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Isomorphism Theorems in the Primary Categories of Krasner Hypermodules

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Abstract: Let R be a Krasner hyperring. In this paper, we prove a factorization theorem in the category of Krasner R-hypermodules with inclusion single-valued R-homomorphisms as its morphisms. Then, we prove various isomorphism theorems for a smaller category, i.e., the category of Krasner R-hypermodules with strong single-valued R-homomorphisms as its morphisms. In addition, we show that the latter category is balanced. Finally, we prove that for every strong single-valued R-homomorphism $f\colon A \to B$ and $a \in A$, we have $Ker(f) + a = a + Ker(f) = \{x \in A \mid f(x) = f(a)\}$.

Keywords: Krasner hyperring; Krasner hypermodule; isomorphism theorem; factorization theorem; category theory

1. Introduction

Algebraic hyperstructure theory addresses the study of algebraic objects endowed with multivalued operations, which are intended to generalize classical algebraic structures as groups, rings, or modules [1–5]. In the framework of that theory, hypergroups play a major role. A hypergroup is basically a set endowed by an associative multivalued binary operation, which fulfills an additional condition called reproducibility. Their inspection may reveal complex relationships among algebra, combinatorics, graphs, and numeric sequences [6,7]. Indeed, hyperstructures are inherently more complicated and bizarre than their classical counterparts. Hence, one of the main research directions in hyperstructure theory consists of identifying a subclass of a rather generic hyperstructure on the basis of a reasonable set of axioms, symmetries, or properties and proceeding with their analysis, in order to construct a theory that is at the same time general, profound, and beautiful.

Here, we consider one of the most important classes of hypergroups, that is the class of canonical hypergroups introduced by Mittas in [8]. This class constitutes the additive hyperstructure of many other hyperstructures, for example some types of hyperrings, hyperfields, and hypermodules. Notably, hyperrings and hyperfields, whose additive hyperstructure is a canonical hypergroup, were firstly introduced by Krasner [9]. Later, various authors defined and studied many other kinds of hyperrings and hypermodules; see, e.g., [5,10–12]. In the context of canonical hypergroups, Madanshekaf [11] and Massouros [12] studied hypermodules whose additive structure is a canonical hypergroup equipped with a single-valued external multiplication. We call *Krasner hypermodule* a hypermodule equipped with a canonical hypergroup as its additive hyperstructure and an external single-valued multiplication, in order to distinguish it from other types of hypermodules. In fact, the name "Krasner" has been given to this kind of hypermodule in [13], inspired by the structure of the Krasner hyperring [9], even though such a hyperstructure has been previously considered by other authors in [11,12]. In fact, Krasner hypermodules are meant to generalize the concept of the Krasner hyperring.

In this work, our aim is to take a detailed look at isomorphism theorems for Krasner hypermodules. The relevance of isomorphism theorems is undoubtable, in all algebraic studies. In fact, in every category of algebraic structures, homomorphisms describe the relationship between objects. However, due to the multivalued nature of hyperstructure algebra, the analysis of isomorphisms on hypermodules is very involved. In fact, one encounters various kinds of homomorphisms when studying Krasner hypermodules. In [14–16], both single-valued and multi-valued homomorphisms were introduced; and in both classes, at least three different kinds of homomorphisms can be considered, that is the so-called inclusion homomorphisms, strong homomorphisms, and weak homomorphisms, depending on their behavior with respect to the multivalued addition. We mention that Davvaz proved in [17] that for a strong hyperring homomorphism, a first isomorphism theoremholds provided that its kernel is normal. Furthermore, Verlajan and Asokkumar proved in [18] similar results without the latter condition for a strong hyperring homomorphism in a different and more general class of hyperrings.

Here, by fixing a Krasner hyperring R, we consider the class of Krasner hypermodules over R together with inclusion, strong and weak single-valued R-homomorphisms among them (with the usual composition of mappings), and the so-called *primary categories* of Krasner R-hypermodules [19], which are denoted by ${}_R h\mathbf{mod}$, ${}_{R_s} h\mathbf{mod}$, and ${}_{R_w} h\mathbf{mod}$, respectively. When dealing with multivalued R-homomorphisms, the composition of morphisms is defined in a more general way, and we obtain different categories whose morphisms are multivalued R-homomorphisms. The corresponding categories, which are called *secondary categories*, fall outside the scope of the present work. We note in passing that in [12], Massouros worked with multivalued R-homomorphisms, while Madanshekaf in [11] considered only strong single-valued R-homomorphisms.

This paper is organized as follows. In Section 2, we state some basic concepts, definitions, and basic results needed to develop our work. Krasner hypermodules are presented in Section 3 together with their main properties. Section 4 contains the main results of this paper. In particular, by considering a generic Krasner hyperring R, Theorem 1 provides a factorization theorem in the category ${}_R h\mathbf{mod}$. Subsequently, we prove various isomorphism theorems in ${}_{R_s} h\mathbf{mod}$. The last section contains concluding remarks and a suggestion for possible further research.

2. Preliminaries

Throughout this paper, we use a few basic concepts and definitions that belong to standard terminology in hyperstructure theory. For more details about hyperstructures, the interested reader can refer to the classical references [1–5].

Let H be a non-empty set; let $P(H)$ denote the set of all subsets of H; and let $P^*(H) = P(H) \setminus \{\emptyset\}$. Then, H together with a map:

$$\begin{aligned} + : H \times H &\mapsto P^*(H) \\ (a, b) &\mapsto a + b, \end{aligned}$$

is called a *hypergroupoid* and is denoted by $(H, +)$. The operation $+$ is called the *hyperoperation* on H. Let $A, B \subseteq H$. The hyperoperation $A + B$ is defined as:

$$A + B = \bigcup_{(a,b) \in A \times B} a + b.$$

If there is no confusion, for simplicity $\{a\}$, $A + \{b\}$, and $\{a\} + B$ are denoted by a, $A + b$, and $a + B$, respectively.

Definition 1. *A non-empty set S together with the hyperoperation $+$, denoted by $(S, +)$, is called a* semihypergroup *if for all $x, y, z \in S$, $(x + y) + z = x + (y + z)$, that is, the hyperoperation is associative.*

Definition 2. *A semihypergroup $(H,+)$ satisfying the* reproducibility condition $x+H=H+x=H$ *for every $x \in H$ is called a* hypergroup.

Let e be an element of a semihypergroup $(H,+)$ such that $e+x=x$ for all $x \in H$. Then, e is called a *left scalar identity*. A right scalar identity is defined analogously. Moreover, an element x of a semihypergroup $(H,+)$ is called a scalar identity if it is both a left and right scalar identity. A scalar identity in a semihypergroup H is unique, if it exists. In that case, we denote the scalar identity of H by 0_H. Let 0_H be the scalar identity of hypergroup $(H,+)$ and $x \in H$. An element $x' \in H$ is called an *inverse* of x in $(H,+)$ if $0_H \in (x+x') \cap (x'+x)$.

Definition 3. *A non-empty set M together with the hyperoperation $+$ is called a* canonical hypergroup *if the following axioms hold:*

1. *$(M,+)$ is a semihypergroup (associativity);*
2. *$(M,+)$ is commutative (commutativity);*
3. *there is a scalar identity 0_M (existence of scalar identity);*
4. *for every $x \in M$, there is a unique element denoted by $-x$ called the inverse of x such that $0_M \in x+(-x)$, which for simplicity, we write as $0_H \in x-x$ (existence of inverse);*
5. *for all $x,y,z \in M$, it holds that $x \in y+z \Longrightarrow y \in x-z$ (reversibility).*

In the sequel, $-x$ denotes the inverse of x in the hypergroup $(M,+)$, and we write $x-y$ instead of $x+(-y)$. If there is no confusion, sometimes we omit the indication of the hyperoperation in a hypergroup, and for simplicity, we write M instead of $(M,+)$.

Definition 4. *Let M be a hypergroup. A non-empty subset N of M is called a canonical subhypergroup of M, denoted by $N \leq M$, if it is a canonical hypergroup itself.*

It is easy to verify that $N \leq M$ if and only if $N \neq \emptyset$ and $x-y \subseteq N$ for all $x,y \in N$. Clearly, it follows that $0_M \in N$. Hereafter, we recall some results discussed in [20] concerning the structure of the quotient of a canonical hypergroup with respect to a canonical subhypergroup.

Let $(M,+)$ be a canonical hypergroup; let N be an arbitrary canonical subhypergroup of M; and set $\frac{M}{N} := \{x+N \mid x \in M\}$. Consider the hyperoperation $+'$ on $\frac{M}{N}$ defined as:

$$(x+N)+'(y+N) = \{t+N \mid t \in x+y\}. \tag{1}$$

For notational convenience, we may write $\bar{x}$ instead of $x+N$.

Lemma 1. *([20], Lemma 3.1) $\bar{x} \cap \bar{x'} \neq \emptyset$ implies $\bar{x} = \bar{x'}$.*

Proposition 1. *([20], Proposition 3.2) For every canonical hypergroup M, if $N \leq M$, then $(\frac{M}{N},+')$ is a canonical hypergroup.*

Proposition 2. *([20], Proposition 3.3) The hyperoperation $+'$ on $\frac{M}{N}$ is the same as $+''$ defined by:*

$$(x+N)+''(y+N) := \{t+N \mid t \in \bar{x}+\bar{y}\}.$$

Finally, we recall some concepts from category theory; see, e.g., [21].

Definition 5. *In every category $\mathcal{C}$,*

1. *a morphism $f\colon B \mapsto C$ is said to be a mono if for every $g,h\colon A \mapsto B$ the following implication holds:*

$$f \circ g = f \circ h \Longrightarrow g = h.$$

2. *a morphism $f\colon A \mapsto B$ is said to be an epi if for every $g, h\colon B \mapsto C$ the following implication holds:*

$$g \circ f = h \circ f \Longrightarrow g = h.$$

3. *A morphism $f\colon A \mapsto B$ of a category $\mathcal{C}$ is called an iso in $\mathcal{C}$ if there exists some $g\colon B \mapsto A$ (in $\mathcal{C}$) such that $f \circ g = id_B$ and $g \circ f = id_A$. In that case, g is denoted by f^{-1}.*

3. Krasner Hypermodules

We start this section with a notable generalization of classical rings introduced by Krasner in [9] and called "Krasner hyperrings" by many authors; see [4,13–16,19].

Definition 6. *A non-empty set R together with the hyperoperation $+$ and the operation $\cdot$ is called a Krasner hyperring if the following axioms hold:*

1. *$(R, +)$ is a canonical hypergroup;*
2. *$(R, \cdot)$ is a semigroup including 0_R as a bilaterally-absorbing element, that is $0_R \cdot x = x \cdot 0_R = 0_R$ for all $x \in R$;*
3. *$(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.*

Definition 7. *Let $(R, +, \cdot)$ be a hyperring, and let S be a non-empty subset of R that is closed under the maps $+$ and $\cdot$ in R. If S is itself a hyperring under these maps, then S is called a subhyperring of R. A subhyperring I of R is called a left (resp., right) hyperideal if $r \in R$ and $i \in I$ implies $r \cdot i \in I$ (resp., $i \cdot r \in I$). A left and right hyperideal I is simply called a hyperideal of R.*

We use the notation $I \trianglelefteq R$ when the set I is a hyperideal of R. Moreover, we write $I \triangleleft R$ if $I \trianglelefteq R$ and $I \subsetneq R$, that is I is a proper hyperideal.

From now on, we overlook the name "Krasner" and simply use "hyperring". Furthermore, by R, we mean a (Krasner) hyperring. Usually, R is said to have a unit element if there exists $y \in R$ such that $x \cdot y = y \cdot x = x$ for every $x \in R$. If the element y exists, then it is unique, and we use the notation 1_R to denote it. In that case, we say that R is unitary.

Definition 8. *Let X and Y be two non-empty sets. A map $* : X \times Y \mapsto Y$ sending (x, y) to $x * y \in Y$ is called a left external multiplication on Y. That operation is naturally extended to any $U \subseteq X$ and $V \subseteq Y$ by $u * V := \{u * v \mid v \in V\}$ and $U * v := \{u * v \mid u \in U\}$.*

Analogously to the previous definition, a right external multiplication on Y is defined by $* : Y \times X \mapsto Y$ sending (y, x) to $y * x \in Y$.

Definition 9. *Let $(R, +, \cdot)$ be a hyperring. A canonical hypergroup $(A, +)$ together with the left external multiplication $* : R \times A \mapsto A$ is called a left Krasner hypermodule over R if for all $r_1, r_2 \in R$ and for all $a_1, a_2 \in A$, the following axioms hold:*

1. *$r_1 * (a_1 + a_2) = r_1 * a_1 + r_1 * a_2$;*
2. *$(r_1 + r_2) * a_1 = r_1 * a_1 + r_2 * a_1$;*
3. *$(r_1 \cdot r_2) * a_1 = r_1 * (r_2 * a_1)$;*
4. *$0_R * a_1 = 0_A$.*

Remark 1.

(i) *If A is a left Krasner hypermodule over a Krasner hyperring R, then we say that A is a left Krasner R-hypermodule. Similarly, the right Krasner R-hypermodule is defined by the map $* : A \times R \mapsto A$ satisfying the similar properties mentioned in Definition 9 with affection on the right.*

(ii) *If R is a hyperring with the unit element 1_R and A is a Krasner R-hypermodule satisfying $1_R * a = a$ (resp. $a * 1_R = a$) for all $a \in A$, then A is said to be a unitary left (resp. right) Krasner R-hypermodule.*

(iii) *From now on, for convenience, every hyperring R is assumed to have the unit element 1_R and by "an R-hypermodule A", we mean a unitary left Krasner R-hypermodule, unless otherwise stated.*

Definition 10. *A non-empty subset B of an R-hypermodule A is said to be an R-subhypermodule of A, denoted by $B \leq A$, if B is an R-hypermodule itself, that is for all $x, y \in B$ and all $r \in R$, $x - y \subseteq B$ and $r * x \in B$.*

Remark 2. *We list here below some examples and properties related to R-subhypermodules, whose simple proofs are omitted for brevity:*

1. *Every hyperring R is an R-hypermodule, and every $I \trianglelefteq R$ is an R-subhypermodule of R.*
2. *Let R be a hyperring. Every canonical hypergroup A can be considered as an R-hypermodule with the trivial external multiplication $r * a = 0_A$ for every $r \in R$ and $a \in A$.*
3. *Let A be an R-hypermodule and $\emptyset \neq B \subseteq A$. For every $I \trianglelefteq R$,*

$$IB = \{a \in \sum_{i=1}^{m} r_i * n_i \mid r_i \in I,\ n_i \in B,\ m \in \mathbb{Z}^+\}$$

is an R-subhypermodule of A.
4. *Let $\{A_i\}_{i \in I}$ be a family of R-subhypermodules of A. Then, $\cap_{i \in I} A_i \leq A$.*

Definition 11. *Let A and B be R-hypermodules. A function $f : A \mapsto B$ that satisfies the conditions:*

1. $f(x + y) \subseteq f(x) + f(y)$;
2. $f(r * x) = r * f(x)$

for all $r \in R$ and all $x, y \in A$, is said to be an (inclusion) R-homomorphism from A into B. Moreover, if the equality holds in Point 1, then f is called a strong (or good) R-homomorphism; and if $f(x + y) \cap f(x) + f(y) \neq \emptyset$ holds instead of 1, then f is called a weak R-homomorphism.

The category whose objects are all R-hypermodules and whose morphisms are all R-homomorphisms is denoted by ${}_R h\mathbf{mod}$. The class of all R-homomorphisms from A into B is denoted by $hom_R(A, B)$. Moreover, we denote by ${}_{R_s} h\mathbf{mod}$ (resp., ${}_{R_w} h\mathbf{mod}$) the category of all R-hypermodules whose morphisms are strong (resp., weak) R-homomorphisms. The class of all strong (resp., weak) R-homomorphisms from A into B is denoted by $hom^s_R(A, B)$ (resp., $hom^w_R(A, B)$). It is easy to see that ${}_{R_s} h\mathbf{mod}$ is a subcategory of ${}_R h\mathbf{mod}$, and ${}_R h\mathbf{mod}$ is a subcategory of ${}_{R_w} h\mathbf{mod}$. Using standard notations, we express this as:

$${}_{R_s} h\mathbf{mod} \preceq {}_R h\mathbf{mod} \preceq {}_{R_w} h\mathbf{mod}.$$

The categories are usually called the *primary categories* of R-hypermodules; see, e.g., [19].

Now, let $f \in hom_R(A, B)$, and define:

$$Ker(f) := \{x \in A \mid f(x) = 0_B\},$$
$$Im(f) := \{y \in B \mid \exists x \in A : \quad f(x) = y\}.$$

For further reference, we gather hereafter some information about $Ker(f)$ and $Im(f)$ from [13]:

- $Ker(f)$ is an R-subhypermodule of A.
- Clearly, $Im(f)$ may not be an R-subhypermodule of B.
- For every morphism f in ${}_{R_s} h\mathbf{mod}$, $Im(f)$ is always an R-subhypermodule of the codomain of f.

Recall that if A is an R-hypermodule and B is a non-empty subset of A such that B is itself a hypermodule over R, then B is said to be an R-subhypermodule of B denoted by $B \leq A$. Clearly, for every R-hypermodule A, $\{0_A\}$ and A are two R-subhypermodules of A. In the following, we construct another R-hypermodule from A and $B \leq A$ called the quotient R-hypermodule.

Proposition 3. *Let $B \leq A$ be an R-hypermodule. Then, the canonical hypergroup $(\frac{A}{B}, +')$ with the hyperoperation $+'$ defined as in (1) is an R-hypermodule with the external multiplication $\odot$ defined by:*

$$r \odot (x + B) = r * x + B,$$

for $x, y \in A$ and $r \in R$.

Proof. As mentioned in Proposition 1, $(\frac{A}{B}, +')$ is a canonical hypergroup. First, we show that $\odot$ is well defined. Hence, let $r_1 = r_2 \in R$ and $x + B = y + B$. We prove that $r_1 * x + B = r_2 * y + B$. In fact, since $x \in y + b$ for $b \in B$, we have $r_1 * x \in r_1 * y + r_1 * b \subseteq r_1 * y + B$, so $r_1 * x + B \subseteq r_1 * y + B$. Consequently $r_1 * x + B \subseteq r_2 * y + B$. Analogously, $r_1 * x + B \supseteq r_2 * y + B$. Thus, $r_1 \odot (x + B) = r_2 \odot (y + B)$ and $\odot$ is well defined.

Next, we check the axioms mentioned in Definition 9. Let $r_1, r_2 \in R$ and $x + B, y + B \in \frac{A}{B}$. Clearly, $r_1 * 0_A = 0_A$, $0_R * x = 0_A$ in A. Moreover, the zero element of $\frac{A}{B}$ is $0_A + B$ (or B). Therefore, $r_1 \odot (0_A + B) = r_1 * 0_A + B = 0_A + B$ and $0_R \odot (x + B) = 0_R * x + B = 0_A + B$ imply that Axiom 4 of Definition 9 holds true. In order to prove the first axiom of Definition 9, note that:

$$\begin{aligned}
r_1 \odot [(x + B) +' (y + B)] &= r_1 \odot \{z + B \mid z \in x + B + y + B\} \\
&= \{r_1 * z + B \mid z \in x + B + y + B\} \\
&= \{a + B \mid a = r_1 * z, \quad z \in x + B + y + B\} \\
&= \{a + B \mid a \in r_1 * (x + B + y + B)\} \\
&= \{a + B \mid a \in r_1 * x + B + r_1 * y + B\} \\
&= (r_1 * x + B) +' (r_1 * y + B) = [r_1 \odot (x + B)] +' [r_1 \odot (y + B)].
\end{aligned}$$

Furthermore, the second axiom of Definition 9 holds. Indeed,

$$\begin{aligned}
(r_1 + r_2) \odot (x + B) &= \{r \mid r \in r_1 + r_2\} \odot (x + B) \\
&= \{r * x + B \mid r \in r_1 + r_2\} \\
&= \{a + B \mid a = r * x, \quad r \in r_1 + r_2\} \\
&= \{a + B \mid a \in (r_1 + r_2) * x\} = \{a + B \mid a \in r_1 * x + r_2 * x\}.
\end{aligned}$$

By Proposition 2,

$$\begin{aligned}
\{a + B \mid a \in r_1 * x + r_2 * x\} &= \{a + B \mid a \in r_1 * x + B + r_2 * x + B\} \\
&= (r_1 * x + B) +' (r_2 * x + B) = [r_1 \odot (x + B)] +' [r_2 \odot (x + B)].
\end{aligned}$$

Thus:

$$(r_1 + r_2) \odot (x + B) = r_1 \odot (x + B)] +' [r_2 \odot (x + B)].$$

Finally, as concerns the third axiom of Definition 9, we have:

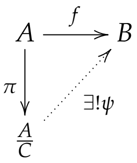
$$\begin{aligned}
(r_1 \cdot r_2) \odot (x + B) &= ((r_1 \cdot r_2) * x) + B \\
&= (r_1 * (r_2 * x)) + B \\
&= r_1 \odot [(r_2 * x) + B] = r_1 \odot [r_2 \odot (x + B)],
\end{aligned}$$

and the proof is complete. □

4. Main Results

Factorization theorems are keystone results in abstract algebra. Indeed, these kinds of theorems relate the structure of two objects between which a homomorphism is given, in terms of the kernel and the image of the homomorphism. Moreover, they are preliminary steps toward more stringent results, where homomorphisms are replaced by isomorphisms. We start this section with a factorization theorem for an R-homomorphism between R-hypermodules.

Theorem 1. *(Factorization theorem) Let $f \in hom_R(A, B)$. If C is any R-subhypermodule included in $Ker(f)$, then there exists a unique R-homomorphism $\psi \in hom_R(\frac{A}{C}, B)$ such that $f = \psi \circ \pi$, where $\pi\colon A \mapsto \frac{A}{C}$ is the canonical quotient map. Hence, the diagram:*



commutes.

Proof. By assumption, $C \subseteq Ker(f) \subseteq A$. Define $\psi\colon \frac{A}{C} \mapsto B$ by $\psi(x + C) = f(x)$. Clearly, ψ makes the diagram commute. Now, we need to check that it is a unique well-defined R-homomorphism.

Suppose that $x + C = y + C$. Then, $(x - y) \cap C \neq \emptyset$. Clearly, $C \subseteq Ker(f)$ implies $(x - y) \cap Ker(f) \neq \emptyset$. Then, there is $z \in x - y$ with $f(z) = 0_B$. Thus, $x \in z + y$. Therefore, $f(x) \in f(z + y) \subseteq f(z) + f(y) = f(y)$. Consequently, $f(x) = f(y)$. Therefore, $\psi(x + C) = \psi(y + C)$, and ψ is well defined.

Moreover,

$$\begin{aligned}
\psi((x + C) +' (y + C)) &= \psi(\{z + C \mid z \in x + C + y + C\}) \\
&= \psi(\{z + C \mid z \in x + y\}) \\
&= \{\psi(z + C) \mid z \in x + y\} \\
&= \{f(z) \mid z \in x + y\} \\
&\subseteq f(x) + f(y) = \psi(x + C) + \psi(y + C).
\end{aligned}$$

Now, assume $r \in R$. Then:

$$\psi(r \odot (x + C)) = f(r * x) = r * f(x) = r * \psi(x + C).$$

Suppose there is another map ρ satisfying the conditions. Then, $\rho(\pi(x)) = f(x)$ for all $x \in A$, but $\psi(\pi(x)) = f(x)$ for all $x \in A$. This means $\rho(x + C) = f(x)$ and $\psi(x + C) = f(x)$. Thus, $\rho = \psi$, so that it is unique. □

A mono in the category ${}_R h\mathbf{mod}$ (resp., ${}_{R_s} h\mathbf{mod}$) is called an *R-monomorphism* (resp., *strong R-monomorphism*). An epi in the category ${}_R h\mathbf{mod}$ (resp., ${}_{R_s} h\mathbf{mod}$) is called an *R-epimorphism* (resp., *strong R-epimorphism*). An iso in the category ${}_R h\mathbf{mod}$ (resp., ${}_{R_s} h\mathbf{mod}$) is called an *R-isomorphism* (resp., *strong R-isomorphism*). When we say $f : A \mapsto B$ is an R-isomorphism in ${}_{R_s} h\mathbf{mod}$, automatically, f is a strong R-homomorphism.

Remark 3. *We point out some properties of R-isomorphisms in ${}_R h\mathbf{mod}$ and ${}_{R_s} h\mathbf{mod}$.*

(i) *An iso in the category ${}_R h\mathbf{mod}$ (or an R-isomorphism) is surjective and injective, i.e., bijective. For this, let $f : A \mapsto B$ be an iso in ${}_R h\mathbf{mod}$. Therefore, $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$. Clearly, $f \circ f^{-1} = id_B$ implies f is surjective. Furthermore, if $f(x) = f(y)$, then $f^{-1} \circ f = id_A$ implies that $x = y$.*

(ii) *$f : A \mapsto B$ is an R-isomorphism in ${}_{R_s} h\mathbf{mod}$ if and only if it is bijective. To show this fact, suppose $f : A \mapsto B$ is bijective. Then, $f^{-1}\colon B \mapsto A$ is also an R-homomorphism. Indeed, for every $y_1, y_2 \in B$, there are (unique) $x_1, x_2 \in A$ such that $f(x_i) = y_i$ for $i \in \{1, 2\}$, and since $f(x_1 + x_2) = f(x_1) + f(x_2) =$*

$y_1 + y_2$, we obtain $f^{-1}(y_1 + y_2) = x_1 + x_2 = f^{-1}(y_1) + f^{-1}(y_2)$. Finally, $f(r * x) = r * f(x)$ implies $f^{-1}(r * y) = r * f^{-1}(y)$, and so, f^{-1} is an R-homomorphism. Therefore, $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$, and f is an R-isomorphism. The converse fact follows from (i) since every R-isomorphism in ${}_{R_s}h\boldsymbol{mod}$ is an R-isomorphism in ${}_Rh\boldsymbol{mod}$.

Notation 1. *If there exists an iso between R-hypermodules A and B in ${}_Rh\boldsymbol{mod}$ (resp., ${}_{R_s}h\boldsymbol{mod}$), we use the notation $A \cong B$ (resp., $A \cong_s B$). Moreover, if $B \leq A$ and $x \in A$, for convenience, we use $\bar{x}$ instead of $x + B$.*

Theorem 2. *(First strong isomorphism theorem) If $\varphi \in hom^s_R(A, B)$, then:*

$$\frac{A}{Ker(\varphi)} \cong_s \varphi(A) = Im(\varphi)$$

as R-hypermodules.

Proof. First note that $Im(\varphi)$ is an R-subhypermodule of B. Let $K = Ker(\varphi)$. Define a map $f\colon \frac{A}{K} \mapsto Im(\varphi)$ by $f(\bar{x}) = \varphi(x)$ for all $x \in A$. Suppose that $\bar{x} = \bar{y}$, where $x, y \in A$. Then, $x \in \bar{y}$ and $x \in y + k$ for some $k \in K$. Hence,

$$\varphi(x) \in \varphi(y + k) = \varphi(y) + \varphi(k) = \varphi(y) + 0_B = \varphi(y).$$

Therefore, $\varphi(x) = \varphi(y)$. Thus, $f(\bar{x}) = f(\bar{y})$, and the map f is well defined.
If $x, y \in A$, then:

$$\begin{aligned} f(\bar{x} +' \bar{y}) &= f\left(\{\bar{z} \mid z \in \bar{x} + \bar{y}\}\right) \\ &= f\left(\{\bar{z} \mid z \in x + y\}\right) \\ &= \{f(\bar{z}) \mid z \in x + y\} = \{\varphi(z) \mid z \in x + y\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(\bar{x}) + f(\bar{y}) &= \varphi(x) + \varphi(y) \\ &= \varphi(x + y) = \{\varphi(z) \mid z \in x + y\}. \end{aligned}$$

Hence, $f(\bar{x} +' \bar{y}) = f(\bar{x}) + f(\bar{y})$. Moreover,

$$f(r \odot \bar{x}) = f(\overline{r * x}) = \varphi(r * x) = r * \varphi(x) = r * f(\bar{x}).$$

Thus, f is an R-homomorphism. Let $\bar{x}, \bar{y} \in \frac{A}{K}$ be such that $f(\bar{x}) = f(\bar{y})$. Then, $\varphi(x) = \varphi(y)$. This means that $0_B \in \varphi(x) - \varphi(y) = \varphi(x - y)$, that is $\varphi(z) = 0_B$ for some $z \in x - y$. Therefore, $z \in K$. Now,

$$z \in x - y \Longrightarrow x \in z + y \Longrightarrow x \in y + K.$$

Since $x \in x + K$ and $\{a + K \mid a \in A\}$ is a partition for A, we have $x + K = y + K$, i.e., $\bar{x} = \bar{y}$, and hence, f is injective. Clearly, f is surjective. Thus, according to the part (ii) of Remark 3, we have $\frac{A}{K} \cong_s Im(\varphi)$, and the proof is complete. □

Theorem 3. *(Second strong isomorphism theorem) If A and B are R-subhypermodules of an R-hypermodule H, then $\frac{B}{A \cap B} \cong_s \frac{A+B}{A}$ in ${}_{R_s}h\boldsymbol{mod}$.*

Proof. It is clear that we can consider the R-subhypermodule $A + B$ of the R-hypermodule H as an R-hypermodule $A + B$ for which A is an R-subhypermodule. Similarly, the R-subhypermodule B

of the R-hypermodule H as an R-hypermodule B for which $A \cap B$ is an R-subhypermodule. Hence, define $g : B \mapsto \frac{A+B}{A}$ by $g(b) = b + A$ for every $b \in B$. For all $a, b \in B$,

$$\begin{aligned} g(a+b) &= g(\{x \mid x \in a+b\}) \\ &= \{g(x) \mid x \in a+b\} \\ &= \{x + A \mid x \in a+b\} \\ &= a + A +' b + A = g(a) + g(b). \end{aligned}$$

Furthermore,

$$g(r * x) = r * x + A = r * (x + A) = r * g(x).$$

Thus, g is a strong R-homomorphism. Now, $x + A \in \frac{A+B}{A}$ implies that $x \in y + A$ for some $y \in A + B$. That is, $y \in a + b$ for some $a \in A$, $b \in B$. Since $y \in b + A$, we get $y + A = b + A$ by Lemma 1. Thus, $g(b) = b + A = y + A = x + A$, and g is surjective.

Finally, let $b \in B$. Then,

$$\begin{aligned} b \in Ker(g) &\Longleftrightarrow g(b) = 0_{\frac{A+B}{A}} \\ &\Longleftrightarrow b + A = 0_H + A \Longleftrightarrow b \in A. \end{aligned}$$

Thus, $b \in Ker(g)$ if and only if $b \in A \cap B$. Hence, by the first strong isomorphism theorem, $\frac{B}{A \cap B} \cong_s \frac{A+B}{A}$ and the theorem is proved. □

Theorem 4. *(Third strong isomorphism theorem) If A and B are R-subhypermodules of an R-hypermodule H such that* $B \subseteq A$*, then* $\frac{H}{B} / \frac{A}{B} \cong_s \frac{H}{A}$.

Proof. Define a map $h : \frac{H}{B} \mapsto \frac{H}{A}$ by $h(x + B) = x + A$. Then, h is a strong and surjective R-homomorphism with $Ker(h) = \frac{A}{B}$. Therefore, by the first strong isomorphism theorem, $\frac{H}{B} / \frac{A}{B} \cong_s \frac{H}{A}$, and we have the claim. □

Proposition 4. *(i) Let* $f \in hom_R(A, B)$ *or* $f \in hom^s_R(A, B)$*. Then, f is an R-monomorphism if and only if it is injective. (ii)* $f \in hom^s_R(A, B)$ *is injective if and only if* $Ker(f) = \{0_A\}$.

Proof.

(i) (⇐) It is clear.
(⇒) Let $f \in hom_R(A, B)$ (resp., $f \in hom^s_R(A, B)$) be an R-monomorphism and $f(x) = f(y)$. Then, $f(r * x) = f(r * y)$ for an arbitrary $r \in R$. Now, define $g, h \in hom^s_R(R, A)$ with $g(r) = r * x$ and $h(r) = r * y$ for each $r \in R$. Clearly, $f \circ g = f \circ h$. Since f is monic, we have $g = h$. Therefore, $f(1) = g(1)$ implies $x = y$.

(ii) (⇐) Let $Ker(f) = \{0_A\}$ and $f(x) = f(y)$. Therefore, $0_B \in f(x) - f(y) = f(x - y)$. Thus, there is $z \in x - y$ such that $f(z) = 0_A$. By assumption, $z = 0_A$. Now, $0_A \in x - y$ implies $x = y$.
(⇒) Let $z \in Ker(f)$ and f is injective. Therefore, $f(z) = 0_B$. On the other hand $f(0_A) = 0_B$. Thus, $z = 0_A$ by injectivity. □

Corollary 1. *In the category* ${}_{R_s}$***hmod****,* $f \in hom_R(A, B)$ *is an R-monomorphism if and only if* $Ker(f) = \{0_A\}$.

Proposition 5. *(i) In the category* ${}_R$***hmod****, every surjective R-homomorphism is an R-epimorphism. (ii) In the category* ${}_{R_s}$***hmod****, an R-homomorphism is an R-epimorphism if and only if it is surjective.*

Proof.

(i) Let $f \in hom_R(A,B)$ be surjective, and let $g,h \in hom_R(B,C)$. If $g \circ f = h \circ f$, then for all $a \in A$, we have $g(f((a)) = h(f(a))$. Now, let $b \in B$. Clearly, there is $x \in A$ such that $f(x) = b$. Thus, $g(b) = g(f(x)) = h(f(x)) = h(b)$. Hence, f is an R-epimorphism.

(ii) ($\Leftarrow$) It is clear from (i).
($\Rightarrow$) Let $f \in hom^s_R(A,B)$ be an R-epimorphism and $b \in B$. Suppose f is not surjective. Then, $f(A) \neq B$. Define $g,h\colon B \mapsto \frac{B}{f(A)}$ with $g(b) = 0_{\frac{B}{A}}$ and $h(b) = b + f(A)$. Then, clearly, $g \circ f = h \circ f$, but $g \neq h$. This contradiction shows that f is surjective. □

A morphism is said to be a *bimorphism* if it is a mono, as well as an epi. A category is said to be *balanced* when a morphism is a bimorphism if and only if it is an iso.

Proposition 6. *The category* ${}_{R_s}$***hmod*** *is balanced.*

Proof. $f \in hom^s_R(A,B)$ is a bimorphism if and only if f is bijective by Propositions 4(i) and 5(ii) if and only if f is an R-isomorphism by Remark 3(ii). □

We conclude this section with a result concerning $f \in hom^s_R(A,B)$.

Proposition 7. *Let* $f \in hom^s_R(A,B)$ *and* $a \in A$. *Let*

$$f^{-1}(f(a)) := \{x \in A \mid f(x) = f(a)\}.$$

Then, $f^{-1}(f(a)) = Ker(f) + a = a + Ker(f)$.

Proof. Setting $K := Ker(f)$, we have:

$$\begin{aligned} x \in f^{-1}(f(a)) &\Longleftrightarrow f(x) = f(a) \\ &\Longleftrightarrow 0_B \in f(x) - f(a) \\ &\Longleftrightarrow 0_B \in f(x-a) \\ &\Longleftrightarrow z \in (x-a) \cap K \\ &\Longleftrightarrow x \in z + a,\ f(z) = 0_B \Longleftrightarrow x \in K + a, \end{aligned}$$

and the proof is complete. □

5. Conclusions

Krasner hypermodules have been already considered from the standpoint of category theory by various authors in, e.g., [11–13,15], focusing on the properties of different types of homomorphisms, notably the so-called inclusion homomorphisms, strong homomorphisms, and weak homomorphisms, according to their behavior with respect to the multivalued addition. Since morphisms play an important role in every category, one needs a clear understanding of fundamental theorems concerning homomorphisms, such as factorization and isomorphism theorems, in order to pursue fundamental studies in category theory. In this paper, we first studied the primary categories of Krasner hypermodules over a Krasner hyperring, introduced in [19]. In particular, we proved factorization and isomorphism theorems regarding both inclusion and strong single-valued homomorphisms as their morphisms. Moreover, we focused on strong isomorphism theorems between quotient hypermodules, and we showed that the category of Krasner R-hypermodules with strong single-valued homomorphisms is balanced. Arguably, analogous results may have a different and more complex form in other categories of (general) Krasner hypermodules, depending on multivalued homomorphisms. On the basis of preliminary results presented in [13,15,19], we believe that the exploration of factorization and isomorphism theorems for multivalued homomorphisms between hypermodules is a possible direction for further research.

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