^*(v_{14}) \equiv 4mn - 4n + 82, \dots, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 36, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 28, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 12, f^*(v_{n-1}) \equiv 4n - 8, f^*(v_n) \equiv 4n - 4.$

The labels of the vertices of $C_n^{(2)}, C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$, respectively, are as follows: $f^*(v_{n+i}) \equiv 4i + 2; f^*(v_{2n+i}) \equiv 2mn + 4n + 4i + 2; f^*(v_{3n+i}) \equiv 4n + 4i + 2; \dots; f^*(v_{(m-3)n+i}) \equiv 4mn - 6n + 4i + 2; f^*(v_{(m-2)n+i}) \equiv 2mn - 6n + 4i + 2; f^*(v_{(m-1)n+i}) \equiv 2mn + 2i + 2, 1 \leq i \leq n.$

Case (2): $n \equiv 3 \pmod{12}.$

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ as the same in case (1).

Second, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, and then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}.$

Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}.$

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m - 1) + 2, f(v_2v_3) = 2n(m - 1) + 4, f(v_3v_4) = 2n(m - 1) + 8, f(v_4v_5) = 2n(m - 1) + 6, f(v_5v_6) = 2n(m - 1) + 10, f(v_6v_7) = 2n(m - 1) + 12, f(v_7v_8) = 2n(m - 1) + 16, f(v_8v_9) = 2n(m - 1) + 14, f(v_9v_{10}) = 2n(m - 1) + 18, f(v_{10}v_{11}) = 2n(m - 1) + 20, f(v_{11}v_{12}) = 2n(m - 1) + 24, f(v_{12}v_{13}) = 2n(m - 1) + 22, f(v_{13}v_{14}) = 2n(m - 1) + 26, \dots, f(v_{n-9}v_{n-8}) = 2mn - 18, f(v_{n-8}v_{n-7}) = 2mn - 14, f(v_{n-7}v_{n-6}) = 2mn - 16, f(v_{n-6}v_{n-5}) = 2mn - 12, f(v_{n-5}v_{n-4}) = 2mn - 10, f(v_{n-4}v_{n-3}) = 2mn - 6, f(v_{n-3}v_{n-2}) = 2mn - 8, f(v_{n-2}v_{n-1}) = 2mn, f(v_{n-1}v_n) = 2mn - 2, f(v_nv_1) = 2mn - 4.$

Label the edges of the circle $C_n^{(2)}$ as follows: $f(v_{n+i}v_{n+i+1}) = 4n(m - 1) + 2i, 1 \leq i \leq n - 9, f(v_{2n-9}v_{2n-8}) = 2n(2m - 1) - 18, f(v_{2n-8}v_{2n-7}) = 2n(2m - 1) - 14, f(v_{2n-7}v_{2n-6}) = 2n(2m - 1) - 16, f(v_{2n-6}v_{2n-5}) = 2n(2m - 1) - 10, f(v_{2n-5}v_{2n-4}) = 2n(2m - 1) - 12, f(v_{2n-4}v_{2n-3}) = 2n(2m - 1) - 6, f(v_{2n-3}v_{2n-2}) = 2n(2m - 1) - 8, f(v_{2n-2}v_{2n-1}) = 2n(2m - 1) - 4, f(v_{2n-1}v_{2n}) = 2n(2m - 1) - 2, f(v_{2n}v_{n+1}) = 2n(2m - 1).$

Label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}$ as in case (1).

Thus, the labels of corresponding vertices $\text{mod}(4mn - 2n)$ will be:

The labels of the vertices of $C_n^{(1)}$ are as follows: $f^*(v_1) \equiv 0, f^*(v_2) \equiv 4mn - 4n + 10, f^*(v_3) \equiv 4mn - 4n + 18, f^*(v_4) \equiv 4mn - 4n + 22, f^*(v_5) \equiv 4mn - 4n + 26, f^*(v_6) \equiv 4mn - 4n + 34, f^*(v_7) \equiv 4mn - 4n + 42, f^*(v_8) \equiv 4mn - 4n + 46, f^*(v_9) \equiv 4mn - 4n + 50, f^*(v_{10}) \equiv 4mn - 4n + 58, f^*(v_{11}) \equiv 4mn - 4n + 66, f^*(v_{12}) \equiv 4mn - 4n + 70, f^*(v_{13}) \equiv 4mn - 4n + 74, f^*(v_{14}) \equiv 4mn - 4n + 82, \dots, f^*(v_{n-8}) \equiv 4n - 48, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 40, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 24, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 12, f^*(v_{n-1}) \equiv 4n - 4, f^*(v_n) \equiv 4n - 6.$

The labels of the vertices of the circle $C_n^{(2)}$ are as follows: $f^*(v_{n+i}) \equiv 4i + 2, 1 \leq i \leq n - 9, f^*(v_{2n-8}) \equiv 4n - 28, f^*(v_{2n-7}) \equiv 4n - 26, f^*(v_{2n-6}) \equiv 4n - 22, f^*(v_{2n-5}) \equiv 4n - 18, f^*(v_{2n-4}) \equiv 4n - 14, f^*(v_{2n-3}) \equiv 4n - 10, f^*(v_{2n-2}) \equiv 4n - 8, f^*(v_{2n-1}) \equiv 4n - 2, f^*(v_{2n}) \equiv 4n + 2.$

The labels of the vertices of $C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$, respectively, are the same as in case (1).

Remark 4. In case $n = 3$ and m is odd, $m \geq 3.$

Let the label of edges of the cylinder grid graph $C_{m,3}$ be as in Figure 13.

Thus, the labels of corresponding vertices $\text{mod}(12m - 6)$ are as follows:

The labels of the vertices of the circle $C_3^{(1)}$ are $f^*(v_1) \equiv 8, f^*(v_2) \equiv 12, f^*(v_3) \equiv 16.$

The labels of the vertices of the circle $C_3^{(3)}$ are $f^*(v_{3m-2}) \equiv 6m + 10, f^*(v_{3m-1}) \equiv 6m + 12, f^*(v_{3m}) \equiv 6m + 14.$

The labels of the vertices of the circles $C_3^{(2)}, C_3^{(4)}, \dots, C_3^{(m-1)}$ are $f^*(v_{3k+i}) \equiv 4i + 6k + 4, 1 \leq i \leq 3, 1 \leq k \leq m - 2, k$ is odd.

The labels of the vertices of the circles $C_3^{(3)}, C_3^{(5)}, \dots, C_3^{(m-2)}$ are $f^*(v_{3k+i}) \equiv 6m + 4i + 6k + 10, 1 \leq i \leq 3, 2 \leq k \leq m - 3, k$ is even.

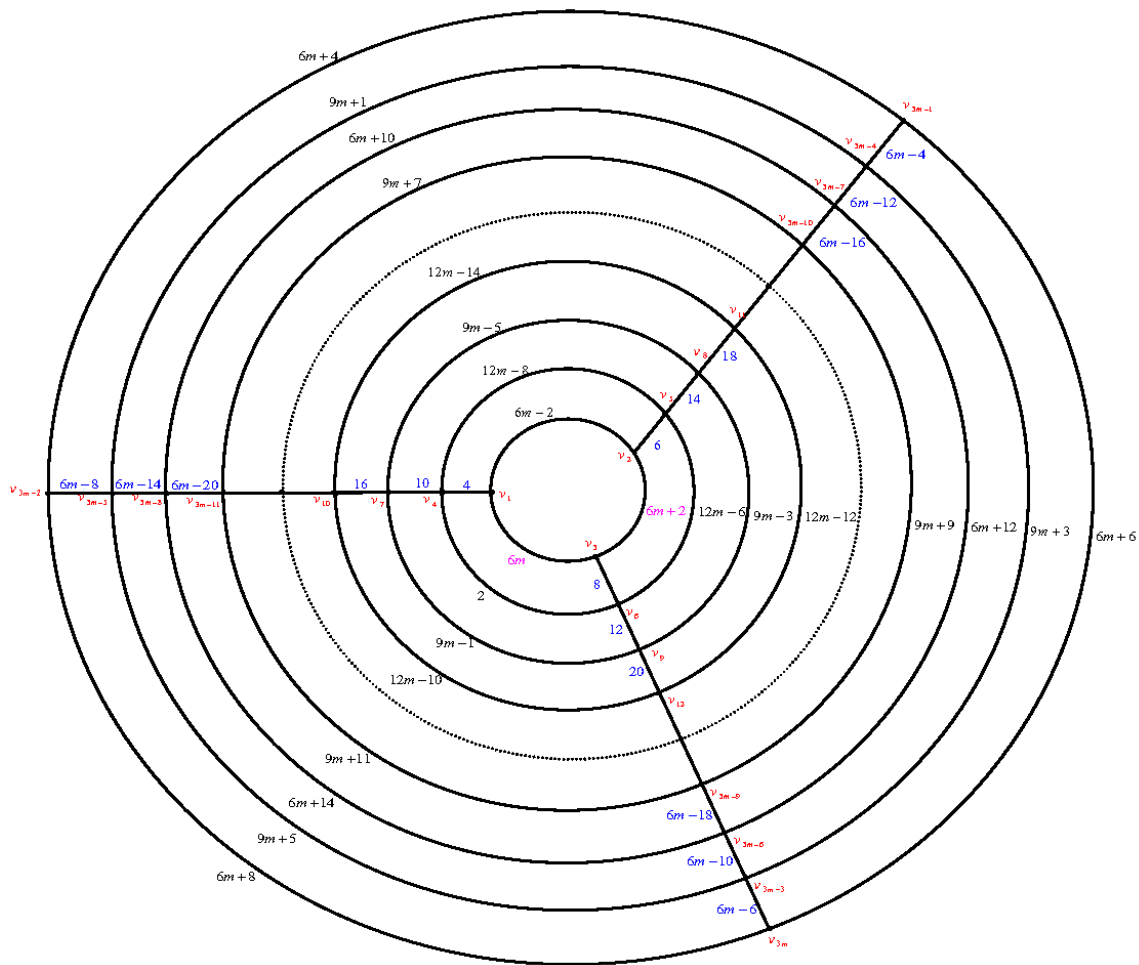


Figure 13. The cylinder grid graph $C_{m,3}$, m is odd, $m \geq 3$.

Case (3): $n \equiv 5 \pmod{12}$.

First, we label the edges of the paths $P_m^{(k)}$, $1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ the same as in case (1).

Second, we label the edges of the circles $C_n^{(k)}$, $1 \leq k \leq m$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, and then the edges of the circles $C_n^{(m-2)}$, $C_n^{(m-4)}$, \dots , $C_n^{(3)}$.

Finally, we label the edges of the circles $C_m^{(m-1)}$, $C_m^{(m-3)}$, \dots , $C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m-1) + 2$, $f(v_2v_3) = 2n(m-1) + 4$, $f(v_3v_4) = 2n(m-1) + 8$, $f(v_4v_5) = 2n(m-1) + 6$, $f(v_5v_6) = 2n(m-1) + 10$, $f(v_6v_7) = 2n(m-1) + 12$, $f(v_7v_8) = 2n(m-1) + 16$, $f(v_8v_9) = 2n(m-1) + 14$, $f(v_9v_{10}) = 2n(m-1) + 18$, $f(v_{10}v_{11}) = 2n(m-1) + 20$, $f(v_{11}v_{12}) = 2n(m-1) + 24$, $f(v_{12}v_{13}) = 2n(m-1) + 22$, $f(v_{13}v_{14}) = 2n(m-1) + 26$, \dots , $f(v_{n-8}v_{n-7}) = 2mn - 16$, $f(v_{n-7}v_{n-6}) = 2mn - 14$, $f(v_{n-6}v_{n-5}) = 2mn - 10$, $f(v_{n-5}v_{n-4}) = 2mn - 12$, $f(v_{n-4}v_{n-3}) = 2mn - 8$, $f(v_{n-3}v_{n-2}) = 2mn - 6$, $f(v_{n-2}v_{n-1}) = 2mn - 2$, $f(v_{n-1}v_n) = 2mn - 4$, $f(v_nv_1) = 2mn$. Label the edges of $C_n^{(m)}$, $C_n^{(m-2)}$, $C_n^{(m-4)}$, \dots , $C_n^{(3)}$ and $C_m^{(m-1)}$, $C_m^{(m-3)}$, $C_m^{(m-5)}$, \dots , $C_m^{(4)}$, $C_m^{(2)}$ as in case (1).

Thus, the labels of corresponding vertices of the circle $C_n^{(1)} \pmod{4mn - 2n}$ will be: $f^*(v_1) \equiv 4$, $f^*(v_2) \equiv 4mn - 4n + 10$, $f^*(v_3) \equiv 4mn - 4n + 18$, $f^*(v_4) \equiv 4mn - 4n + 22$, $f^*(v_5) \equiv 4mn - 4n + 26$, $f^*(v_6) \equiv 4mn - 4n + 34$, $f^*(v_7) \equiv 4mn - 4n + 42$, $f^*(v_8) \equiv 4mn - 4n + 46$, $f^*(v_9) \equiv 4mn - 4n + 50$, $f^*(v_{10}) \equiv 4mn - 4n + 58$, $f^*(v_{11}) \equiv 4mn - 4n + 66$, $f^*(v_{12}) \equiv 4mn - 4n + 70$, $f^*(v_{13}) \equiv 4mn - 4n + 74$, $f^*(v_{14}) \equiv 4mn - 4n + 82$, \dots , $f^*(v_{n-7}) \equiv 4n - 44$, $f^*(v_{n-6}) \equiv 4n - 36$, $f^*(v_{n-5}) \equiv 4n - 32$, $f^*(v_{n-4}) \equiv 4n - 28$, $f^*(v_{n-3}) \equiv 4n - 20$, $f^*(v_{n-2}) \equiv 4n - 12$, $f^*(v_{n-1}) \equiv 4n - 8$, $f^*(v_n) \equiv 4n - 4$.

The labels of the vertices of $C_n^{(2)}, C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$, respectively, are the same as in case (1).

Case (4): $n \equiv 7 \pmod{12}$.

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ the same as in case (1).

Second, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, and then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m-1) + 2, f(v_2v_3) = 2n(m-1) + 4, f(v_3v_4) = 2n(m-1) + 8, f(v_4v_5) = 2n(m-1) + 6, f(v_5v_6) = 2n(m-1) + 10, f(v_6v_7) = 2n(m-1) + 12, f(v_7v_8) = 2n(m-1) + 16, f(v_8v_9) = 2n(m-1) + 14, f(v_9v_{10}) = 2n(m-1) + 18, f(v_{10}v_{11}) = 2n(m-1) + 20, f(v_{11}v_{12}) = 2n(m-1) + 24, f(v_{12}v_{13}) = 2n(m-1) + 22, f(v_{13}v_{14}) = 2n(m-1) + 26, \dots, f(v_{n-10}v_{n-9}) = 2mn - 20, f(v_{n-9}v_{n-8}) = 2mn - 18, f(v_{n-8}v_{n-7}) = 2mn - 14, f(v_{n-7}v_{n-6}) = 2mn - 16, f(v_{n-6}v_{n-5}) = 2mn - 12, f(v_{n-5}v_{n-4}) = 2mn - 10, f(v_{n-4}v_{n-3}) = 2mn - 6, f(v_{n-3}v_{n-2}) = 2mn - 8, f(v_{n-2}v_{n-1}) = 2mn, f(v_{n-1}v_n) = 2mn - 2, f(v_nv_1) = 2mn - 4.$

Label the edges of the circle $C_n^{(2)}$ as follows: $f(v_{n+i}v_{n+i+1}) = 4n(m-1) + 2i, 1 \leq i \leq n-9, f(v_{2n-9}v_{2n-8}) = 2n(2m-1) - 18, f(v_{2n-8}v_{2n-7}) = 2n(2m-1) - 14, f(v_{2n-7}v_{2n-6}) = 2n(2m-1) - 16, f(v_{2n-6}v_{2n-5}) = 2n(2m-1) - 10, f(v_{2n-5}v_{2n-4}) = 2n(2m-1) - 12, f(v_{2n-4}v_{2n-3}) = 2n(2m-1) - 6, f(v_{2n-3}v_{2n-2}) = 2n(2m-1) - 8, f(v_{2n-2}v_{2n-1}) = 2n(2m-1) - 4, f(v_{2n-1}v_{2n}) = 2n(2m-1) - 2, f(v_{2n}v_{n+1}) = 2n(2m-1).$

Label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}$ as in case (1).

Thus, the labels of corresponding vertices $\pmod{4mn - 2n}$ will be:

The labels of the vertices of the circle $C_n^{(1)}$ are as follows: $f^*(v_1) \equiv 0, f^*(v_2) \equiv 4mn - 4n + 10, f^*(v_3) \equiv 4mn - 4n + 18, f^*(v_4) \equiv 4mn - 4n + 22, f^*(v_5) \equiv 4mn - 4n + 26, f^*(v_6) \equiv 4mn - 4n + 34, f^*(v_7) \equiv 4mn - 4n + 42, f^*(v_8) \equiv 4mn - 4n + 46, f^*(v_9) \equiv 4mn - 4n + 50, f^*(v_{10}) \equiv 4mn - 4n + 58, f^*(v_{11}) \equiv 4mn - 4n + 66, f^*(v_{12}) \equiv 4mn - 4n + 70, f^*(v_{13}) \equiv 4mn - 4n + 74, f^*(v_{14}) \equiv 4mn - 4n + 82, \dots, f^*(v_{n-9}) \equiv 4n - 56, f^*(v_{n-8}) \equiv 4n - 48, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 40, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 24, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 12, f^*(v_{n-1}) \equiv 4n - 4, f^*(v_n) \equiv 4n - 6.$

The labels of the vertices of the circle $C_n^{(2)}$ are as follows: $f^*(v_{n+i}) \equiv 4i + 2, 1 \leq i \leq n-9, f^*(v_{2n-8}) \equiv 4n - 28, f^*(v_{2n-7}) \equiv 4n - 26, f^*(v_{2n-6}) \equiv 4n - 22, f^*(v_{2n-5}) \equiv 4n - 18, f^*(v_{2n-4}) \equiv 4n - 14, f^*(v_{2n-3}) \equiv 4n - 10, f^*(v_{2n-2}) \equiv 4n - 8, f^*(v_{2n-1}) \equiv 4n - 2, f^*(v_{2n}) \equiv 4n + 2.$

The labels of the vertices of $C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$, respectively, are as in case (1).

Case (5): $n \equiv 9 \pmod{12}$.

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ the same as in case (1).

Second, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, and then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m-1) + 2, f(v_2v_3) = 2n(m-1) + 4, f(v_3v_4) = 2n(m-1) + 8, f(v_4v_5) = 2n(m-1) + 6, f(v_5v_6) = 2n(m-1) + 10, f(v_6v_7) = 2n(m-1) + 12, f(v_7v_8) = 2n(m-1) + 16, f(v_8v_9) = 2n(m-1) + 14, f(v_9v_{10}) = 2n(m-1) + 18, f(v_{10}v_{11}) = 2n(m-1) + 20, f(v_{11}v_{12}) = 2n(m-1) + 24, f(v_{12}v_{13}) = 2n(m-1) + 22, f(v_{13}v_{14}) = 2n(m-1) + 26, \dots, f(v_{n-11}v_{n-10}) = 2mn - 22, f(v_{n-10}v_{n-9}) = 2mn - 18, f(v_{n-9}v_{n-8}) = 2mn - 20, f(v_{n-8}v_{n-7}) = 2mn - 16, f(v_{n-7}v_{n-6}) = 2mn - 14, f(v_{n-6}v_{n-5}) = 2mn - 10, f(v_{n-5}v_{n-4}) = 2mn - 12, f(v_{n-4}v_{n-3}) = 2mn - 8, f(v_{n-3}v_{n-2}) = 2mn - 6, f(v_{n-2}v_{n-1}) = 2mn, f(v_{n-1}v_n) = 2mn - 2, f(v_nv_1) = 2mn - 4.$

Label the edges of the circle $C_n^{(2)}$ as follows: $f(v_{n+i}v_{n+i+1}) = 4n(m-1) + 2i, 1 \leq i \leq n-8, f(v_{2n-7}v_{2n-6}) = 2n(2m-1) - 12, f(v_{2n-6}v_{2n-5}) = 2n(2m-1) - 14, f(v_{2n-5}v_{2n-4}) = 2n(2m-1) - 6, f(v_{2n-4}v_{2n-3}) = 2n(2m-1) - 10, f(v_{2n-3}v_{2n-2}) = 2n(2m-1) - 8, f(v_{2n-2}v_{2n-1}) = 2n(2m-1) - 4, f(v_{2n}v_{n+1}) = 2n(2m-1).$

Label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}$ as in case (1). Thus, the labels of corresponding vertices mod $(4mn - 2n)$ will be:

The labels of the vertices of the circle $C_n^{(1)}$ are as follows: $f^*(v_1) \equiv 0, f^*(v_2) \equiv 4mn - 4n + 10, f^*(v_3) \equiv 4mn - 4n + 18, f^*(v_4) \equiv 4mn - 4n + 22, f^*(v_5) \equiv 4mn - 4n + 26, f^*(v_6) \equiv 4mn - 4n + 34, f^*(v_7) \equiv 4mn - 4n + 42, f^*(v_8) \equiv 4mn - 4n + 46, f^*(v_9) \equiv 4mn - 4n + 50, f^*(v_{10}) \equiv 4mn - 4n + 58, f^*(v_{11}) \equiv 4mn - 4n + 66, f^*(v_{12}) \equiv 4mn - 4n + 70, f^*(v_{13}) \equiv 4mn - 4n + 74, f^*(v_{14}) \equiv 4mn - 4n + 82, \dots, f^*(v_{n-10}) \equiv 4n - 60, f^*(v_{n-9}) \equiv 4n - 56, f^*(v_{n-8}) \equiv 4n - 52, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 36, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 28, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 10, f^*(v_{n-1}) \equiv 4n - 4, f^*(v_n) \equiv 4n - 6.$

The labels of the vertices of the circle $C_n^{(2)}$ are as follows: $f^*(v_{n+i}) \equiv 4i + 2, 1 \leq i \leq n - 8, f^*(v_{2n-7}) \equiv 4n - 24, f^*(v_{2n-6}) \equiv 4n - 22, f^*(v_{2n-5}) \equiv 4n - 16, f^*(v_{2n-4}) \equiv 4n - 12, f^*(v_{2n-3}) \equiv 4n - 14, f^*(v_{2n-2}) \equiv 4n - 8, f^*(v_{2n-1}) \equiv 4n - 2, f^*(v_{2n}) \equiv 4n + 2.$

The labels of the vertices of $C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$, respectively, are as in case (1).

Case (6): $n \equiv 11 \pmod{12}$.

First, we label the edges of the paths $P_m^{(k)}, 1 \leq k \leq n$ beginning with the edges of the path $P_m^{(1)}$ the same as in case (1).

Second, we label the edges of the circles $C_n^{(k)}, 1 \leq k \leq m$ beginning with the edges of the innermost circle $C_n^{(1)}$, then the edges of outer circle $C_n^{(m)}$, and then the edges of the circles $C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$.

Finally, we label the edges of the circles $C_m^{(m-1)}, C_m^{(m-3)}, \dots, C_m^{(2)}$.

Label the edges of the circle $C_n^{(1)}$ as follows: $f(v_1v_2) = 2n(m - 1) + 4, f(v_2v_3) = 2n(m - 1) + 2, f(v_3v_4) = 2n(m - 1) + 6, f(v_4v_5) = 2n(m - 1) + 8, f(v_5v_6) = 2n(m - 1) + 12, f(v_6v_7) = 2n(m - 1) + 10, f(v_7v_8) = 2n(m - 1) + 14, f(v_8v_9) = 2n(m - 1) + 16, f(v_9v_{10}) = 2n(m - 1) + 20, f(v_{10}v_{11}) = 2n(m - 1) + 18, f(v_{11}v_{12}) = 2n(m - 1) + 22, f(v_{12}v_{13}) = 2n(m - 1) + 24, f(v_{13}v_{14}) = 2n(m - 1) + 28, \dots, f(v_{n-8}v_{n-7}) = 2mn - 16, f(v_{n-7}v_{n-6}) = 2mn - 14, f(v_{n-6}v_{n-5}) = 2mn - 10, f(v_{n-5}v_{n-4}) = 2mn - 12, f(v_{n-4}v_{n-3}) = 2mn - 8, f(v_{n-3}v_{n-2}) = 2mn - 6, f(v_{n-2}v_{n-1}) = 2mn - 2, f(v_{n-1}v_n) = 2mn - 4, f(v_nv_1) = 2mn.$

Label the edges of the circle $C_n^{(2)}$ as follows: $f(v_{n+i}v_{n+i+1}) = 4n(m - 1) + 2i, 1 \leq i \leq n - 2, f(v_{2n-1}v_{2n}) = 4mn, f(v_{2n}v_{n+1}) = 4mn - 2.$

Label the edges of $C_n^{(m)}, C_n^{(m-2)}, C_n^{(m-4)}, \dots, C_n^{(3)}$ and $C_n^{(m-1)}, C_n^{(m-3)}, C_n^{(m-5)}, \dots, C_n^{(4)}$ as in case (1).

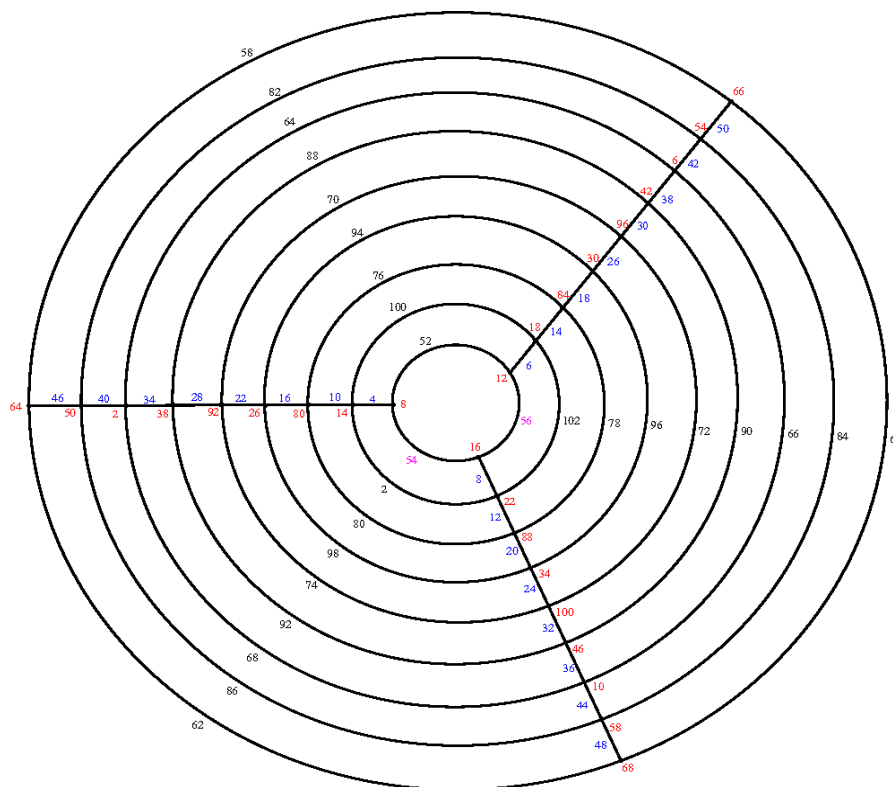
Thus, the labels of corresponding vertices mod $(4mn - 2n)$ will be:

The labels of the vertices of the circle $C_n^{(1)}$ are as follows: $f^*(v_1) \equiv 6, f^*(v_2) \equiv 4mn - 4n + 10, f^*(v_3) \equiv 4mn - 4n + 14, f^*(v_4) \equiv 4mn - 4n + 22, f^*(v_5) \equiv 4mn - 4n + 30, f^*(v_6) \equiv 4mn - 4n + 34, f^*(v_7) \equiv 4mn - 4n + 38, f^*(v_8) \equiv 4mn - 4n + 46, f^*(v_9) \equiv 4mn - 4n + 54, f^*(v_{10}) \equiv 4mn - 4n + 58, f^*(v_{11}) \equiv 4mn - 4n + 62, f^*(v_{12}) \equiv 4mn - 4n + 70, \dots, f^*(v_{n-7}) \equiv 4n - 44, f^*(v_{n-6}) \equiv 4n - 36, f^*(v_{n-5}) \equiv 4n - 32, f^*(v_{n-4}) \equiv 4n - 28, f^*(v_{n-3}) \equiv 4n - 20, f^*(v_{n-2}) \equiv 4n - 12, f^*(v_{n-1}) \equiv 4n - 8, f^*(v_n) \equiv 4n - 4.$

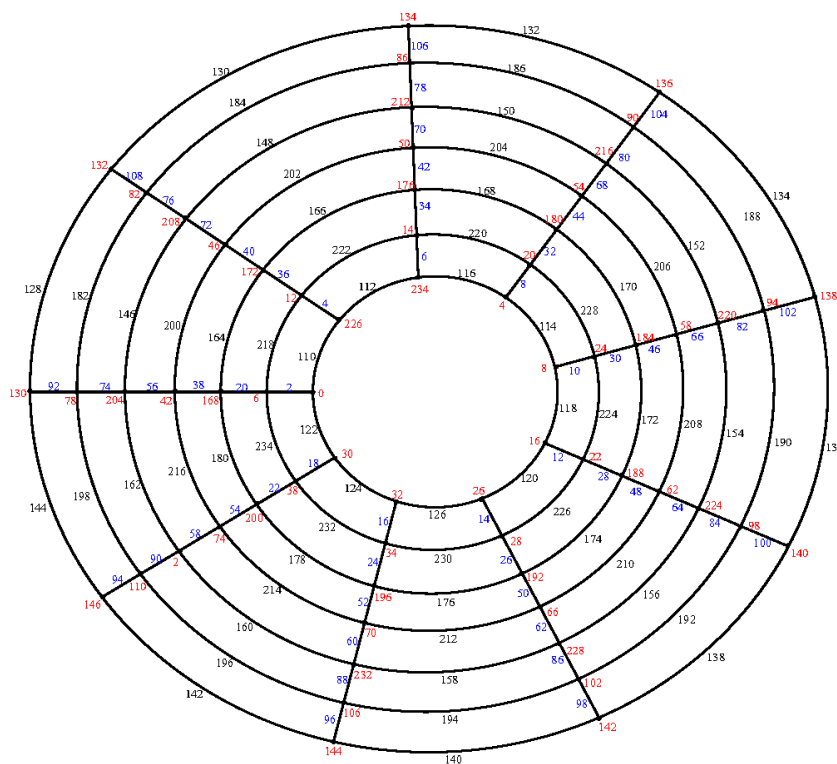
The labels of the vertices of the circle $C_n^{(2)}$ are as follows: $f^*(v_{n+1}) \equiv 4, f^*(v_{n+i}) \equiv 4i + 2, 2 \leq i \leq n - 2, f^*(v_{2n-1}) \equiv 4n, f^*(v_{2n}) \equiv 4n + 2.$

The labels of the vertices of $C_n^{(3)}, C_n^{(4)}, \dots, C_n^{(m-2)}, C_n^{(m-1)}, C_n^{(m)}$, respectively, are as the same as in case (1).

Illustration: An e.e.g.l. of the cylinder grid graphs $C_{9,3}, C_{7,9}, C_{7,11}, C_{7,13}, C_{7,15}, C_{7,17}$ and $C_{7,19}$ is shown in Figure 14.

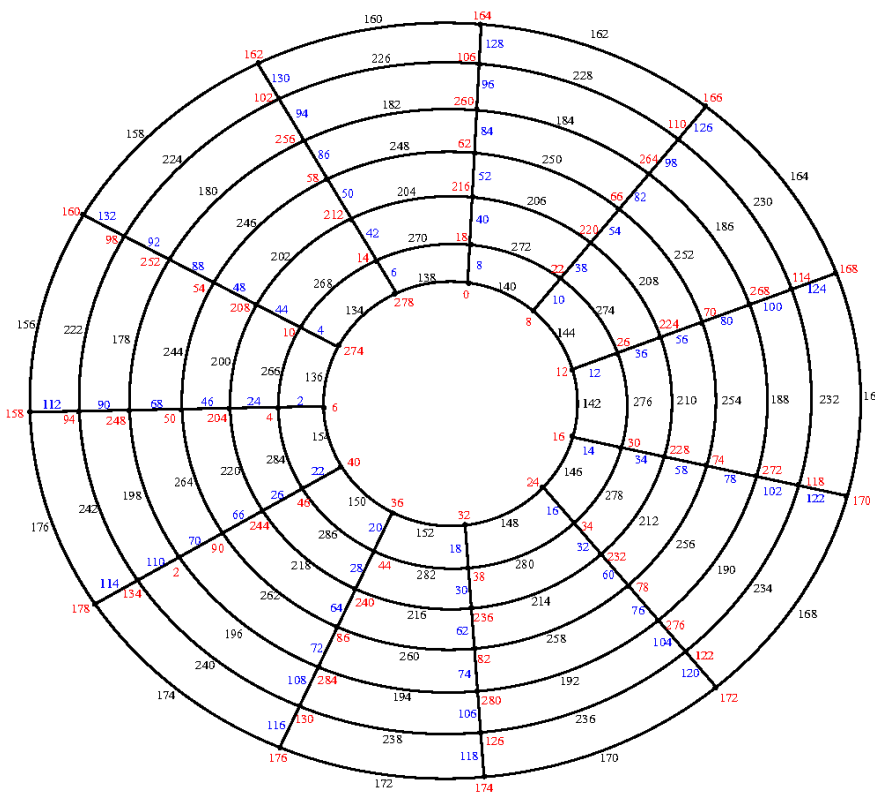


(a) $C_{9,3}$

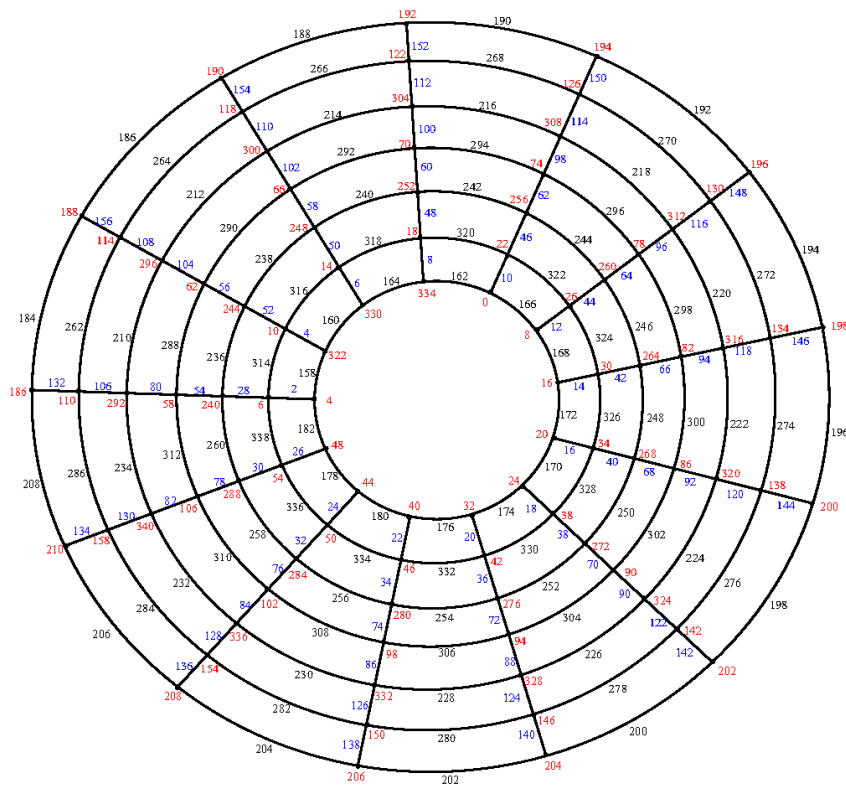


(b) $C_{7,9}$

Figure 14. Cont.

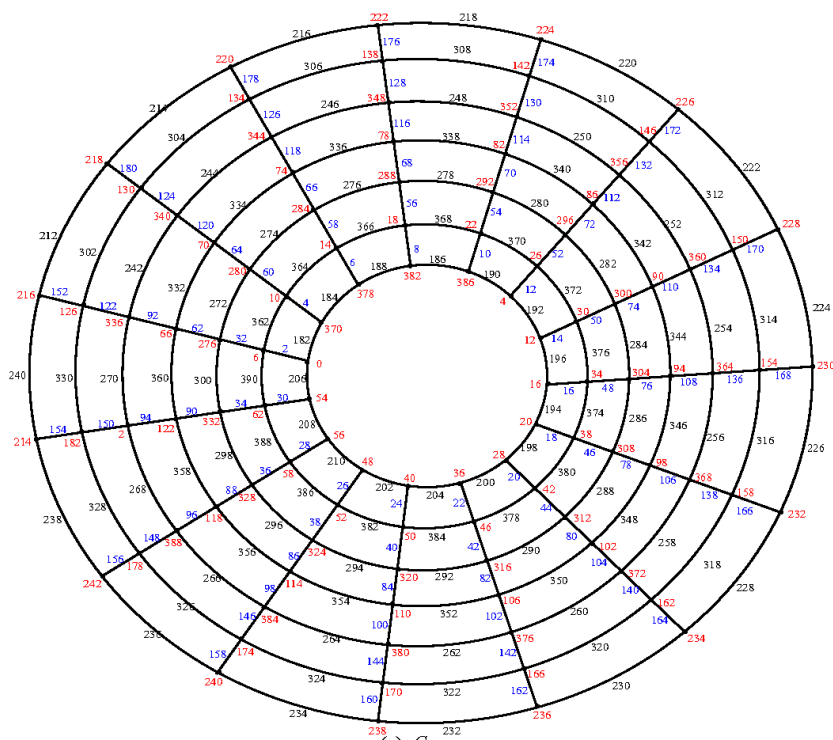


(c) $C_{7,11}$

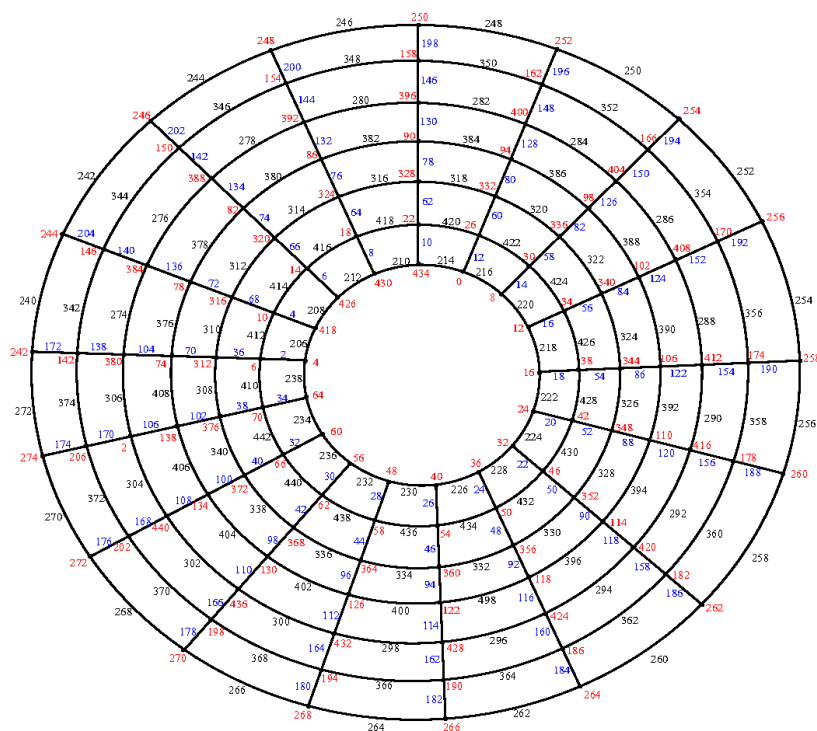


(d) $C_{7,13}$

Figure 14. Cont.

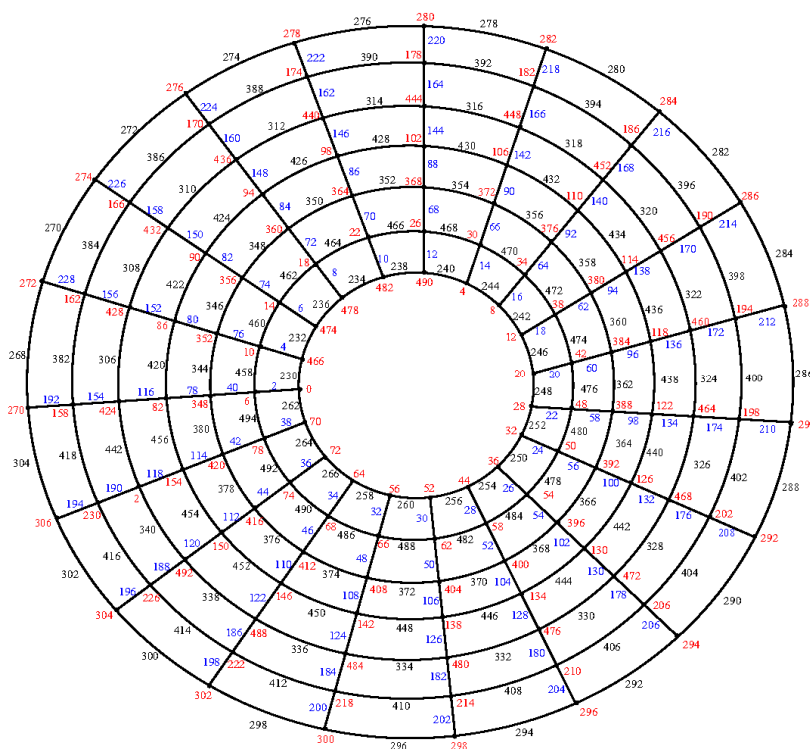


(e) $C_{7,15}$



(f) $C_{7,17}$

Figure 14. Cont.



(g) $C_{7,19}$

Figure 14. An e.e.g.l. of the cylinder grid graphs $C_{9,3}, C_{7,9}, C_{7,11}, C_{7,13}, C_{7,15}, C_{7,17}$ and $C_{7,19}$.

3. Conclusions

In this paper, using the connection of labeling of graphs with modular arithmetic and theory of numbers in general, we give a detailed study for e.e.g., l. of all cases of members of the cylinder grid graphs. The study of necessary and sufficient conditions for e.e.g., l. of other important families including torus $C_m \times C_n$ and rectangular $P_m \times P_n$ grid graphs should be taken into consideration in future studies of e.e.g., l.

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