symmetry

MDPI

Article

Counterintuitive Test Problems for Transformed Fuzzy Number-Based Similarity Measures between Intuitionistic Fuzzy Sets

Hui-Chin Tang * and Kuan-Sheng Cheng

Department of Industrial Engineering and Management, National Kaohsiung University of Science and Technology, Kaohsiung City 80778, Taiwan; 1104403110@nkust.edu.tw

* Correspondence: tang@nkust.edu.tw

Received: 15 April 2019; Accepted: 25 April 2019; Published: 2 May 2019

check for updates

Abstract: This paper analyzes the counterintuitive behaviors of transformed fuzzy number (FN)-based similarity measures between intuitionistic fuzzy sets (IFSs). Among these transformed FN-based similarity measures, Chen and Chang's similarity measure (2015) is a novel one. An algorithm of computing Chen and Chang's similarity measure is proposed. We analyze the counterintuitive behaviors of Chen and Chang's similarity measure for seven general test problems and four test problems with three inclusive IFSs. The results indicate that there are six counterintuitive test problems for Chen and Chang's similarity measure.

Keywords: intuitionistic fuzzy set; similarity; counterintuitive

1. Introduction

Fuzzy sets (FSs) theory, proposed by Zadeh [1], has successfully been applied in various fields. As a generalization of FSs, intuitionistic fuzzy sets (IFSs) proposed by Atanassov [2] are characterized by a membership function and a non-membership function.

A similarity measure between two IFSs represents alignment of the two sets. The degree of similarity measure is an important tool for cluster analysis [3,4], decision-making [5–9], medical diagnosis [10,11], and pattern recognition [12–17]. In the literature, many papers have been dedicated to problems connected with the similarity measures between two IFSs and research on this area is still carrying on [3–22]. Recently, Jiang et al. [16] reviewed fifteen similarity measures of IFSs and proposed a novel similarity measure between two IFSs. The existing similarity measures between two IFSs can be classified into four categories: one minus distance between two vectors [11,15,19–22], transformed fuzzy numbers (FNs) [9,13], centroid points [12,14], and others [17,18]. Among the transformed FN-based similarity measures, Zhang and Yu's similarity measure [9] has the drawback of the division by zero problem, so this paper focuses on Chen and Chang's similarity measure [13].

In the literature, various approaches for the similarity measures between IFSs are inconsistent with our intuition [12–21]. To analyze the counterintuitive behaviors of Chen and Chang's similarity measure, we present an algorithm to compute Chen and Chang's similarity measure. To illustrate the mechanics of the proposed algorithm, we present some examples. Two kinds of test examples are considered to analyze its counterintuitive behaviors. One is the six general test problems proposed by Tang and Yang [22]. Furthermore, we propose a new general test problem. The other one is the four special test problems with three inclusive IFSs.

The organization of this paper is as follows. Section 2 briefly reviews the FSs and IFSs and presents the transformed FN-based similarity measures. Section 3 presents an algorithm for calculating Chen and Chang's similarity measure. We analyze the counterintuitive behaviors of Chen and Chang's similarity measure for seven general test problems in Section 4 and the four special test problems with three inclusive IFNs in Section 5. Finally, some concluding remarks and future research are presented.

2. IFSs and Similarity Measures

We firstly review the basic notations of FSs and IFSs. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a non-empty universal set of real numbers $\mathcal{R}$.

Definition 1. *An FS A over X is defined as*

$$A = \{(x_i, \mu_A(x_i)) \big| 1 \le i \le n\}$$

where the membership function is $\mu_A(x_i) : X \to [0,1]$ *for* $1 \le i \le n$. *We use FS(X) to denote the set of all FSs over X.*

Definition 2. *An IFS A over X is defined as*

$$A = \{(x_i, \mu_A(x_i), v_A(x_i)) \big| 1 \le i \le n\}$$

where the membership function $\mu_A(x_i) : X \to [0,1]$ *and non-membership function* $v_A(x_i) : X \to [0,1]$ *of* x_i *belonging to the set A satisfy*

$$\mu_A(x_i) + v_A(x_i) \le 1 \text{ for } 1 \le i \le n.$$

The degree of hesitancy associated with each x_i *is defined as*

$$\pi_A(x_i) = 1 - \mu_A(x_i) - v_A(x_i) \text{ for } 1 \le i \le n$$

measuring the lack of information or certitude. The set of all IFSs over X is denoted by IFS(X).

We now briefly review some operations involving IFSs.

Definition 3. *Let* $A = \{(x_i, \mu_A(x_i), v_A(x_i)) \big| 1 \le i \le n\}$ *and* $B = \{(x_i, \mu_B(x_i), v_B(x_i)) \big| 1 \le i \le n\}$ *be two IFSs. Then:*

1. $A \subseteq B$ *if and only if* $\mu_A(x_i) \le \mu_B(x_i)$ *and* $v_A(x_i) \ge v_B(x_i)$ *for* $1 \le i \le n$;
2. $A = B$ *if and only if* $\mu_A(x_i) = \mu_B(x_i)$ *and* $v_A(x_i) = v_B(x_i)$ *for* $1 \le i \le n$;
3. *The complement of A is defined as* $A_c = \{(x_i, v_A(x_i), \mu_A(x_i)) \big| 1 \le i \le n\}$;
4. *We denote the pure intuitionistic fuzzy set by* $PI = \{(x_i, 0, 0) \big| 1 \le i \le n\}$.

For the fuzzy set $A \in FS(X)$, $\pi_A(x_i) = 0$ *for* $1 \le i \le n$, *and for the pure intuitionistic fuzzy set,* $\pi_{PI}(x_i) = 1$ *for* $1 \le i \le n$.

We now recall the definition of similarity measures between two IFSs.

Definition 4. *A similarity measure* $S : IFS(X)^2 \to [0,1]$ *should satisfy the following properties:*

1. $1S(A, B) \in [0, 1]$;
2. $S(A, B) = 1$ *if and only if* $A = B$;
3. $S(A, B) = S(B, A)$;
4. *If* $A \subseteq B \subseteq C$, *then* $S(A, C) \le S(A, B)$ *and* $S(A, C) \le S(B, C)$.

In the literature, the existing transformed FN-based similarity measures $S(A, B)$ *between two IFSs* $A(\mu_A(x_i), v_A(x_i))$ *and* $B(\mu_B(x_i), v_B(x_i))$, *for* $1 \le i \le n$, *are reviewed as follows.*

Zhang and Yu's similarity measure [9] is

$$S_{ZY}(A,\ B) = 1 - \sum_{i=1}^{n} w_i \times (U_i - I_i)$$

where w_i *is the weight of element* x_i, $w_i \in [0,1]$, *for* $1 \le i \le n$, $\sum_{i=1}^{n} w_i = 1$,

$$I_i = \int_0^1 \min\Big(\mu_{\overline{A}_{x_i}}(t), \mu_{\overline{B}_{x_i}}(t)\Big)dt,$$

$$U_i = \int_0^{m_A(x_i)} \max\Big(\mu_{\overline{A}_{x_i}}(t), \mu_{\overline{B}_{x_i}}(t)\Big)dt + \big|m_B(x_i) - m_A(x_i)\big| + \int_{m_B(x_i)}^{1} \max\Big(\mu_{\overline{A}_{x_i}}(t), \mu_{\overline{B}_{x_i}}(t)\Big)dt,$$

$$m_A(x_i) = \frac{\mu_A(x_i) + 1 - v_A(x_i)}{2},$$

$$m_B(x_i) = \frac{\mu_B(x_i) + 1 - v_B(x_i)}{2}$$

and two symmetric triangular FNs

$$\mu_{\overline{A}_{x_i}}(t) = \begin{cases} \frac{t-\mu_A(x_i)}{m_A(x_i)-\mu_A(x_i)}, & if\ \ t \in [\mu_A(x_i), m_A(x_i)] \\ \frac{1-v_A(x_i)-t}{1-v_A(x_i)-m_A(x_i)}, & if\ \ t \in [m_A(x_i), 1 - v_A(x_i)] \\ 0, & otherwise \end{cases}$$

and

$$\mu_{\overline{B}_{x_i}}(t) = \begin{cases} \frac{t-\mu_B(x_i)}{m_B(x_i)-\mu_B(x_i)}, & if\ \ t \in [\mu_B(x_i), m_B(x_i)] \\ \frac{1-v_B(x_i)-t}{1-v_B(x_i)-m_B(x_i)}, & if\ \ t \in [m_B(x_i), 1 - v_B(x_i)] \\ 0, & otherwise \end{cases}$$

for $1 \le i \le n$.

Chen and Chang's similarity measure [13] is

$$S_{CC}(A,\ B) = \sum_{i=1}^{n} w_i \times s(A_{x_i}, B_{x_i})$$

where w_i *is the weight of element* x_i, $w_i \in [0,1]$, *for* $1 \le i \le n$, $\sum_{i=1}^{n} w_i = 1$ *and*

$$s\Big(A_{x_i}, B_{x_i}\Big) = rs\Big(A_{x_i}, B_{x_i}\Big) - us\Big(A_{x_i}, B_{x_i}\Big).$$

Define two membership functions of the transformed right-angled triangular FNs A_{x_i} *and* B_{x_i} *obtained from the IFS* $(\mu_A(x_i),\ 1 - v_A(x_i))$ *of element* x_i *belonging to the IFS A as follows*

$$\mu_{A_{x_i}}(z) = \begin{cases} 1, & if\ \ z = \mu_A(x_i) = 1 - v_A(x_i) \\ \frac{1-v_A(x_i)-z}{1-\mu_A(x_i)-v_A(x_i)}, & if\ \ z \in [\mu_A(x_i), 1 - v_A(x_i)] \\ 0, & otherwise \end{cases}$$

and

$$\mu_{B_{x_i}}(z) = \begin{cases} 1, & if\ \ z = \mu_B(x_i) = 1 - v_B(x_i) \\ \frac{1-v_B(x_i)-z}{1-\mu_B(x_i)-v_B(x_i)}, & if\ \ z \in [\mu_B(x_i), 1 - v_B(x_i)] \\ 0, & otherwise \end{cases}$$

Then, the degree of similarity between A_{x_i} and B_{x_i} and the difference of the areas between A_{x_i} and B_{x_i} are respectively

$$rs(A_{x_i}, B_{x_i}) = 1 - \left|\mu_A(x_i) - \mu_B(x_i)\right| \times (1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2})$$

and

$$us(A_{x_i}, B_{x_i}) = \int_0^1 \left|\mu_{A_{x_i}}(z) - \mu_{B_{x_i}}(z)\right| dz \times \frac{\pi_A(x_i) + \pi_B(x_i)}{2}$$

where $\frac{\pi_A(x_i)+\pi_B(x_i)}{2} \in [0,1]$ *for* $1 \le i \le n$.

Among these transformed FN-based similarity measures, the most distinctive is that the form of the transformation technique is symmetric triangular FN for $S_{ZY}(A,\ B)$ and right-angled triangular FN for $S_{CC}(A,\ B)$. Chen and Chang indicated that if $\mu_A(x_i) + v_A(x_i) = 1$ or $\mu_B(x_i) + v_B(x_i) = 1$ for some $1 \le i \le n$, then Zhang and Yu's similarity measure $S_{ZY}(A,\ B)$ has the division by zero problem. Therefore, this paper focuses on Chen and Chang's similarity measure $S_{CC}(A,\ B)$.

3. Chen and Chang's Similarity Measure

This section will present an algorithm of computing $S_{CC}(A,\ B)$ for $A(\mu_A(x_i), v_A(x_i))$ and $B(\mu_B(x_i), v_B(x_i))$ for $1 \le i \le n$. The major part of computing $S_{CC}(A,\ B)$ is $s\left(A_{x_i}, B_{x_i}\right)$. For simplicity, assume that $n = 1$, and the abbreviated notations of $\mu_A(x_1)$, A_{x_1}, and $\mu_{A_{x_1}}(z)$ are then denoted by μ_A, A_x, and $\mu_{A_x}(z)$, respectively.

Without loss of generality, assume that

$$\mu_A \le \mu_B.$$

If $\mu_A > \mu_B$, since $S_{CC}(A,\ B) = S_{CC}(B, A)$, we can swap A and B. Four cases are considered as follows.

Case 1: $\mu_A \le 1 - v_A \le \mu_B \le 1 - v_B$.

From

$$rs(A_x, B_x) = 1 - \left|\mu_A - \mu_B\right| \times \left(1 - \frac{\pi_A + \pi_B}{2}\right) = 1 - (\mu_B - \mu_A)\left(1 - \frac{\pi_A + \pi_B}{2}\right)$$

and

$$us(A_x, B_x) = \int_0^1 \left|\mu_{A_x}(z) - \mu_{B_x}(z)\right| dz \times \frac{\pi_A + \pi_B}{2}$$
$$= \left(\frac{1 - v_A - \mu_A}{2} + \frac{1 - v_B - \mu_B}{2}\right)\frac{\pi_A + \pi_B}{2} = \left(\frac{\pi_A + \pi_B}{2}\right)^2,$$

it implies that

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)(1 - \frac{\pi_A + \pi_B}{2}) - (\frac{\pi_A + \pi_B}{2})^2.$$

Case 2: $\mu_A \le \mu_B \le 1 - v_A \le 1 - v_B$.

We compute $\int_0^1 \left|\mu_{A_x}(z) - \mu_{B_x}(z)\right| dz$ by decomposing it into three parts:

$$\int_0^1 \left|\mu_{A_x}(z) - \mu_{B_x}(z)\right| dz = \int_{\mu_A}^{\mu_B} \frac{1-v_A-z}{1-\mu_A-v_A} dz + \int_{\mu_B}^{1-v_A} (\frac{1-v_B-z}{1-\mu_B-v_B} - \frac{1-v_A-z}{1-\mu_A-v_A}) dz + \int_{1-v_A}^{1-v_B} \frac{1-v_B-z}{1-\mu_B-v_B} dz,$$

and it follows that

$$rs(A_x, B_x) = 1 - (\mu_B - \mu_A)(1 - \frac{\pi_A + \pi_B}{2})$$

and

$$us(A_x, B_x) = \left(\int_{\mu_A}^{\mu_B} \frac{1-v_A-z}{1-\mu_A-v_A} dz + \int_{\mu_B}^{1-v_A} \left(\frac{1-v_B-z}{1-\mu_B-v_B} - \frac{1-v_A-z}{1-\mu_A-v_A}\right) dz + \int_{1-v_A}^{1-v_B} \frac{1-v_B-z}{1-\mu_B-v_B} dz\right) \times \frac{\pi_A+\pi_B}{2},$$

so

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)\left(1 + \frac{\pi_A + \pi_B}{2}\right) + \frac{\pi_A^2 - \pi_B^2}{4} + \frac{\pi_A + \pi_B}{2\pi_A}(\mu_B - \mu_A)^2.$$

Case 3: $\mu_A \le \mu_B \le 1 - v_B \le 1 - v_A$.

We calculate $\int_0^1 \left|\mu_{A_x}(z) - \mu_{B_x}(z)\right| dz$ by decomposing it into four parts, as follows:

$$\begin{aligned}\int_0^1 \left|\mu_{A_x}(z) - \mu_{B_x}(z)\right| dz = \ & \int_{\mu_A}^{\mu_B} \frac{1-v_A-z}{1-\mu_A-v_A} dz + \int_{\mu_B}^{\bar{z}} \left(\frac{1-v_B-z}{1-\mu_B-v_B} - \frac{1-v_A-z}{1-\mu_A-v_A}\right) dz \\ & + \int_{\bar{z}}^{1-v_B} \left(\frac{1-v_A-z}{1-\mu_A-v_A} - \frac{1-v_B-z}{1-\mu_B-v_B}\right) dz + \int_{1-v_B}^{1-v_A} \frac{1-v_A-z}{1-\mu_A-v_A} dz\end{aligned}$$

where $\bar{z} = \frac{\mu_B(\pi_A+\mu_A) - \mu_A(\pi_B+\mu_B)}{\pi_A - \pi_B}$ obtained from $\frac{1-v_A-\bar{z}}{1-\mu_A-v_A} = \frac{1-v_B-\bar{z}}{1-\mu_B-v_B}$.

It follows that

$$rs(A_x, B_x) = 1 - (\mu_B - \mu_A)\left(1 - \frac{\pi_A + \pi_B}{2}\right),$$

$$\begin{aligned}us(A_x, B_x) = \ & \left(\int_{\mu_A}^{\mu_B} \frac{1-v_A-z}{1-\mu_A-v_A} dz + \int_{\mu_B}^{\bar{z}} \left(\frac{1-v_B-z}{1-\mu_B-v_B} - \frac{1-v_A-z}{1-\mu_A-v_A}\right) dz + \int_{\bar{z}}^{1-v_B} \left(\frac{1-v_A-z}{1-\mu_A-v_A} - \frac{1-v_B-z}{1-\mu_B-v_B}\right) dz \right. \\ & \left. + \int_{1-v_B}^{1-v_A} \frac{1-v_A-z}{1-\mu_A-v_A} dz\right) \times \frac{\pi_A+\pi_B}{2}\end{aligned}$$

and

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)\left(1 - \frac{\pi_A + \pi_B}{2}\right) - \frac{\pi_A^2 - \pi_B^2}{4} - \frac{\pi_B(\pi_A + \pi_B)}{2\pi_A(\pi_A - \pi_B)}(\mu_B - \mu_A)^2.$$

Case 4: $\pi_A = 0$ or $\pi_B = 0$.

If $\pi_A = 0$, then

$$rs(A_x, B_x) = 1 - (\mu_B - \mu_A)\left(1 - \frac{\pi_B}{2}\right)$$

and

$$us(A_x, B_x) = \frac{1 - v_B - \mu_B}{2} \times \frac{\pi_B}{2} = \left(\frac{\pi_B}{2}\right)^2,$$

so

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)\left(1 - \frac{\pi_B}{2}\right) - \left(\frac{\pi_B}{2}\right)^2.$$

If $\pi_B = 0$, a similar argument shows that

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)\left(1 - \frac{\pi_A}{2}\right) - \left(\frac{\pi_A}{2}\right)^2.$$

We observe that case 4 is a special one of case 1. The mechanics of computing $S_{CC}(A,\ B)$ are listed as follows (Algorithm 1).

Algorithm 1

Input $A(\mu_A, v_A)$ and $B(\mu_B, v_B)$. Output $S_{CC}(A,\ B)$.
Step 0 If $\mu_A > \mu_B$, swap A and B.
Step 1 Set $\pi_A = 1 - \mu_A - v_A$ and $\pi_B = 1 - \mu_B - v_B$.
Step 2 If $1 - v_A \le \mu_B$ or $\pi_A = 0$ or $\pi_B = 0$, then

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)(1 - \frac{\pi_A + \pi_B}{2}) - (\frac{\pi_A + \pi_B}{2})^2;$$

else if $v_B \le v_A$, then

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)(1 + \frac{\pi_A + \pi_B}{2}) + \frac{\pi_A^2 - \pi_B^2}{4} + \frac{\pi_A + \pi_B}{2\pi_A}(\mu_B - \mu_A)^2;$$

Else

$$S_{CC}(A,\ B) = 1 - (\mu_B - \mu_A)(1 - \frac{\pi_A + \pi_B}{2}) - \frac{\pi_A^2 - \pi_B^2}{4} - \frac{\pi_B(\pi_A + \pi_B)}{2\pi_A(\pi_A - \pi_B)}(\mu_B - \mu_A)^2.$$

We illustrate some concrete examples with various $A(\mu_A, v_A)$ and $B(\mu_B, v_B)$.

Example 1. *$A(a, a)$ and $B(a+\alpha, a+\alpha)$, $a+\alpha \le 0.5$, $a, \alpha \ge 0$. From the definition of $S_{CC}(A,\ B)$, it follows that*

$$rs(A_x, B_x) = 1 - |a + \alpha - a|\left(1 - \frac{1 - 2a + 1 - 2a - 2\alpha}{2}\right) = 1 - \alpha(2a + \alpha)$$

and

$$us(A_x, B_x) = \left(\int_a^{a+\alpha} \tfrac{1-a-z}{1-2a} dz + \int_{a+\alpha}^{0.5} \tfrac{1-a-\alpha-z}{1-2a-2\alpha} - \tfrac{1-a-z}{1-2a} dz + \int_{0.5}^{1-a-\alpha} \tfrac{1-a-z}{1-2a} - \tfrac{1-a-\alpha-z}{1-2a-2\alpha} dz + \int_{1-a-\alpha}^{1-a} \tfrac{1-a-z}{1-2a} dz\right)(1 - 2a - \alpha)$$
$$= \alpha\left(1.5 - \tfrac{0.5\alpha}{0.5-\alpha}\right)(1 - 2a - \alpha),$$

so

$$S_{CC}(A,\ B) = 1 - \alpha(2a + \alpha) - \alpha\left(1.5 - \frac{0.5\alpha}{0.5 - \alpha}\right)(1 - 2a - \alpha).$$

This result coincides with that of Algorithm 1. From $\frac{\partial}{\partial a} S_{CC}(A,\ B) \ge 0$ and $\frac{\partial}{\partial \alpha} S_{CC}(A,\ B) \le 0$, it implies that $S_{CC}(A,\ B)$ is an increasing function of a and a decreasing function of α for $a + \alpha \le 0.5$. For $\alpha = 0$, associated $A(a, a)$ and $B(a, a)$ can attain the maximum value $S_{CC}(A,\ B) = 1$.

Example 2. *$C(a, a+\alpha)$ and $D(a+\alpha, a)$, $2a + \alpha \le 1$, $a, \alpha \ge 0$. The definition of $S_{CC}(C,\ D)$ gives*

$$rs(C_x, D_x) = 1 - |a + \alpha - a|\left(1 - \frac{1 - 2a - \alpha + 1 - 2a - \alpha}{2}\right) = 1 - \alpha(2a + \alpha)$$

and

$$us(C_x, D_x) = \left(\int_a^{a+\alpha} \tfrac{1-a-\alpha-z}{1-2a-\alpha} dz + \int_{a+\alpha}^{1-a-\alpha} \tfrac{1-a-z}{1-2a-\alpha} - \tfrac{1-a-\alpha-z}{1-2a-\alpha} dz + \int_{1-a-\alpha}^{1-a} \tfrac{1-a-z}{1-2a-\alpha} dz\right)(1 - 2a - \alpha) = \alpha \tfrac{2-4a-3\alpha}{1-2a-\alpha}(1 - 2a - \alpha),$$

so

$$S_{CC}(C,\ D) = 1 - 2\alpha(1 - a) + 2\alpha^2.$$

This result coincides with that of Algorithm 1. Comparedwith that of example 1,

$$S_{CC}(A,\ B) - S_{CC}(C,\ D) = \frac{1}{2(1 - 2a)}(1 - 2a + \alpha)\alpha(1 - 2a - 2\alpha) \ge 0$$

which is consistent with our intuition, for $a + \alpha \le 0.5$, $a, \alpha \ge 0$. For the case of $a + \alpha = 0.5$, we get

$$S_{CC}(A,\ B) = S_{CC}(C,\ D)\ for\ A(a,a), B(0.5,0.5), C(a,0.5)\ and\ D(0.5,a).$$

Example 3. *$A(\mu,v)$, $B(\mu+\alpha,v+\alpha)$ and $C(\mu+\alpha,v)$, $\mu+v+2\alpha \le 1$, $\mu,v,\alpha \ge 0$. Using Algorithm 1, we obtain*

$$S_{CC}(A,\ B) = 1 - \alpha(\mu+v+\alpha) - (1.5\alpha - \frac{\alpha^2}{1-\mu-v})(1-\mu-v-\alpha)$$

and

$$S_{CC}(A,\ C) = 1 - \alpha(\mu+v+0.5\alpha) - (1.5\alpha - \frac{\alpha^2}{1-\mu-v})(1-\mu-v-0.5\alpha).$$

$S_{CC}(A,\ B)$ is an increasing function of μ and v and a decreasing function of α from $\frac{\partial}{\partial\mu}S_{CC}(A,\ B) \ge 0$, $\frac{\partial}{\partial v}S_{CC}(A,\ B) \ge 0$, and $\frac{\partial}{\partial\alpha}S_{CC}(A,\ B) \le 0$. For $\alpha = 0$, we can attain the maximum value $S_{CC}(A,\ B) = 1$ with $A(\mu,v)$ and $B(\mu,v)$. For $\mu+v+2\alpha = 1$, we have $S_{CC}(A,\ B) = 1-\alpha$ with $A(\mu,v)$ and $B\left(\frac{1+\mu-v}{2}, \frac{1-\mu+v}{2}\right)$. From $\frac{\partial}{\partial\mu}S_{CC}(A,\ C) \ge 0$, $\frac{\partial}{\partial v}S_{CC}(A,\ C) \ge 0$, and $\frac{\partial}{\partial\alpha}S_{CC}(A,\ C) \le 0$, the behaviors of $S_{CC}(A,\ C)$ are the same as those of $S_{CC}(A,\ B)$. We also have

$$S_{CC}(A,\ B) - S_{CC}(A,\ C) = \frac{\alpha^2(1-\mu-v-2\alpha)}{4(1-\mu-v)} \ge 0,$$

which is consistent with our intuition.

Example 4. *$A(a,a)$, $B(0,0)$, and $C(a+\alpha,a-\alpha)$, $0 \le \alpha \le a \le 0.5$. Applying Algorithm 1, we get*

$$S_{CC}(A,\ B) = 1 - a^2 - (3-2a)a(1-a)/2,$$

$$S_{CC}(B,C) = 1 - a(a+\alpha) - \frac{1-a}{2a}(2a^3 - (3-4\alpha)a^2 - 2a(1-a)\alpha - a^2)$$

and

$$S_{CC}(A,\ B) - S_{CC}(B,\ C) = \frac{\alpha}{2a}(2(-2+2a+\alpha)a^2 + a(2-3\alpha) + \alpha).$$

Applying an exhaustive search for $\alpha \in \{0, 0.01, 0.02, \ldots, a\}$ and $a \in \{0.01, 0.02, \ldots, 0.5\}$, we get $S_{CC}(A,\ B) - S_{CC}(B,\ C) \ge 0$, which is consistent with our intuition. Additionally, $S_{CC}(A,\ B)$ is a decreasing function of a and $S_{CC}(B,\ C)$ is a decreasing function of a and α.

Example 5. *$A(\mu,v)$, $B(\mu+\alpha,v-\alpha)$, $C(\mu+\beta,v+\beta)$, and $D(\mu+\beta+\alpha,v+\beta-\alpha)$, $\mu+v+2\beta \le 1$, $\mu+\alpha+\beta \le 1$, $\alpha \le v+\beta$, $\mu,v,\alpha,\beta \ge 0$. Applying Algorithm 1 yields*

$$S_{CC}(A,\ B) = 1 - \alpha(2-\mu-v-\alpha),$$

$$S_{CC}(C,\ D) = 1 - \alpha(2-\mu-v-\alpha-2\beta)$$

and

$$S_{CC}(A,\ B) - S_{CC}(C,\ D) = -2\alpha\beta \le 0$$

which is consistent with our intuition. For $v = 1-\mu$, we have $S_{CC}(A,\ B) = 1-\alpha$. Additionally, $S_{CC}(A,\ B)$ is an increasing function of μ and v and a decreasing function of α. $S_{CC}(C,\ D)$ is an increasing function of μ, v and β and a decreasing function of α.

Example 6. *$A(\mu,1-\mu)$ and $B(0,0)$, $0 \le \mu \le 1$. We have*

$$S_{CC}(A,\ B) = 0.75 - \mu/2.$$

$S_{CC}(A, B)$ is a decreasing function of μ. We can attain the maximum value $S_{CC}(A, B) = 0.75$ for $\mu = 0$ and minimum value $S_{CC}(A, B) = 0.25$ for $\mu = 1$, which are inconsistent with our intuition.

4. General Counterintuitive Test Problems

Much literature has been written on the counterintuitive examples for the similarity measures between two IFSs. Two typical counterintuitive examples are (I) $S(A, B) = 1$ for $A \neq B$, $A, B \in IFS(X)$ and (II) $S(P_1, Q) = S(P_2, Q)$ for $P_1 \neq P_2$, $P_1, P_2, Q \in IFS(X)$. Tang and Yang [22] proposed the following six general test problems to analyze the counterintuitive behaviors of similarity measures.

Test problem 1 (T1)

$A(a, a)$, $B(b, b)$, $a \neq b$, $0 \leq a, b \leq 1/2$ satisfying $S(A, B) = 1$.

Test problem 2 (T2)

$A(a, a)$, $B(b, b)$, $C(a, b)$, $D(b, a)$, $a \neq b$, $0 \leq a, b \leq 1/2$ satisfying $S(A, B) = S(C, D)$.

Test problem 3 (T3)

$P_1(a, b)$, $P_2(\frac{a+b}{2}, \frac{a+b}{2})$, $Q(b, b)$, $a \neq b$, $0 \leq a, b, a+b \leq 1$, $b \leq 1/2$ satisfying $S(P_1, Q) = S(P_2, Q)$.

Test problem 4 (T4)

$P_1(1, 0)$, $P_2(a, 1-a)$, $Q(0, 0)$, $0 \leq a \leq 1$ satisfying $S(P_1, Q) = S(P_2, Q)$.

Test problem 5 (T5)

$P_1(b, b-\alpha)$, $P_2(b, a-\alpha)$, $Q(a, a-\alpha)$, $a \neq b$, $\alpha \leq a, b, a+b-\alpha \leq 1$, $2a-\alpha \leq 1$, $2b-\alpha \leq 1$ satisfying $S(P_1, Q) = S(P_2, Q)$.

Test problem 6 (T6)

$P_1\{(a+\alpha, b), (a, b+\alpha)\}$, $P_2\{(a+\alpha, b+\alpha), (a+\alpha, b+\alpha)\}$, $Q\{(a, b), (a+\alpha, b)\}$, $a \neq b$, $0 \leq a, b, \alpha, a+b+2\alpha \leq 1$ satisfying $S(P_1, Q) = S(P_2, Q)$.

For the specific $S_{CC}(A, B)$, we also propose the following general test problem to analyze its counterintuitive behaviors.

Test problem 7 (T7)

$P_1(\mu+\alpha, v-\alpha)$, $P_2(\mu-\alpha, v+\alpha)$, $Q(\mu, v)$, $\alpha \leq \mu, v, \mu+v \leq 1$ satisfying $S(P_1, Q) = S(P_2, Q)$.

We now analyze the counterintuitive behaviors of similarity measure $S_{CC}(A, B)$ for seven general test problems. For test problems T1 and T2 with $a < b$, we apply Algorithm 1 to establish

$$S_{CC}(A, B) = 1 - (b-a)(a+b) - \frac{(b-a)(3-4a-2b)}{2-4a}(1-a-b)$$

and

$$S_{CC}(C, D) = 1 - (b-a)(a+b) - \frac{(b-a)(2-a-3b)}{1-a-b}(1-a-b).$$

Since

$$S_{CC}(A, B) \neq 1$$

and

$$S_{CC}(A, B) - S_{CC}(C, D) = \frac{(a-b)(-1+3a-b)(-1+2b)}{-2+4a} \neq 0,$$

T1 and T2 are not counterintuitive test problems for $S_{CC}(A, B)$ with $a < b$ and $b \neq 0.5$. A symmetric argument shows that T1 and T2 are not counterintuitive ones for $S_{CC}(A, B)$ with $a > b$ and $b \neq 0.5$. Therefore, T1 and T2 are not counterintuitive test problems for $S_{CC}(A, B)$ with $a \neq b$ and $b \neq 0.5$.

For test problem T3 with $a < b$, we get

$$S_{CC}(P_1,\ Q) = 1-(b-a)(0.5a+1.5b) - \frac{(b-a)(3-a-5b)}{2(1-a-b)}(1-0.5a-1.5b),$$

$$S_{CC}(P_2,\ Q) = 1-0.5(b-a)(0.5a+1.5b) - \frac{(b-a)(3-2a-4b)}{4(1-a-b)}(1-0.5a-1.5b)$$

and

$$S_{CC}(P_1,\ Q) - S_{CC}(P_2,\ Q) = \frac{(b-a)(2a^2+a+2ab-6+15b-12b^2)}{8(1-a-b)}.$$

It follows that $S_{CC}(P_1,\ Q) \neq S_{CC}(P_2,\ Q)$ for $a < b$. A similar argument shows that $S_{CC}(P_1,\ Q) \neq S_{CC}(P_2,\ Q)$ for $a > b$. Therefore, T3 is not a counterintuitive test problem for $S_{CC}(A,\ B)$ with $a \neq b$.

For test problem T4,

$$S_{CC}(P_1,\ Q) = 0.25,$$

$$S_{CC}(P_2,\ Q) = 0.75-0.5a$$

and

$$S_{CC}(P_2,\ Q) - S_{CC}(P_1,\ Q) = 0.5-0.5a,$$

imply that T4 with $a \neq 1$ is not a counterintuitive test problem for $S_{CC}(A,\ B)$.

For test problem T5 with $a \leq b$, from

$$S_{CC}(P_1,\ Q) = 1-(b-a)(a+b-\alpha) - \frac{(b-a)(3+3\alpha-4a-2b)}{2(1-2a+\alpha)}(1-a-b+\alpha),$$

$$S_{CC}(P_2,\ Q) = 1-(b-a)(1.5a+0.5b-\alpha) - \frac{(b-a)(3+3\alpha-4a-2b)}{2(1-2a+\alpha)}(1-1.5a-0.5b+\alpha)$$

and

$$S_{CC}(P_1,\ Q) - S_{CC}(P_2,\ Q) = \frac{(b-a)^2(1+\alpha-2b)}{4(1-2a+\alpha)},$$

it follows that $S_{CC}(P_1,\ Q) = S_{CC}(P_2,\ Q)$ for $a = b$ or $\alpha = 2b-1$. The case of $a = b$ is a trivial one. For $\alpha = 2b-1$, we can see that $P_1(b,\ 1-b)$, $P_2(b,\ 1+a-2b)$, and $Q(a,\ 1+a-2b)$ for $0.5 \leq b$, $2b-1 \leq a \leq b \leq 1$ satisfying $S_{CC}(P_1,\ Q) = S_{CC}(P_2,\ Q)$ is a counterintuitive test problem for $S_{CC}(A,\ B)$.

For test problem T6 with $\alpha > 0$, we now use Algorithm 1 to obtain

$$S_{CC}(P_1,\ Q) = 1+\frac{1}{8}\Big\{\alpha(-14+6a+6b)+\alpha^2(13-\frac{2\alpha}{1-a-b})\Big\},$$

$$S_{CC}(P_2,\ Q) = 1+\frac{1}{2}\Big\{\alpha(-2+a+b)+\alpha^2(2-\frac{\alpha}{1-a-b})\Big\}$$

and

$$S_{CC}(P_1,\ Q) - S_{CC}(P_2,\ Q) = \frac{1}{8}\Big\{\alpha(-6+2a+2b)+\alpha^2(5+\frac{2\alpha}{1-a-b})\Big\}.$$

Since $S_{CC}(P_1,\ Q) \neq S_{CC}(P_2,\ Q)$ for $\alpha > 0$, it follows that T6 is not a counterintuitive test problem for $S_{CC}(A,\ B)$.

We now apply Algorithm 1 with test problem T7 to deduce that

$$S_{CC}(P_1,\ Q) = 1-\alpha(\mu+v)-\alpha(2-\frac{\alpha}{1-\mu-v})(1-\mu-v)$$

and

$$S_{CC}(P_2,\ Q) = 1-\alpha(\mu+v)-\alpha(2-\frac{\alpha}{1-\mu-v})(1-\mu-v),$$

so

$$S_{CC}(P_1,\ Q) = S_{CC}(P_2,\ Q).$$

Then, T7 is a counterintuitive test problem for $S_{CC}(A,\ B)$.

Therefore, the counterintuitive test problems for Chen and Chang's similarity measure $S_{CC}(A,\ B)$ are (1) $P_1(b,\ 1-b)$, $P_2(b,\ 1+a-2b)$, and $Q(a,\ 1+a-2b)$ for $0.5 \le b,\ 2b-1 \le a \le b \le 1$ satisfying $S_{CC}(P_1,\ Q) = S_{CC}(P_2,\ Q)$, and (2) $P_1(\mu+\alpha,\ v-\alpha)$, $P_2(\mu-\alpha,\ v+\alpha)$, and $Q(\mu,\ v)$ for $\alpha \le \mu,\ v,\ \mu+v \le 1$ satisfying $S_{CC}(P_1,\ Q) = S_{CC}(P_2,\ Q)$.

5. Counterintuitive Test Problem with $A \subseteq B \subseteq C$

This section presents the counterintuitive test problems of Chen and Chang's similarity measure $S_{CC}(A,\ B)$ for the case that $A \subseteq B \subseteq C$, $A, B, C \in IFS(X)$. Chen, Cheng and Lan [14] proposed some counterexamples with $A \subseteq B \subseteq C$ for $S_{CC}(A,\ B)$. More precisely, given $A \subseteq B \subseteq C$, we have $S_{CC}(A,\ C) \nleq S_{CC}(A,\ B)$ or $S_{CC}(A,\ C) \nleq S_{CC}(B,\ C)$, contradicting Definition 4. For test problems with $A \subseteq B \subseteq C$, this section proposes some general counterexamples satisfying $S_{CC}(A,\ C) \nleq S_{CC}(A,\ B)$. A symmetric argument shows the similar results of $S_{CC}(A,\ C) \nleq S_{CC}(B,\ C)$ which are omitted in this paper.

From $A \subseteq B \subseteq C$, we have $\mu_A \le \mu_B \le \mu_C$ and $v_A \ge v_B \ge v_C$. Without loss of generality, assume that

$$A(a,\ c+\gamma+\delta), B(a+\alpha,\ c+\gamma) \text{ and } C(a+\alpha+\beta, c) \text{ for } a+c+\alpha+\beta \le 1, a+c+\alpha+\gamma \le 1,$$

$$a+c+\gamma+\delta \le 1, a, c, \alpha, \beta, \gamma, \delta \ge 0.$$

For simplicity, we assume that

$$1-c-\gamma-\delta \le a+\alpha.$$

Using Algorithm 1 yields

$$S_{CC}(A,\ B) = 1-\alpha(a+c+\gamma+\frac{\alpha+\delta}{2}) - (1-a-c-\gamma-\frac{\alpha+\delta}{2})^2,$$

$$S_{CC}(A,\ C) = 1-(\alpha+\beta)\left(a+c+\frac{\alpha+\beta+\gamma+\delta}{2}\right) - (1-a-c-\frac{\alpha+\beta+\gamma+\delta}{2})^2$$

and

$$S_{CC}(A,\ C) - S_{CC}(A,\ B) = \tfrac{1}{4}\left\{-3\beta^2 - 2\beta(4a+4c+3\alpha-2(1-\gamma-\delta)) + \gamma(-4+4a+4c+4\alpha+3\gamma+2\delta)\right\}$$

Four cases of $A(a,\ c+\gamma+\delta)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha+\beta, c)$ satisfying $S_{CC}(A,\ C) > S_{CC}(A,\ B)$ are distinguished: (I) $\beta = 0$ and $\delta = \alpha$, (II) $\beta = 0$, (III) $\beta = \gamma$, and (IV) $\beta > 0$.

First, consider the case I: $\beta = 0$ and $\delta = \alpha$, so we have $A(a,\ c+\gamma+\alpha)$, $B(a+\alpha,\ c+\gamma)$ and $C(a+\alpha, c)$ for $a+c+2\alpha+\gamma \ge 1$, $a+c+\alpha+\gamma \le 1$, $a, c, \alpha, \gamma \ge 0$. Applying Algorithm 1 gives

$$S_{CC}(A,\ B) = 1-\alpha(a+c+\alpha+\gamma) - (1-a-c-\alpha-\gamma)^2,$$

$$S_{CC}(A,\ C) = 1-\alpha(a+c+\alpha+0.5\gamma) - (1-a-c-\alpha-0.5\gamma)^2$$

and

$$S_{CC}(A,\ B) - S_{CC}(A,\ C) = -\frac{\gamma}{4}(-4+4a+4c+6\alpha+3\gamma).$$

It implies that if

$$a+c+\frac{3}{4}(2\alpha+\gamma) > 1,$$

then $S_{CC}(A,\ B) < S_{CC}(A,\ C)$. Therefore, if $A(a,\ c+\gamma+\alpha)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha, c)$ for $a+c+\frac{3}{4}(2\alpha+\gamma) > 1$, $a+c+\alpha+\gamma \le 1$, $a, c, \alpha, \gamma \le 0$, we have $S_{CC}(A,\ B) < S_{CC}(A,\ C)$.

In the case of II: $\beta = 0$, we get

$$S_{CC}(A,\ C) - S_{CC}(A,\ B) = \frac{1}{4}\gamma(-4 + 4a + 4c + 4\alpha + 3\gamma + 2\delta).$$

It implies that if

$$a + c + \alpha + 0.75\gamma + 0.5\delta > 1, a + c + \alpha + \gamma + \delta \geq 1,\ a + c + \alpha + \gamma \leq 1, a + c + \gamma + \delta \leq 1,$$

$$a, c, \alpha, \gamma, \delta \geq 0$$

then $S_{CC}(A,\ B) < S_{CC}(A,\ C)$. If $\alpha = 0$, then $a + c + 0.75\gamma + 0.5\delta > 1$, in contradiction to $a + c + \gamma + \delta \leq 1$. If $\delta = 0$, then $a + c + \alpha + 0.75\gamma > 1$, in contradiction to $a + c + \alpha + \gamma \leq 1$. Also if $a + c + \alpha + 0.75\gamma + 0.5\delta > 1$, then $a + c + \alpha + \gamma + \delta \geq 1$. Therefore, if

$$A(a,\ c + \gamma + \delta), B(a + \alpha,\ c + \gamma) \text{ and } C(a + \alpha, c) \text{ for } a + c + \alpha + 0.75\gamma + 0.5\delta > 1, a + c + \alpha + \gamma \leq 1,$$
$$a + c + \gamma + \delta \leq 1, \alpha, \gamma, \delta > 0, a, c \geq 0,$$

then $S_{CC}(A,\ B) < S_{CC}(A,\ C)$.

For the case of III: $\beta = \gamma$, we have

$$S_{CC}(A,\ C) - S_{CC}(A,\ B) = -\frac{\gamma}{2}(2a + 2c + \alpha + 2\gamma + \delta) \leq 0.$$

It implies that if

$$A(a,\ c + \gamma + \delta), B(a + \alpha,\ c + \gamma) \text{ and } C(a + \alpha + \gamma, c) \text{ for } a + c + \alpha + \gamma + \delta \geq 1, a + c + \alpha + \gamma \leq 1,$$
$$a + c + \gamma + \delta \leq 1, a, c, \alpha, \gamma, \delta \geq 0,$$

then $S_{CC}(A,\ B) < S_{CC}(A,\ C)$.

We now analyze case IV. Substituting

$$\alpha = 1 - a - c - \gamma \text{ and } \delta = 1 - a - c - \gamma$$

into $S_{CC}(A,\ C) - S_{CC}(A,\ B)$ gives

$$\begin{aligned} S_{CC}(A,\ C) - S_{CC}(A,\ B) = \tfrac{1}{4}\Big\{&-3\beta^2 - 2\beta(4a + 4c + 3(1 - a - c - \gamma) - 2(1 - \gamma - (1 - a - c - \gamma))) \\ &+\gamma(-4 + 4a + 4c + 4(1 - a - c - \gamma) + 3\gamma + 2(1 - a - c - \gamma))\} \end{aligned}$$

$$= \frac{1}{4}\Big\{-3\gamma^2 + \gamma(2 - 2a - 2c + 6\beta) - 3\beta^2 - 2\beta(3 - a - c)\Big\}.$$

From

$$\frac{\partial}{\partial\gamma}[S_{CC}(A,\ C) - S_{CC}(A,\ B)] = \frac{1}{4}\{-6\gamma + 2 - 2a - 2c + 6\beta\} = 0$$

and

$$\frac{\partial^2}{\partial\gamma^2}[S_{CC}(A,\ C) - S_{CC}(A,\ B)] = -\frac{3}{2} < 0,$$

it follows that

$$\gamma = \beta + \frac{1 - a - c}{3}$$

attains the maximum value

$$S_{CC}(A,\ C) - S_{CC}(A,\ B) = -\beta + \frac{(1 - a - c)^2}{12}.$$

So if

$$\beta < \frac{(1-a-c)^2}{12},\ \gamma = \beta + \frac{1-a-c}{3},\ \alpha = \frac{2(1-a-c)}{3} - \beta \text{ and } \delta = \frac{2(1-a-c)}{3} - \beta,$$

then

$A(a,\ 1-a)$, $B(\frac{2+a-2c}{3} - \beta,\ \frac{1-a+2c}{3} + \beta)$ and $C(\frac{2+a-2c}{3}, c)$ for $\beta < \frac{(1-a-c)^2}{12}$, $a+c \leq 1$, $a, c, \beta \geq 0$ satisfying $S_{CC}(A,\ C) > S_{CC}(A,\ B)$.

For case IV, we illustrate some concrete examples with various values of (a, c).

Example 7. *Consider four subcases (1) $a = 0$ and $c = 0$; (2) $a = 0$ and $c \neq 0$; (3) $a \neq 0$ and $c = 0$; and (4) $a \neq 0$ and $c \neq 0$. Some sample values of $(\alpha, \beta, \gamma, \delta)$ and $S_{CC}(A,\ C) - S_{CC}(A,\ B)$ are given as follows for each subcase. The values of $(\alpha, \beta, \gamma, \delta)$ and $S_{CC}(A,\ C) - S_{CC}(A,\ B)$ are (0.6567, 0.01, 0.3433, 0.6567) and 0.0183 for $a = 0$ and $c = 0$, (0.59, 0.01, 0.31, 0.59) and 0.0144 for $a = 0$ and $c = 0.1$, (0.39, 0.01, 0.21, 0.39) and 0.02 for $a = 0.4$ and $c = 0$, and (0.39, 0.01, 0.21, 0.39) and 0.005 for $a = 0.1$ and $c = 0.3$. Therefore, these four values of $(\alpha, \beta, \gamma, \delta)$ are counterintuitive ones satisfying $S_{CC}(A,\ C) > S_{CC}(A,\ B)$ for case IV.*

Therefore, consider three IFNs $A(a,\ c+\gamma+\delta)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha+\beta, c)$ satisfying $A \subseteq B \subseteq C$ for $a+c+\alpha+\gamma+\delta \geq 1$, $a+c+\alpha+\beta \leq 1$, $a+c+\alpha+\gamma \leq 1$, $a+c+\gamma+\delta \leq 1$, $a, c, \alpha, \beta, \gamma, \delta \geq 0$. Four cases of A, B, and C satisfying $S_{CC}(A,\ C) > S_{CC}(A,\ B)$ are distinguished: (I) $\beta = 0$ and $\delta = \alpha$; (II) $\beta = 0l$ (III) $\beta = \gamma$; and (IV) $0 < \beta < \frac{(1-a-c)^2}{12}$, $\gamma = \beta + \frac{1-a-c}{3}$, $\alpha = \frac{2(1-a-c)}{3} - \beta$, and $\delta = \frac{2(1-a-c)}{3} - \beta$. The corresponding counterexamples with $A \subseteq B \subseteq C$ satisfying $S_{CC}(A,\ C) > S_{CC}(A,\ B)$ are (I) $A(a,\ c+\gamma+\alpha)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha, c)$ for $a+c+\frac{3}{4}(2\alpha+\gamma) > 1$, $a+c+\alpha+\gamma \leq 1$, $a, c, \alpha, \gamma \geq 0$; (II) $A(a,\ c+\gamma+\delta)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha, c)$ for $a+c+\alpha+0.75\gamma+0.5\delta > 1$, $a+c+\alpha+\gamma \leq 1$, $a+c+\gamma+\delta \leq 1$, $\alpha, \gamma, \delta > 0$, $a, c \geq 0$; (III) $A(a,\ c+\gamma+\delta)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha+\gamma, c)$ for $a+c+\alpha+\gamma+\delta \geq 1$, $a+c+\alpha+\gamma \leq 1$, $a+c+\gamma+\delta \leq 1$, $a, c, \alpha, \gamma, \delta \geq 0$; and (IV) $A(a,\ 1-a)$, $B(\frac{2+a-2c}{3} - \beta,\ \frac{1-a+2c}{3} + \beta)$, and $C(\frac{2+a-2c}{3}, c)$ for $\beta < \frac{(1-a-c)^2}{12}$, $a+c \leq 1$, $a, c, \beta \geq 0$.

6. Conclusion and Future Research

This paper analyzes the counterintuitive behaviors of Chen and Chang's similarity measure $S_{CC}(A,\ B)$ between IFSs. Algorithm 1 is proposed to compute Chen and Chang's similarity measure. Applying Algorithm 1, we illustrate some concrete examples with various IFSs. Two kinds of test examples are considered to analyze the counterintuitive behaviors of $S_{CC}(A,\ B)$. One is the six general test problems proposed by Tang and Yang [22]. We also propose a new general test problem. The other one is the four special test problems with $A \subseteq B \subseteq C$. For the general test problems, the counterintuitive ones are (1) $P_1(b,\ 1-b)$, $P_2(b,\ 1+a-2b)$, and $Q(a,\ 1+a-2b)$ for $0.5 \leq b$, $2b-1 \leq a \leq b \leq 1$ satisfying $S_{CC}(P_1,\ Q) = S_{CC}(P_2,\ Q)$, and (2) $P_1(\mu+\alpha,\ v-\alpha)$, $P_2(\mu-\alpha,\ v+\alpha)$ and $Q(\mu,\ v)$ for $\alpha \leq \mu,\ v$, $\mu + v \leq 1$ satisfying $S_{CC}(P_1,\ Q) = S_{CC}(P_2,\ Q)$. For $A \subseteq B \subseteq C$, consider three IFNs $A(a,\ c+\gamma+\delta)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha+\beta, c)$ satisfying for $a+c+\alpha+\gamma+\delta \geq 1$, $a+c+\alpha+\beta \leq 1$, $a+c+\alpha+\gamma \leq 1$, $a+c+\gamma+\delta \leq 1$, $a, c, \alpha, \beta, \gamma, \delta \geq 0$. The corresponding counterexamples are (1) $A(a,\ c+\gamma+\alpha)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha, c)$ for $a+c+\frac{3}{4}(2\alpha+\gamma) > 1$, $a+c+\alpha+\gamma \leq 1$, $a, c, \alpha, \gamma \geq 0$; (2) $A(a,\ c+\gamma+\delta)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha, c)$ for $a+c+\alpha+0.75\gamma+0.5\delta > 1$, $a+c+\alpha+\gamma \leq 1$, $a+c+\gamma+\delta \leq 1$, $\alpha, \gamma, \delta > 0$, $a, c \geq 0$; (3) $A(a,\ c+\gamma+\delta)$, $B(a+\alpha,\ c+\gamma)$, and $C(a+\alpha+\gamma, c)$ for $a+c+\alpha+\gamma+\delta \geq 1$, $a+c+\alpha+\gamma \leq 1$, $a+c+\gamma+\delta \leq 1$, $a, c, \alpha, \gamma, \delta \geq 0$; and (4) $A(a,\ 1-a)$, $B(\frac{2+a-2c}{3} - \beta,\ \frac{1-a+2c}{3} + \beta)$ and $C(\frac{2+a-2c}{3}, c)$ for $\beta < \frac{(1-a-c)^2}{12}$, $a+c \leq 1$, $a, c, \beta \geq 0$. Therefore, the counterintuitive behaviors are inevitable for the Chen and Chang's similarity measure between IFNs.

In the future, we will try to analyze the counterintuitive behaviors of similarity measures for the generalization of IFSs. In particular, the analysis can be extended to the hesitant fuzzy sets and neutrosophic sets. Thus, the counterintuitive analyses of similarity measures for the hesitant fuzzy sets and neutrosophic sets are a subject of considerable ongoing research.

Author Contributions: H.-C.T. analyzed the method and wrote the paper. K.-S.C. performed the experiments.

Funding: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–356. [CrossRef]
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Hwang, C.M.; Yang, M.S.; Hung, W.L. New similarity measures of intuitionistic fuzzy sets based on the Jaccard index with its application to clustering. *Int. J. Intell. Syst.* **2018**, *33*, 1672–1688. [CrossRef]
4. Wang, Z.; Xu, Z.; Liu, S.; Yao, Z. Direct clustering analysis based on intuitionistic fuzzy implication. *Appl. Soft Comput.* **2014**, *23*, 1–8. [CrossRef]
5. Garg, H.; Kumar, K. An advanced study on the similarity measures of intuitionistic fuzzy, sets based on the set pair analysis theory and their application in decision making. *Soft Comput.* **2018**, *22*, 4959–4970. [CrossRef]
6. He, Y.; Chen, H.; Zhou, L.; Liu, J.; Tao, Z. Intuitionistic fuzzy geometric interaction averaging operators and their application to multi-criteria decision making. *Inf. Sci.* **2014**, *269*, 142–159. [CrossRef]
7. Pedrycz, W.; Chen, S.M. *Granular Computing and Decision-Making: Interactive and Iterative Approaches*; Springer: Heidelberg, Germany, 2015.
8. Wang, J.Q.; Zhang, H.Y. Multicriteria decision-making approach based on Atanassov's intuitionistic fuzzy sets with incomplete certain information on weights. *IEEE Trans. Fuzzy Syst.* **2013**, *2*, 510–515. [CrossRef]
9. Zhang, H.; Yu, L. New distance measures between intuitionitic fuzzy sets and interval-valued fuzzy sets. *Inf. Sci.* **2013**, *245*, 181–196. [CrossRef]
10. De, S.K.; Biswas, R.; Roy, A.R. An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy Sets Syst.* **2001**, *117*, 209–213. [CrossRef]
11. Papacostas, G.A.; Hatzimichaillidis, A.G.; Kaburlasos, V.G. Distance and similarity measures between intuitionistic fuzzy sets: A comparative analysis from a pattern recognition point view. *Pattern Recognit. Lett.* **2013**, *34*, 1609–1622. [CrossRef]
12. Boran, F.E.; Akay, D. A biparametric similarity measure on intuitionistic fuzzy sets with applications to pattern recognition. *Inf. Sci.* **2014**, *255*, 45–57. [CrossRef]
13. Chen, S.M.; Chang, C.H. A novel similarity measure between Atanassov's intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition. *Inf. Sci.* **2015**, *291*, 96–114. [CrossRef]
14. Chen, S.M.; Cheng, S.H.; Lan, T.C. A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition. *Inf. Sci.* **2016**, *343–344*, 15–40. [CrossRef]
15. Chen, S.M.; Randyanto, Y. A novel similarity measure between intuitionistic fuzzy sets and its applications. *Int. J. Pattern Recognit. Artif. Intell.* **2013**, *27*. [CrossRef]
16. Jiang, Q.; Jin, X.; Lee, S.J.; Yao, S. A new similarity/distance measure between intuitionistic fuzzy sets based on the transformed isosceles triangles and its applications to pattern recognition. *Expert Sys. Appli.* **2019**, *116*, 439–453. [CrossRef]
17. Ye, J. Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Math. Comput. Model.* **2011**, *53*, 91–97. [CrossRef]
18. Beliakov, G.; Pagola, M.; Wilkin, T. Vector valued similarity measures for Atanassov's intuitionistic fuzzy sets. *Inf. Sci.* **2014**, *280*, 352–367. [CrossRef]
19. Grzegorzewski, P. Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. *Fuzzy Sets Syst.* **2004**, *148*, 319–328. [CrossRef]
20. Li, Y.; Olson, D.L.; Qin, Z. Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis. *Pattern Recognit. Lett.* **2007**, *28*, 278–285. [CrossRef]

21. Liang, Z.; Shi, P. Similarity measures on intuitionistic fuzzy sets. *Pattern Recognit. Lett.* **2003**, *24*, 2687–2693. [CrossRef]
22. Tang, H.C.; Yang, S.T. Counterintuitive test problems for distance based similarity measures between intuitionistic fuzzy sets, working paper. 2019.