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Derivative Free Fourth Order Solvers of Equations with Applications in Applied Disciplines

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Abstract: This paper develops efficient equation solvers for real- and complex-valued functions. An earlier study by Lee and Kim, used the Taylor-type expansions and hypotheses on higher than first order derivatives, but no derivatives appeared in the suggested method. However, we have many cases where the calculations of the fourth derivative are expensive, or the result is unbounded, or even does not exist. We only use the first order derivative of function Ω in the proposed convergence analysis. Hence, we expand the utilization of the earlier scheme, and we study the computable radii of convergence and error bounds based on the Lipschitz constants. Furthermore, the range of starting points is also explored to know how close the initial guess should be considered for assuring convergence. Several numerical examples where earlier studies cannot be applied illustrate the new technique.

Keywords: divided difference; radius of convergence; Kung–Traub method; local convergence; Lipschitz constant; Banach space

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1. Introduction

We look for a unique root p_* of the equation:

$$\Omega(v) = 0, \tag{1}$$

where Ω is a continuous operator defined on a convex subset $\mathbb{P}$ of $\mathbb{S}$ with values in $\mathbb{S}$, and $\mathbb{S} = \mathbb{R}$ or $\mathbb{S} = \mathbb{C}$. This is a relevant issue since several problems from mathematics, physics, chemistry, and engineering can be reduced to Equation (1).

In general, either the lack, or the intractability of analytic solutions force researchers to adopt iterative techniques. However, when using that type of approach, we find problems such as slow convergence, converge to undesired root, divergence, computational inefficiency, or failure (see Traub [1] and Petkovíc et al. [2]). The study of the convergence of iterative algorithms can be classified into two categories, namely the semi-local and local convergence analysis. The first case is based on the information in the neighborhood of the starting point. This also gives criteria for guaranteeing the convergence of iteration algorithms. Therefore, a relevant issue is the convergence domain, as well as the radii of convergence of the algorithm.

Herein, we deal with the second case, that is the local convergence analysis. Let us consider a fourth order algorithm defined for $n = 0,\ 1,\ 2,\ \dots,$ as:

$$\begin{aligned} \lambda_s &= \delta_s + \beta\Omega(\delta_s)^k, \text{ with } \beta \neq 0 \in \mathbb{R}, \\ \mu_s &= \lambda_s - \frac{\Omega(\lambda_s)}{[\delta_s,\ \lambda_s;\Omega]}, \\ \delta_{s+1} &= \mu_s - H(v_s,\ w_s)\frac{\Omega(\mu_s)}{[\delta_s,\ \lambda_s;\Omega]}, \end{aligned} \tag{2}$$

where $\lambda_0 \in \mathbb{P}$ is an initial point, $k \in \mathbb{N}$ (k is an arbitrary natural number), $[\delta_s,\ \lambda_s;\Omega] : \mathbb{P} \times \mathbb{P} \to L(\mathbb{S},\ \mathbb{S})$ satisfies $[\delta_s,\ \lambda_s;\Omega] = \frac{\Omega(x)-\Omega(y)}{x-y}$ for $x \neq y$, $v_s = \frac{\Omega(\mu_s)}{\Omega(\lambda_s)}$, $w_s = \frac{\Omega(\mu_s)}{\Omega(\delta_s)}$, and $H : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ is a continuous function. The fourth order convergence for Method (2) was studied by Lee and Kim [3] with Taylor series, hypotheses up to the fourth order derivative of function Ω, and hypotheses on the first and second partial derivatives of function H. However, only the divided difference of the first order appears in (2). Favorable computations were also given with related Kung–Traub methods [1] of the form:

$$\begin{aligned} \lambda_s &= \delta_s + \beta\Omega(\delta_s)^4, \text{ with } \beta \neq 0 \in \mathbb{R}, \\ \mu_s &= \lambda_s - \frac{\Omega(\lambda_s)}{[\delta_s,\ \lambda_s;\Omega]}, \\ \delta_{s+1} &= \mu_s - \frac{\Omega(\delta_s)}{\Omega(\delta_s) - 2\Omega(\mu_s)}\frac{\Omega(\mu_s)}{[\lambda_s,\ \mu_s;\Omega]}. \end{aligned} \tag{3}$$

Notice that (3) is obtained from (2), if we define function H as $H(v,\ w) = \frac{1}{1-2w}$. The assumptions on the derivatives of Ω and H restrict the suitability of Algorithms (2) and (3). For instance, let us consider Ω on $\mathbb{P} = \mathbb{S} = \mathbb{R}$, $\mathbb{P}_1 = [-\frac{1}{\pi},\ \frac{2}{\pi}]$ as:

$$\Omega(v) = \begin{cases} v^3\log(\pi^2 v^2) + v^5 \sin\left(\frac{1}{v}\right), & v \neq 0 \\ 0, & v = 0 \end{cases}.$$

From this expression, we obtain:

$$\Omega'(v) = 2v^2 - v^3\cos\left(\frac{1}{v}\right) + 3v^2\log(\pi^2v^2) + 5v^4\sin\left(\frac{1}{v}\right),$$

$$\Omega''(v) = -8v^2\cos\left(\frac{1}{v}\right) + 2v(5 + 3\log(\pi^2v^2)) + v(20v^2-1)\sin\left(\frac{1}{v}\right),$$

$$\Omega'''(v) = \frac{1}{v}\left[(1-36v^2)\cos\left(\frac{1}{v}\right) + v\left(22 + 6\log(\pi^2v^2) + (60v^2-9)\sin\left(\frac{1}{v}\right)\right)\right].$$

We find that $\Omega'''(v)$ is unbounded on $\mathbb{P}_1$ at the point $v = 0$. Therefore, the results in [3] cannot be applied for the analysis of the convergence of Methods (2) or (3). Notice that there are numerous algorithms and convergence results available in the literature [1–15]. Nonetheless, practice shows that the initial prediction must be in the neighborhood of the root for achieving convergence. However, how close must it be to the starting point? Indeed, local results do not give any information about the ball convergence radii.

We broaden the suitability of Methods (2) and (3) by using only assumptions on the first derivative of function Ω. Moreover, we estimate the computable radii of convergence and the error bounds from Lipschitz constants. Additionally, we discuss the range of initial estimate p_* that tells us how close it must be to achieve a granted convergence of (2). This problem was not addressed in [3], but is of capital importance in practical applications.

In what follows: Section 2 addresses the study of local convergence (2) and (3). Section 3 contains three numerical examples that illustrate the theoretical formulation. Finally, Section 4 gives the concluding remarks.

2. Convergence Analysis

Let $b > 0,\ \alpha > 0,\ \gamma > 0, \beta \in \mathbb{S},\ k \in \mathbb{N}$ and $M \geq 1$ be given constants. Furthermore, we consider that $H : \mathbb{S} \times \mathbb{S} \to \mathbb{S}, h : [0, \infty) \to [0, \infty)$ are continuous functions such that:

$$|H(v,\ \eta)| \leq |H(|v|,\ |\eta|)| \leq h(v), \tag{4}$$

for each $v, \eta \in \mathbb{S}$ with $|\eta| \leq v$, and that $|H|$ and h are nondecreasing functions on the interval $\left[0,\ \frac{1}{\gamma}\right)^2$, $\left[0,\ \frac{1}{\gamma}\right)^2$, respectively. For the local convergence analysis of (2), we need to introduce a few functions and parameters. Let us define the parameters R_0 and R_1 given by:

$$R_0 = \frac{1}{(1+\alpha)\gamma},\ R_1 = \frac{1}{(1+\alpha)\gamma + \gamma\alpha(b|\beta|M + \alpha)}, \tag{5}$$

and function g_1 on the interval $[0, R_1)$ by:

$$g_1(v) = \frac{\gamma\alpha(b|\beta|M + \alpha)v}{1 - (1+\alpha)\gamma v}. \tag{6}$$

From the above functions, it is easy to see that $R_1 < R_0 < \frac{1}{\gamma}$, $g_1(R_1) = 1$ and $0 \leq g_1(v) < 1$, for $v \in [0,\ R_1)$. Moreover, we consider the functions q and $\bar{q}$ on $[0,\ R_1)$ as:

$$q(v) = \gamma(\alpha + g_1(v))v \quad \text{and} \quad \bar{q}(v) = q(v) - 1.$$

It is straightforward to find that $\bar{q}(0) = -1 < 0$ and that $\bar{q}(v) \to +\infty$ as $v \to r_1^-$. By the intermediate value theorem, we know that $\bar{q}$ has zeros in the interval $(0,\ R_1)$. Let us assume that R_q is the smallest zero of function $\bar{q}$ on $(0,\ R_1)$, and set:

$$\bar{r} = \min\{R_1,\ R_q\}. \tag{7}$$

Furthermore, let us define functions g_2 and $\bar{g}_2$ on $[0,\ \bar{r})$ such that:

$$g_2(v) = \left(1 + \frac{Mh(v)}{1 - q(v)}\right) g_1(v) \tag{8}$$

and:

$$\bar{g}_2(v) = g_2(v) - 1. \tag{9}$$

Suppose that:

$$\bar{g}_2(v) \to \ \text{a positive number or} + \infty,\ \text{as}\ v \to \bar{r}^-. \tag{10}$$

From (8), we have that $\bar{g}_2(0) < 0$ and from (10) that $\bar{g}_2(v) > 0$ as $v \to \bar{r}^{-1}$. Further, we assume that R is the smallest zero of function $\bar{g}_2$ on $(0,\ \bar{r})$. Therefore, we have that for each $v \in [0,\ r)$:

$$0 \leq g_1(v) < 1, \tag{11}$$

$$0 \leq g_2(v) < 1, \tag{12}$$

$$0 \leq q(v) < 1. \tag{13}$$

Let us denote by $U(\mu,\ r)$ and $\bar{U}(\mu,\ r)$ the open and closed balls in $\mathbb{S}$ with center $\mu \in S$ and of radius $r > 0$, respectively.

Theorem 1. *Let us assume that $\Omega : \mathbb{P} \subset \mathbb{S} \to \mathbb{S}$ is a differentiable function and $[\cdot, \cdot\,; \Omega] : \mathbb{P} \times \mathbb{P} \to L(\mathbb{S},\ \mathbb{S})$ is a divided difference of first order of Ω. Furthermore, we consider that h and H are functions satisfying (4), (9), $p_* \in \mathbb{P},\ b > 0,\ \alpha > 0,\ \gamma > 0,\ M \geq 1, k \in \mathbb{N}, \beta \in S$ and that for each $x,\ y \in \mathbb{P}$, we have:*

$$\Omega(p_*) = 0, \quad \Omega'(p_*) \neq 0, |\Omega'(p_*)| \leq b, \tag{14}$$

$$|\Omega'(p_*)^{-1}([x,\ y,\ \Omega] - \Omega'(p_*)| \leq \gamma(|x - p_*| + |y - p_*|), \tag{15}$$

$$h(v) = H\left(\frac{M\gamma(|\beta|Mb + \alpha)v}{(1 - \gamma\alpha v)(1 - \gamma(1 + \alpha)v)}, \frac{Mg_1(v)}{1 - \gamma v}\right) \tag{16}$$

$$|I + \beta[x,\ p_*; \Omega]^k (x - p_*)^{k-1}| \leq \alpha, \tag{17}$$

$$|\Omega'(p_*)^{-1}[x,\ p_*,\ \Omega]| \leq M, \tag{18}$$

$$\bar{U}(p_*, \alpha r) \subseteq \mathbb{P}. \tag{19}$$

Then, the sequence $\{\delta_s\}$ obtained for $\lambda_0 \in U(p_*,\ R) - \{x^*\}$ by (2) is well defined, remains in $U(p_*,\ R)$ for each $n = 0, 1, 2, \ldots$, and converges to p_* so that:

$$|\lambda_s - p_*| \leq \alpha|\delta_s - p_*| < R, \tag{20}$$

$$|\mu_s - p_*| \leq g_1(|\delta_s - p_*|)|\delta_s - p_*| \leq |\delta_s - p_*| < R, \tag{21}$$

$$|\delta_{s+1} - p_*| \leq g_2(|\delta_s - p_*|)|\delta_s - p_*| < |\delta_s - p_*|, \tag{22}$$

and $G \in [R, \frac{1}{\gamma})$. Moreover, the limit point p_* is the unique root of equation $\Omega(x) = 0$ in $\bar{U}(p_*,\ G) \cap \mathbb{P}$.

Proof. By hypotheses $\lambda_0 \in U(p_*, r) - \{x^*\}$, (14), (17) and (19), we further obtain:

$$\begin{aligned}\delta_0 - p_* &= \lambda_0 - p_* + \beta\left(\Omega(\lambda_0) - \Omega(p_*)\right)^k \\ &= \left(I + \beta[\lambda_0,\ p_*; \Omega]^k (\lambda_0 - p_*)^{k-1}\right)(\lambda_0 - p_*),\end{aligned}$$

so that:

$$\begin{aligned}|\delta_0 - p_*| &= \left|I + \beta[\lambda_0,\ p_*;\ \Omega]^k (\lambda_0 - p_*)^{k-1}\right| |\lambda_0 - p_*| \\ &\leq \alpha r,\end{aligned} \tag{23}$$

which leads to (20) for $s = 0$ and $\delta_0 \in U(p_*,\ \alpha r)$. We need to show that $[\lambda_0,\ \delta_0;\ \Omega] \neq 0$. Using (15) and the definition of R, we obtain:

$$\begin{aligned}\left|\Omega'(p_*)^{-1}([\lambda_0,\ \delta_0;\ \Omega] - \Omega'(p_*)\right| &\leq \gamma\left(|\lambda_0 - p_*| + |\delta_0 - p_*|\right) \\ &\leq \gamma\left(|\lambda_0 - p_*| + \alpha|\lambda_0 - p_*|\right) \\ &\leq \gamma(1 + \alpha)|\lambda_0 - p_*| < \gamma(1 + \alpha)R < 1.\end{aligned} \tag{24}$$

From the Banach lemma on invertible functions [7,14], it follows that $[\lambda_0,\ \delta_0;\ \Omega] \neq 0$ and:

$$\left|[\lambda_0,\ \delta_0;\ \Omega]^{-1}\Omega'(p_*)\right| \leq \frac{1}{1 - \gamma(1 + \alpha)|\lambda_0 - p_*|}. \tag{25}$$

In view of (14) and (18), we have:

$$\begin{aligned} \left|\Omega'(p_*)^{-1}\Omega(\lambda_0)\right| &= \left|\Omega'(p_*)^{-1}\left(\Omega(\lambda_0)-\Omega(p_*)\right)\right| \\ &= \left|\Omega'(p_*)^{-1}[\lambda_0,\ p_*,\ \Omega](\lambda_0-p_*)\right| \\ &\leq M|\lambda_0-p_*| \end{aligned} \tag{26}$$

and similarly:

$$\left|\Omega'(p_*)^{-1}\Omega(\delta_0)\right| \leq M|\delta_0-p_*|, \tag{27}$$

since $\delta_0 \in \mathbb{P}$. Then, using the second substep of Methods (2), (11), (14), (16), (25) and (27), we obtain:

$$\begin{aligned} |\mu_0-p_*| &= \left|\delta_0-p_*-[\lambda_0,\ \delta_0,\ \Omega]^{-1}\Omega(\delta_0)\right| \\ &\leq \left|[\lambda_0,\ \delta_0,\ \Omega]^{-1}\Omega'(p_*)\right|\left|\Omega'(p_*)^{-1}\left([\lambda_0,\ \delta_0,\ \Omega](\delta_0-p_*)-(\Omega(\delta_0)-\Omega(p_*))\right)\right| \\ &\leq \left|[\lambda_0,\ \delta_0,\ \Omega]^{-1}\Omega'(p_*)\right|\left|\Omega'(p_*)^{-1}\left([\lambda_0,\ \delta_0,\ \Omega]-[\delta_0,\ p_*,\ \Omega]\right)(\delta_0-p_*)\right| \\ &\leq \frac{\gamma\left(|\lambda_0-\delta_0|+|\delta_0-p_*|\right)|\delta_0-p_*|}{1-\gamma(1+\alpha)|\lambda_0-p_*|} \\ &\leq \frac{\gamma\left(|\beta|bM|\lambda_0-p_*|+\alpha|\lambda_0-p_*|\right)\alpha|\lambda_0-p_*|}{1-\gamma(1+\alpha)|\lambda_0-p_*|} \\ &\leq \frac{\gamma\alpha\left(|\beta|bM+\alpha\right)|\lambda_0-p_*|^2}{1-\gamma(1+\alpha)|\lambda_0-p_*|} \\ &= g_1(|\lambda_0-p_*|)|\lambda_0-p_*| < |\lambda_0-p_*| < R, \end{aligned} \tag{28}$$

and so, (21) is true for $s=0$ and $\mu_0 \in U(p_*,\ R)$. Next, we need to show that $\Omega(\lambda_0) \neq 0$ and $\Omega(\delta_0) \neq 0$, for $\delta_0 \neq p_*$. Using (14) and (15), and the definition of R, we obtain:

$$\begin{aligned} &\left|\left((\lambda_0-p_*)\Omega'(p_*)\right)^{-1}\left[\Omega(\lambda_0)-\Omega(p_*)-\Omega'(p_*)(\lambda_0-p_*)\right]\right| \\ &\leq |\lambda_0-p_*|^{-1}\left|\Omega'(p_*)^{-1}\left([\lambda_0,\ p_*;\ \Omega]-\Omega'(p_*)(\lambda_0-p_*)\right)\right| \\ &\leq \gamma|\lambda_0-p_*|^{-1}|\lambda_0-p_*|^2 = \gamma|\lambda_0-p_*| < \gamma R < 1. \end{aligned} \tag{29}$$

Hence, $\Omega(\lambda_0) \neq 0$ and:

$$|\Omega'(\lambda_0)^{-1}\Omega'(p_*)| \leq \frac{1}{|\lambda_0-p_*|(1-\gamma|\lambda_0-p_*|)}. \tag{30}$$

Similarly, we have that:

$$|\Omega'(\delta_0)^{-1}\Omega'(p_*)| \leq \frac{1}{|\delta_0-p_*|(1-\gamma|\delta_0-p_*|)} \leq \frac{1}{|\delta_0-p_*|(1-\alpha\gamma|\lambda_0-p_*|)}. \tag{31}$$

Then, by using (4) and (12) (for $\delta_0=\mu_0$), (16), (27), (28), (30) and (31), we have:

$$\begin{aligned} |H(v_0,\ \eta_0)| &\leq |H(|v_0|,\ |\eta_0|)| \\ &\leq \left|H\left(\frac{M|\mu_0-p_*|}{|\delta_0-p_*|(1-\gamma|\delta_0-p_*|)},\ \frac{M|\mu_0-p_*|}{|\lambda_0-p_*|(1-\gamma|\lambda_0-p_*|)}\right)\right| \\ &\leq \left|H\left(\frac{M\gamma(|\beta|Mb+\alpha)|\lambda_0-p_*||\delta_0-p_*|}{|\delta_0-p_*|(1-\alpha\gamma|\lambda_0-p_*|)(1-\gamma(1+\alpha)|\lambda_0-p_*|)},\ \frac{Mg_1(|\lambda_0-p_*|)|\lambda_0-p_*|}{|\lambda_0-p_*|(1-\gamma|\lambda_0-p_*|)}\right)\right| \\ &\leq \left|H\left(\frac{M\gamma(|\beta|Mb+\alpha)|\lambda_0-p_*|}{(1-\gamma(1+\alpha)|\lambda_0-p_*|)(1-\alpha\gamma|\lambda_0-p_*|)},\ \frac{Mg_1(|\lambda_0-p_*|)}{1-\gamma|\lambda_0-p_*|}\right)\right| \\ &= h(|\lambda_0-p_*|). \end{aligned} \tag{32}$$

Adopting (13), we get:

$$\begin{aligned}\left|\Omega'(p_*)^{-1}\left([\delta_0,\ \mu_0;\ \Omega]-\Omega'(p_*)\right)\right| &\le \gamma\left(|\delta_0-p_*|+|\mu_0-p_*|\right)\\ &\le \gamma\left(\alpha|\lambda_0-p_*|+g_1(|\lambda_0-p_*|)|\lambda_0-p_*|\right)\\ &= q(|\lambda_0-p_*|)<q(R)<1.\end{aligned} \tag{33}$$

Hence, we have:

$$\left|[\delta_0,\ \mu_0;\ \Omega]^{-1}\Omega'(p_*)\right| \le \frac{1}{1-q(|\lambda_0-p_*|)}. \tag{34}$$

Furthermore, λ_1 is well defined by (24), (32) and (34). Using the third substep of (2), (12), (27) (for $\delta_0=\mu_0$), (28), (32) and (34), we get:

$$\begin{aligned}|\lambda_1-p_*| &\le |\mu_0-p_*|+|H(v_0,\ \eta_0)|\left|[\lambda_0,\ \delta_0;\ \Omega]^{-1}\Omega'(p_*)\right|\left|\Omega'(p_*)^{-1}\Omega(\mu_0)\right|\\ &\le \left[1+\frac{Mh(|\lambda_0-p_*|)}{1-q(|\lambda_0-p_*|)}\right]|\mu_0-p_*|\\ &\le \left[1+\frac{Mh(|\lambda_0-p_*|)}{1-q(|\lambda_0-p_*|)}\right]g_1(|\lambda_0-p_*|)|\lambda_0-p_*|\\ &\le g_2(|\lambda_0-p_*|)|\lambda_0-p_*|<|\lambda_0-p_*|<R,\end{aligned} \tag{35}$$

showing that (22) is true for $s=0$ and $\lambda_1\in U(p_*,\ R)$. Replacing λ_0, δ_0, and μ_0 by λ_s, δ_s, and μ_s, respectively, in the preceding estimates, we arrive at (20)–(22). From the estimates $\|\delta_{s+1}-p_*\|<\|\delta_s-p_*\|<r$, we conclude that $\lim_{s\to\infty}\delta_s=p_*$ and $x_{s+1}\in U(p_*,\ R)$. Finally, to illustrate the uniqueness, let $p_{**}\in\bar{U}(p_*,T)$ such that $\Omega(p_{**})=0$. We assume $Q=[p_*,p_{**};\Omega]$. Adopting (15), we get:

$$\begin{aligned}\left|\Omega'(p_*)^{-1}(Q-\Omega'(p_*))\right| &\le \gamma\left(|p_*-p_*|+|p_{**}-p_*|\right)\\ &= \gamma T<1.\end{aligned} \tag{36}$$

Therefore, $Q\neq 0$, and in view of the identity $\Omega(p_*)-\Omega(p_{**})=Q(p_*-p_{**})$, we conclude that $p_*=p_{**}$. □

Remark 1.

(a) *It follows from condition (15) and the estimate:*

$$\begin{aligned}\left|\Omega'(p_*)^{-1}[x,\ p_*;\ \Omega]\right| &= |\Omega'(p_*)^{-1}([x,\ p_*;\ \Omega]-\Omega'(p_*)-\Omega'(p_*))+I|\\ &\le 1+\left|\Omega'(p_*)^{-1}([x,\ p_*;\ \Omega]-\Omega'(p_*))\right|\\ &\le 1+\gamma|\lambda_0-p_*|\end{aligned}$$

and Condition (14) can be discarded and M substituted by:

$$M=M(v)=1+\gamma v$$

or $M=2$, since $v\in[0,\frac{1}{\gamma})$.

(b) *We note that (2) does not change if we adopt the conditions of Theorem 1 instead of the stronger ones given in [3]. In practice, for the error bounds, we can consider the computational order of convergence (COC) [10]:*

$$\xi = \frac{ln\frac{|\delta_{s+2} - p_*|}{|\delta_{s+1} - p_*|}}{ln\frac{|\delta_{s+1} - p_*|}{|\delta_s - p_*|}}, \quad \text{for each } s = 0, 1, 2, \ldots \tag{37}$$

or the approximated computational order of convergence (ACOC) [10]:

$$\xi^* = \frac{ln\frac{|\delta_{s+2} - \delta_{s+1}|}{|\delta_{s+1} - \delta_s|}}{ln\frac{|\delta_{s+1} - \delta_s|}{|\delta_s - \delta_{s-1}|}}, \quad \text{for each } s = 1, 2, \ldots \tag{38}$$

In practice, we obtain the order of convergence that, avoiding the bounds, involves estimates higher than the first Fréchet derivative.

3. Numerical Examples

We consider some of the weight functions to solve a variety of univariate problems that are depicted in Examples 1–3.

Tables 1–3 display the minimum number of iterations necessary to obtain the required accuracy for the zeros of the functions $\Omega(x)$ in Examples 1–3. Moreover, we include also the initial guess, the radius of convergence of the corresponding function, and the theoretical order of convergence. Additionally, we calculate the *COC* approximated by means of (37) and (38).

All computations used the package *Mathematica* 9 with multiple precision arithmetic, adopting $\epsilon = 10^{-50}$ as a tolerance error and the stopping criteria:

(i) $|\delta_{s+1} - \delta_s| < \epsilon$ and (ii) $|\Omega(\delta_s)| < \epsilon$.

Example 1. *Let* $\mathbb{S} = \mathbb{R}$, $\mathbb{P} = [-\pi, \ \pi], x^* = 0$. *Let us define function* Ω *on* $\mathbb{P}$ *by:*

$$\Omega(x) = \cos x - x - 1. \tag{39}$$

Consequently, it results $\alpha = 1 + \frac{|\beta| + M^k |\Omega'(p_*)|^{k-1}}{\gamma^{k-1}}$, $\gamma = \frac{1}{2}$, $b = |\Omega'(p_*)| = 1$ *and* $M = 2$. *We obtain a different radius of convergence when using distinct types of weight functions (for details, please see [3]), COC (ξ) and s presented in Table 1.*

Table 1. Radii of convergence according to the adopted weight function.

Cases	Different Values of the Parameters that Satisfy Theorem 1								
	β	k	$H(v, \eta)$	R_1	R_q	R	λ_0	s	ξ
1.	-1	1	$\frac{1+v}{1-\eta}$	0.10526	0.27008	0.02535	0.024	4	4
2.	3	2	$1+2v$	0.00250	0.03749	0.00082	0.0007	3	4
3.	-3	3	$1+2\eta$	0.00020	0.01013	0.00004	0.0003	3	4
4.	0.1	4	$\frac{1}{1-2\eta}$	0.00962	0.07090	0.00160	0.0005	3	4

Example 2. *Let* $\mathbb{S} = \mathbb{R}, \mathbb{P} = [-1, \ 1], x^* = 0.714806$ *(approximated root), and let us assume function* Ω *on* $\mathbb{P}$ *by*

$$\Omega(x) = e^x - 4x^2. \tag{40}$$

As a consequence, we get $\alpha = 1 + \frac{|\beta|+M^k|\Omega'(p_*)|^{k-1}}{\gamma^{k-1}}$, $\gamma = 2$, $b = |\Omega'(p_*)| = |e^{x^*} - 8p_*| \approx 3.67466$ *and* $M = 2$. *We have the distinct radius of convergence when using several weight functions (for details, please see [3]), COC* (ξ) *and s listed in Table 2.*

Table 2. Radii of convergence according to the adopted weight function.

Cases	Different Values of the Parameters that Satisfy Theorem 1							
	β	k	$H(v,\eta)$	R_1	R_q	R	λ_0	s	ξ
1.	-1	1	$\frac{1+v}{1-\eta}$	0.01427	0.05498	0.00318	0.713	4	4
2.	3	2	$1+2v$	0.00047	0.00896	0.00015	0.7417	4	4
3.	-3	3	$1+2\eta$	0.00006	0.00286	0.00001	0.7418	3	4
4.	0.1	4	$\frac{1}{1-2\eta}$	0.00359	0.02201	0.00060	0.7413	4	4

Example 3. *Using the example of the introduction, we have* $\alpha = 1 + \frac{|\beta|+M^k|\Omega'(p_*)|^{k-1}}{\gamma^{k-1}}$, $\gamma = 2$, $b = |\Omega'(p_*)| = \frac{2\pi+1}{\pi^3} \approx 0.23489$, $M = 2$, *and the required zero is* $p_* = \frac{1}{\pi} \approx 0.318309886$. *We have different radii of convergence by adopting distinct types of weight functions (for details, please see [3]), COC* (ξ) *and s in Table 3.*

Table 3. Radii of convergence according to the adopted weight function.

Cases	Different Values of the Parameters that Satisfy Theorem 1							
	β	k	$H(v,\eta)$	R_1	R_q	R	λ_0	s	ξ
1.	-1	1	$\frac{1+v}{1-\eta}$	0.03470	0.07391	0.00884	0.325	4	4
2.	3	2	$1+2v$	0.03965	0.08356	0.01225	0.329	4	4
3.	-3	3	$1+2\eta$	0.08363	0.13140	0.02437	0.298	5	4
4.	0.1	4	$\frac{1}{1-2\eta}$	0.16367	0.18912	0.05268	0.358	5	4

4. Conclusions

Locating the range or interval of the required root that provides sure convergence of an iterative method is one of the difficult problems in computational analysis. This paper addressed this problem and expanded the applicability of Methods (2) and (3) using hypotheses only on the functions appearing in these techniques. Further, we provided the radii of ball convergence and error bounds using Lipschitz conditions. This type of study was not addressed in the earlier work. With the help of the radius of convergence, we can find the range of initial estimate p_* that tells us how close it must be for granting the convergence of Methods (2) and (3). Finally, the applicability of new approach was illustrated with several numerical examples.

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