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Some Refinements of Ostrowski's Inequality and an Extension to a 2-Inner Product Space

Nicuşor Minculete

Faculty of Mathematics and Computer Science, Transilvania University of Braşov, Iuliu Maniu Street, No. 50, 500091 Braşov, Romania; minculeten@yahoo.com

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Abstract: The purpose of this paper is to prove certain refinements of Ostrowski's inequality in an inner product space. We study extensions of Ostrowski type inequalities in a 2-inner product space. Finally, some applications which are related to the Chebyshev function and the Grüss inequality are presented.

Keywords: 2-inner product space; Cauchy–Schwarz inequality; Ostrowski inequality

MSC: Primary 46C05; secondary 26D15; 26D10

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1. Introduction

In various fields of Mathematics the important applications are given using the inequalities in inner product spaces. Among these, a classic inequality in a complex inner product space X is the inequality of Cauchy–Schwarz [1], given by:

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|, \tag{1}$$

for all $\mathbf{x}, \mathbf{y} \in X$.

If $X = \mathbb{R}^n$ and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then inequality (1) becomes in the quadratic form, thus:

$$(x_1y_1 + ... + x_ny_n)^2 \leq (x_1^2 + ... + x_n^2)(y_1^2 + ... + y_n^2). \tag{2}$$

The equality holds if and only if $\mathbf{x} = \lambda\mathbf{y}$ or $\mathbf{y} = \lambda\mathbf{x}$. This inequality is called the Cauchy–Bunyakovsky–Schwarz inequality (C-B-S inequality).

Many refinements and generalizations for the C-B-S inequality can be found in the theory of inequalities (see [2–6]). An improvement of the C-B-S inequality is given by Ostrowski [7], which proved the following: If $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ and $\mathbf{u} = (u_1, \ldots, u_n)$ are n-tuples of real numbers such that the vectors $\mathbf{x}$ and $\mathbf{y}$ from the space $\mathbb{R}^n$ are linearly independent and

$$\sum_{k=1}^{n} y_k u_k = 0 \quad \text{and} \quad \sum_{k=1}^{n} x_k u_k = 1, \quad \text{then}$$

$$\sum_{k=1}^{n} y_k^2 / \sum_{k=1}^{n} u_k^2 \leq \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 - \left(\sum_{k=1}^{n} x_k y_k\right)^2. \tag{3}$$

The inequality of Ostrowski in an inner product space over the field of complex numbers can be written as:

$$0 \leq \frac{\|\mathbf{y}\|^2}{\|\mathbf{u}\|^2} \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - |\langle \mathbf{x}, \mathbf{y}\rangle|^2, \tag{4}$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{u} \in X$, $\mathbf{u} \neq 0$, and $\langle \mathbf{x}, \mathbf{u}\rangle = 1$, $\langle \mathbf{y}, \mathbf{u}\rangle = 0$.

This inequality represents an improvement of inequality (1). In [8], Dragomir showed some improvements of the celebrated Cauchy–Schwarz inequality in complex inner product spaces.

Next, we make a connection with combinations of several vectors. Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space over the field of real numbers (X is also called *Euclidean space*) and let $\{e_1, e_2, ..., e_n\}$ be an orthonormal system of vectors in X.

For $x \in X$, we take

$$\tilde{x} = x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k \quad \text{and} \quad S_n(x, y) = \langle x, y\rangle - \sum_{k=1}^{n} \langle x, e_k\rangle\langle e_k, y\rangle,$$

where $x, y \in X$.

In [9], Dragomir proved the following inequality:

$$[S_n(x, y)]^2 \leq S_n(x, x)S_n(y, y) \tag{5}$$

where $x, y \in X$. This inequality can also be found in [10].

In Relation (5) the equality holds if and only if $\{x, y, e_1, ..., e_n\}$ are linearly dependent.

From [10] we obtain the following identity:

$$\langle \tilde{x}, \tilde{y}\rangle = \langle x, y\rangle - \sum_{k=1}^{n} \langle x, e_k\rangle\langle e_k, y\rangle = S_n(x, y). \tag{6}$$

In addition, we easily deduce

$$\|\tilde{x}\|^2 = S_n(x, x) = \|x\|^2 - \sum_{k=1}^{n} \langle x, e_k\rangle^2.$$

Inequality (5) is in fact the Cauchy–Schwarz inequality given by the vectors $\tilde{x}, \tilde{y}$, i.e.,

$$|\langle \tilde{x}, \tilde{y}\rangle|^2 \leq \|\tilde{x}\|^2\|\tilde{y}\|^2.$$

Next, we recall a result that will be used in the next section.

Theorem 1. *([4]) In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$|\alpha|^2\|y\|^2\|z\|^2 + \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \geq 2Re\big(\overline{\alpha}\big(\langle x, y\rangle\langle y, z\rangle - \langle x, z\rangle\|y\|^2\big)\big),$$

for all $x, y, z \in X$ and for every $\alpha \in \mathbb{C}$.

The purpose of this paper is to prove certain refinements of Ostrowski's inequality in an inner product space. Therefore, in Section 2 we give some of them. In Section 3 we establish an extension of Ostrowski's inequality in a 2-inner product space. We also show certain types of Ostrowski's inequality in a 2-inner product space. In Section 4 we present some applications which are related to the Chebyshev function and the Grüss inequality. Finally, we mention several conclusions about the development of other inequalities similar to Ostrowski's inequality.

2. Some Refinements of Ostrowski's Inequality

In this section we formulate and prove some extensions of Ostrowski's inequality.

Theorem 2. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an Euclidean space with $\dim_{\mathbb{R}} X \geq n+1$ and let $\{e_1, e_2, ..., e_n\}$ be an orthonormal system of vectors in X. The following inequality*

$$0 \leq \frac{S_n(y,y)}{S_n(u,u)} \leq S_n(x,x)S_n(y,y) - S_n^2(x,y), \tag{7}$$

holds, for all $x, y, u \in X$, with $\langle x,u\rangle = 1 + \sum_{k=1}^n \langle x,e_k\rangle\langle e_k,u\rangle$, $\langle y,u\rangle = \sum_{k=1}^n \langle y,e_k\rangle\langle e_k,u\rangle$ and $\{u, e_1, e_2, ..., e_n\}$ are linearly independent.

Proof. Using Taylor type development, we take the vectors $\tilde{x} = x - \sum_{k=1}^n \langle x,e_k\rangle e_k$, $\tilde{y} = y - \sum_{k=1}^n \langle y,e_k\rangle e_k$ and $\tilde{u} = u - \sum_{k=1}^n \langle u,e_k\rangle e_k$, where $x, y, u \in X$. Since $\langle \tilde{x}, \tilde{u}\rangle = \langle x,u\rangle - \sum_{k=1}^n \langle x,e_k\rangle\langle e_k,u\rangle = 1$ and $\langle \tilde{y}, \tilde{u}\rangle = \langle y,u\rangle - \sum_{k=1}^n \langle y,e_k\rangle\langle e_k,u\rangle = 0$, we apply the inequality of Ostrowski in the Euclidean space X to the vectors $\tilde{x}, \tilde{y}, \tilde{u} \in X$. Therefore, we obtain the statement. □

Remark 1. *Inequality (7) is an improvement of Inequality (5) in an inner product space X over the field of real numbers $\mathbb{R}$ with $\dim_{\mathbb{R}} X \geq n+1$. If $\dim_{\mathbb{R}} X = n$, then we have the Parseval identity $\|x\|^2 = \sum_{k=1}^n \langle x,e_k\rangle^2$, for all $x \in X$ (see [6]). Therefore, we cannot use Ostrowski's inequality, because $\|\tilde{u}\| = 0$.*

Theorem 3. *In an Euclidean space $X = (X, \langle\cdot,\cdot\rangle)$, with $\dim_{\mathbb{R}} X \geq 2$. The following inequality*

$$0 \leq \frac{\|y\|^2\|z\|^2 - \langle y,z\rangle^2}{\|u\|^2\|z\|^2 - \langle u,z\rangle^2}\|z\|^4 \leq \left(\|x\|^2\|z\|^2 - \langle x,z\rangle^2\right)\left(\|y\|^2\|z\|^2 - \langle y,z\rangle^2\right) - \left(\langle x,y\rangle\|z\|^2 - \langle x,z\rangle\langle z,y\rangle\right)^2 \tag{8}$$

holds for all $x, y, z, u \in X$, with $\langle x,u\rangle\|z\|^2 = \|z\|^2 + \langle x,z\rangle\langle z,u\rangle$, $\langle y,u\rangle\|z\|^2 = \langle y,z\rangle\langle z,u\rangle$ and $\{u, z\}$ are linearly independent.

Proof. If we take $e = \frac{z}{\|z\|}$, then we have $\|e\| = 1$ and the inequality of the statement becomes

$$0 \leq \frac{\|y\|^2 - \langle y,e\rangle^2}{\|u\|^2 - \langle u,e\rangle^2} \leq \left(\|x\|^2 - \langle x,e\rangle^2\right)\left(\|y\|^2 - \langle y,e\rangle^2\right) - \left(\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right)^2 \tag{9}$$

which holds for all $x, y, z, u \in X$, with $\langle x,u\rangle = 1 + \langle x,e\rangle\langle e,u\rangle$, $\langle y,u\rangle = \langle y,e\rangle\langle e,u\rangle$, and $\{u, e\}$ are linearly independent. This inequality is in fact Inequality (7), for $n = 1$. Therefore, the statement is true. □

Theorem 4. *In an inner product space X over the field of complex numbers $\mathbb{C}$, the following inequality*

$$0 \leq \frac{\|y\|^2}{\|z\|^2}\left|\frac{\langle x,y\rangle\langle y,z\rangle}{\|y\|^2} - \langle x,z\rangle\right|^2 \leq \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2, \tag{10}$$

holds for all $x, y, z \in X$, $y, z \neq 0$.

Proof. In Theorem 1, if we take $\alpha = \dfrac{\langle x,y\rangle\langle y,z\rangle - \langle x,z\rangle\|y\|^2}{\|y\|^2\|z\|^2}$, where $x, y, z \in X$, $y, z \neq 0$, then we obtain

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq |\alpha|^2\|y\|^2\|z\|^2.$$

Therefore, the statement in Theorem 4 follows. □

Remark 2. (a) *Because for a number $\alpha \in \mathbb{C}$, we have $|\alpha| \geq Re(\alpha)$, then using Inequality (10), we deduce the following inequality*

$$0 \leq \frac{\|y\|^2}{\|z\|^2}\left(Re\left(\frac{\langle x,y\rangle\langle y,z\rangle}{\|y\|^2} - \langle x,z\rangle\right)\right)^2 \leq \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2, \tag{11}$$

which holds for all $x, y, z \in X$, $y, z \neq 0$.
(b) *In an inner product space X over the field of real numbers $\mathbb{R}$, we have*

$$0 \leq \frac{\|y\|^2}{\|z\|^2}\left(\frac{\langle x,y\rangle\langle y,z\rangle}{\|y\|^2} - \langle x,z\rangle\right)^2 \leq \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2, \tag{12}$$

for all $x, y, z \in X$, $y, z \neq 0$. This is an inequality given by Dragomir in [3].
(c) *If we take $\langle x,z\rangle = 1$ and $\langle y,z\rangle = 0$ in Inequality (11), then we proved the inequality of Ostrowski for an inner product space,*

$$0 \leq \frac{\|y\|^2}{\|z\|^2} \leq \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2, \tag{13}$$

for all $x, y, z \in X$.
(d) *If in Inequality (10) we replace z by $z - \dfrac{\langle y,z\rangle}{\|y\|^2}y$, then we deduce another result of Dragomir [3], given in an inner product space X over the field of complex numbers $\mathbb{C}$, that is:*

$$\left|\langle x,y\rangle\langle y,z\rangle - \langle x,z\rangle\|y\|^2\right|^2 \leq \left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right)\left(\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2\right), \tag{14}$$

for all $x, y, z \in X$, $y, z \neq 0$. This inequality is also checked for $y = 0$ or $z = 0$. Therefore, Inequality (14) is true for any $x, y, z \in X$.

Inequalities (10)–(14) are improvements of the Cauchy–Schwarz inequality for an inner product space. In [11], Liu and Gao used a companion of Ostrowski's inequality for functions of bounded variation.

Next, as a working method, we want to study how these inequalities behave in other types of vector spaces. Therefore, we prove new results related to several inequalities in a 2-inner product space. We will present some results regarding the Cauchy–Schwarz inequality and an inequality of Ostrowski type in a 2-inner product space. We will also present some characterizations of the relationship between the two inequalities.

3. Extensions of Several Inequalities to 2-Inner Product Spaces

We present the basic definitions of 2-inner product spaces and of linear 2-normed spaces and we enumerate several elementary properties of these spaces.

In [12], Gähler investigated the concept of linear 2-normed spaces and 2-metric spaces. In [13,14], Diminnie, Gähler and White studied the 2-inner product spaces and their properties.

A classification of results which are related to the theory of 2-inner product spaces can be found in books [15,16]. Here, several properties of 2-inner product spaces are given. In [17] Dragomir et al. showed the corresponding version of Boas-Bellman inequality in 2-inner product spaces. Najati et al. [18] showed the generalized Dunkl-Williams inequality in 2-normed spaces.

Let X be a linear space over the field $\mathbb{K}$ such that $dim_{\mathbb{K}} X \geq 1$, where $\mathbb{K}$ is the set of the real or the complex numbers. A $\mathbb{K}$-valued function $(\cdot,\cdot|\cdot)$ defined on $X \times X \times X$ which verifies the conditions:

(a) $(x,x|z) \geq 0$ and $(x,x|z) = 0$ if and only if x and z are linearly dependent;
(b) $(x,x|z) = (z,z|x)$;
(c) $(x,y|z) = \overline{(y,x|z)}$;
(d) $(\alpha x,y|z) = \alpha(x,y|z)$, for any $\alpha \in \mathbb{K}$;
(e) $(x_1 + x_2, y|z) = (x_1,y|z) + (x_2,y|z)$,

is called a 2-inner product on X. In this context, $(X,(\cdot,\cdot|\cdot))$ is called a 2-inner product space (or 2-pre-Hilbert space).

Some elementary properties of 2-inner products $(\cdot,\cdot|\cdot)$ can be obtained from [15,19], namely:

$$(0,y|z) = (x,0|z) = (x,y|0) = 0\,,\ (z,y|z) = (y,z|z) = 0,$$

$$Re(x,y|z) = \frac{1}{4}[(z,z|x+y) - (z,z|x-y)],$$

$$(x,\alpha y|z) = \overline{\alpha}(x,y|z),\ (x,y|\alpha z) = |\alpha|^2(x,y|z),$$

for all $x,y,z \in X$ and $\alpha \in \mathbb{K}$.

Example 1. *Let $X = (X,\langle\cdot,\cdot\rangle)$ be an inner product space. The standard 2-inner product $(\cdot,\cdot|\cdot)$ is defined on X by:*

$$(x,y|z) := \det\begin{pmatrix} \langle x,y\rangle & \langle x,z\rangle \\ \langle z,y\rangle & \langle z,z\rangle \end{pmatrix} = \langle x,y\rangle\langle z,z\rangle - \langle x,z\rangle\langle z,y\rangle, \tag{15}$$

for all $x,y,x \in X$.

If $(X,(\cdot,\cdot|\cdot))$ is a 2-inner product space, then we can define a function $\|\cdot|\cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x,x|z)},$$

for all $x,z \in X$. This function verifies the following conditions:

(a) (Positivity) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent;
(b) (Symmetry) $\|x|z\| = \|z|x\|$;
(c) (Homogeneity) $\|\alpha x|z\| = |\alpha|\|x|z\|$, for any $\alpha \in \mathbb{K}$;
(d) (Triangle inequality) $\|x_1 + x_2|z\| \leq \|x_1|z\| + \|x_2|z\|$, for all $x_1, x_2, z \in X$.

A function $\|\cdot|\cdot\|$ defined on $X \times X$ and satisfying the above conditions is called a 2-norm on X. In this context, $(X, \|\cdot|\cdot\|)$ is called a linear 2-normed space.

If $X = (X,(\cdot,\cdot|\cdot))$ is a 2-inner product space over the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, then (X, $\|\cdot|\cdot\|$) is a linear 2-normed space and the 2-norm $\|\cdot|\cdot\|$ is generated by a 2-inner product $(\cdot,\cdot|\cdot)$. We remark that the parallelogram law in this space is true:

$$\|x+y|z\|^2+\|x-y|z\|^2=2\|x|z\|^2+2\|y|z\|^2, \tag{16}$$

for all $x,y,z\in X$.

Ehret [20] showed that, if (X, $\|\cdot|\cdot\|$) is a linear 2-normed space, such that (15) holds for every $x,y,z\in X$, then the following:

$$(x,y|z)=\frac{1}{4}(\|x+y|z\|^2-\|x-y|z\|^2),$$

defines a 2-inner product on X.

Using the above properties, we can demonstrate the Cauchy–Schwarz inequality

$$|(x,y|z)|\leq\|x|z\|\|y|z\|, \tag{17}$$

for all $x,y,z\in X$. In relation to (17) the equality holds if and only if x, y and z are linearly dependent.

If $X=(X,\langle\cdot,\cdot\rangle)$ is an inner product space, then Inequality (17) can be written as in [21] or [22], that is:

$$\begin{aligned}&|\langle x,y\rangle\|z\|^2-\langle x,z\rangle\langle z,y\rangle|\\&\leq\sqrt{(\|x\|^2\|z\|^2-|\langle x,z\rangle|^2)(\|y\|^2\|z\|^2-|\langle y,z\rangle|^2)}.\end{aligned} \tag{18}$$

From [23] we get a converse of the Cauchy–Schwarz inequality in 2-inner product spaces, namely: If $x,y,z\in X$ and $a,A\in\mathbb{K}$ verifies the property

$$Re(Ay-x,x-ay|z)\geq 0,$$

or equivalently if

$$\|x-\frac{a+A}{2}y|z\|\leq\frac{1}{2}|A-a|\|y|z\|$$

holds, then

$$0\leq\|x|z\|^2\|y|z\|^2-|(x,y|z)^2|\leq\frac{1}{4}|A-a|^2\|y|z\|^4. \tag{19}$$

The constant $\frac{1}{4}$ is the best possible.

The span of $S\subseteq X$ may be defined as the set of all finite linear combinations of elements of S, thus:

$$\text{span}(S)=\left\{\sum_{k=1}^{n}\lambda_i x_i|\lambda_i\in K,x_i\in S,i\in\{1,...,n\}\right\}.$$

If $X=(X,\langle\cdot,\cdot\rangle)$ is an inner product space and $(\cdot,\cdot|\cdot)$ is the standard 2-inner product then, from the above example, it is easy to see that

$$(x,y|z)=\langle x,y\rangle\langle z,z\rangle-\langle x,z\rangle\langle z,y\rangle=\|z\|^2(\langle x,y\rangle-\frac{1}{\|z\|^2}\langle\langle x,z\rangle z,y\rangle)=$$

$$\|z\|^2(\langle x,y\rangle-\langle\frac{\langle x,z\rangle}{\|z\|^2}z,y\rangle)=\|z\|^2\langle x-\frac{\langle x,z\rangle}{\|z\|^2}z,y\rangle.$$

Therefore, we deduce that the relation

$$(x,y|z) = \|z\|^2\langle x - \frac{\langle x,z\rangle}{\|z\|^2}z, y\rangle \tag{20}$$

is true.

Remark 3. *Let vector $e \in X$ be such that $\|e\| = 1$. Then*

$$(x,y|e) = \langle x - \langle x,e\rangle e, y\rangle.$$

In Relation (17) if we apply the Cauchy–Schwarz inequality, then we obtain the following inequality

$$|(x,y|z)| \le \|y\|\|z\|\sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2}.$$

Theorem 5. *Let X be an inner product space over the field of complex numbers $\mathbb{C}$. The following identity*

$$\begin{aligned} &\left\| x - \frac{(x,y|z)}{\|y|z\|}y + \lambda u|z\right\|^2 \|y|z\|^2 = \lambda^2\|y|z\|^2\|u|z\|^2 \\ &+ 2\lambda Re((x,u|z)\|y|z\|^2 - (x,y|z)(y,u|z)) + \|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2 \end{aligned} \tag{21}$$

holds, for all $x,y,z,u \in X$, $y,z \ne 0$, $y \notin span\{z\}$, $\lambda \in \mathbb{R}$.

Proof. In the case $y,z \ne 0$, y and z are linearly independent and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \|x - \frac{(x,y|z)}{\|y|z\|^2}y + \lambda u|z\|^2 &= \|x + \lambda u|z\|^2 - 2\lambda\frac{Re(x,y|z)(y,u|z)}{\|y|z\|^2} - \frac{|(x,y|z)|^2}{\|y|z\|^2} \\ = \lambda^2\|u|z\|^2 + 2\lambda Re(x,u|z) &- 2\lambda\frac{Re(x,y|z)(y,u|z)}{\|y|z\|^2} + \frac{\|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2}{\|y|z\|^2}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\|x - \frac{(x,y|z)}{\|y|z\|^2}y + \lambda u|z\|^2 = \frac{1}{\|y|z\|^2}(\lambda^2\|y|z\|^2\|u|z\|^2 \\ &+ 2\lambda Re((x,u|z)\|y|z\|^2 - (x,y|z)(y,u|z)) + \|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2), \end{aligned}$$

which implies the statement. □

Remark 4. *If μ is a real number and we replace in Relation (21) $u \in X$ by $u = \frac{1}{\lambda}\left(-\mu + \frac{(x,y|z)}{\|y|z\|^2}\right)y$, with $\lambda,\mu \in \mathbb{R}, \lambda \ne 0$, then we find the equality given in Theorem 3.4 from [24], thus:*

$$\|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2 = \|x - \mu y|z\|^2\|y|z\|^2 - |(x - \mu y, y|z)|^2.$$

Corollary 1. *In an inner product space X over the field of complex numbers $\mathbb{C}$, the following inequality*

$$0 \le \frac{\|y|z\|^2}{\|u|z\|^2}\left(Re\left(\frac{(x,y|z)(y,u|z)}{\|y|z\|^2} - (x,u|z)\right)\right)^2 \le \|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2 \tag{22}$$

holds for all $x,y,z,u \in X$, $u,z \ne 0$, $u,y \notin span\{z\}$.

Proof. If $u \neq 0, y \neq 0, z \neq 0$, y and z are linearly independent and u and z are also linearly independent, then we apply Theorem 4, and we deduce the following:

$$\lambda^2 \|y|z\|^2 \|u|z\|^2 + 2\lambda Re((x,u|z)\|y|z\|^2 - (x,y|z)(y,u|z)) \\ + \|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2 = \|x - \frac{(x,y|z)}{\|y|z\|}y + \lambda u|z\|^2 \|y|z\|^2 \geq 0,$$

for all $x,y,z,u \in X$, and for any $\lambda \in \mathbb{R}$. Because we have $\|y|z\|^2\|u|z\|^2 > 0$, then the discriminant is negative. Therefore, the statement in Corollary 1 follows. □

Remark 5. *If we take $(x,u|z) = 1$ and $(y,u|z) = 0$ in Inequality (22), then we obtain the inequality of Ostrowski type for 2-inner product spaces over the field of complex numbers,*

$$0 \leq \frac{\|y|z\|^2}{\|u|z\|^2} \leq \|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2, \tag{23}$$

for all $x,y,z,u \in X$, $u,z \neq 0$, $u \notin span\{z\}$.

Inequalities (22) and (23) are improvements of the Cauchy–Schwarz inequality for 2–inner product space. Let $\{e_1,...,e_n\}$ be a system of vectors in a 2–inner product space $X = (X,(\cdot,\cdot|\cdot))$ over the field of real numbers for which $(e_i,e_j|z) = \delta_{ij}$, for any $i,i \in \{1,...,n\}$, and δ_{ij} is the Kronecker delta function, $z \notin \text{span}\{e_i\}, i \in \{1,...,n\}$.

For $x \in X$, we put

$$\tilde{x} = x - \sum_{k=1}^{n}(x,e_k|z)e_k \quad \text{and} \quad S_n(x,y|z) = (x,y|z) - \sum_{k=1}^{n}(x,e_k|z)(e_k,y|z),$$

where $x,y \in X$. It is easy to see the following identity:

$$(\tilde{x},\tilde{y}|z) = (x,y|z) - \sum_{k=1}^{n}(x,e_k|z)(e_k,y|z) = S_n(x,y|z).$$

However, we deduce

$$\|\tilde{x}|z\|^2 = S_n(x,x|z) = \|x|z\|^2 - \sum_{k=1}^{n}|(x,e_k|z)|^2,$$

Since $\|\tilde{x}|z\|^2 \geq 0$ implies

$$\sum_{k=1}^{n}|(x,e_k|z)|^2 \leq \|x|z\|^2, \tag{24}$$

which is an inequality of Bessel type in a 2-inner product space (see [19]).

Using the Cauchy–Schwarz inequality for vectors $\tilde{x},\tilde{y}$, i.e.,

$$|(\tilde{x},\tilde{y}|z)|^2 \leq \|\tilde{x}|z\|^2\|\tilde{y}|z\|^2,$$

we obtain an inequality of Dragomir type:

$$[S_n(x,y|z)]^2 \leq S_n(x,x|z)S_n(y,y|z) \tag{25}$$

where $x, y, z \in X$.

If $z \in \text{span}\{e_1\}$, then $(x, e_1|z) = 0$ and $(e_1, y|z) = 0$, so $S_1(x, y|z) = (x, y|z)$.

Proposition 1. *With the above notations, the following inequality*

$$0 \leq \frac{S_n(y,y|z)}{S_n(u,u|z)} \left(\frac{S_n(x,y|z)S_n(y,u|z)}{S_n(y,y|z)} - S_n(x,u|z) \right)^2 \leq \\ \leq S_n(x,x|z)S_n(y,y|z) - [S_n(x,y|z)]^2 \tag{26}$$

holds for all $x, y, z, u \in X$, $z \neq 0$, $y - \sum_{k=1}^{n} (y, e_k|z)e_k$, $u - \sum_{k=1}^{n} (u, e_k|z)e_k \notin span\{z\}$.

Proof. Using Corollary 1, we have

$$0 \leq \frac{\|\tilde{y}|z\|^2}{\|\tilde{u}|z\|^2} (Re(\frac{(\tilde{x},\tilde{y}|z)(\tilde{y},\tilde{u}|z)}{\|\tilde{y}|z\|^2} - (\tilde{x},\tilde{u}|z)))^2 \leq \|\tilde{x}|z\|^2\|\tilde{y}|z\|^2 - |(\tilde{x},\tilde{y}|z)|^2,$$

for all $x, y, z, u \in X$, $z \neq 0$, $y - \sum_{k=1}^{n} (y, e_k|z)e_k$, $u - \sum_{k=1}^{n} (u, e_k|z)e_k \notin \text{span}\{z\}$. By substitution we deduce the statement. □

Remark 6. *Obviously, Inequality (26) represents an improvement of Inequality (25).*

4. Applications

(a) Let $X = (X, \langle \cdot, \cdot \rangle)$ be an inner product space over the field of real numbers i.e., X is an Euclidean space. For $n = 1$ in Inequality (25) and vector $e \in X$ with $\|e|z\| = 1$, we have

$$[(x,y|z) - (x,e|z)(e,y|z)]^2 \leq (\|x|z\|^2 - (x,e|z)^2)(\|y|z\|^2 - (y,e|z)^2). \tag{27}$$

Using Proposition 1 for $n = 1$, we obtain a refinement of Inequality (27), given by:

$$0 \leq A \leq (\|x|z\|^2 - (x,e|z)^2)(\|y|z\|^2 - (y,e|z)^2) - [(x,y|z) - (x,e|z)(e,y|z)]^2, \tag{28}$$

where $A = BC^2$, $B = \dfrac{\|y|z\|^2 - (y,e|z)^2}{\|u|z\|^2 - (u,e|z)^2}$ and

$$C = \frac{((x,y|z) - (x,e|z)(e,y|z))((y,u|z) - (y,e|z)(e,u|z))}{\|y|z\|^2 - (y,e|z)^2} \\ -(x,u|z) - (x,e|z)(e,u|z),$$

for all $e, x, y, z \in X, y, u \neq 0$ with $\|e|z\| = 1$ and $\{e, y, z\}, \{e, u, z\}$ are linearly independent.

If $e, x, y, z \in X$ with $\|e|z\| = 1$ and $a, b, A, B \in \mathbb{K}$ are such that

$$Re(Ae - x, x - ae|z) \geq 0 \text{ and } Re(Be - y, y - be|z) \geq 0$$

hold, then using Inequalities (19) and (27) we obtain the following inequality:

$$|(x,y|z) - (x,e|z)(e,y|z)| \leq \frac{1}{4}|A - a||B - b|. \tag{29}$$

(b) In the inner product space $(C^0[a,b], \langle\cdot,\cdot\rangle)$, for $f,g \in C^0[a,b]$, we have $\langle f,g\rangle = \int_a^b f(x)g(x)dx$ and $\|f\| = \sqrt{\int_a^b f^2(x)dx}$.

For $n=1$, we apply Inequality (5) on $C^0[a,b]$ for

$$\left\{ e = \frac{1}{\sqrt{(b-a)}}, x = \frac{1}{\sqrt{(b-a)}} f, y = \frac{1}{\sqrt{b-a}} g \right\},$$

where $f,g \in C^0[a,b]$, and we obtain an inequality in terms of the Chebyshev functional, as follows:

$$[T(f,g)]^2 \leq T(f,f)T(g,g), \tag{30}$$

where $f,g \in C^0[a,b]$ and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx.$$

This inequality in (30) proves the Grüss inequality in [9], which says the following. If f,g belong to $C^0[a,b]$, and if the four constants $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are such that $\gamma_1 \leq f(x) \leq \Gamma_1$ and $\gamma_2 \leq g(x) \leq \Gamma_2$, then we have:

$$|T(f,g)| \leq \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2).$$

But, it is easy to see that $T(f,f) \leq \frac{1}{4}(\Gamma_1 - \gamma_1)^2$, and using inequality (30), we obtain the Grüss inequality.

We apply Inequality (22) on $C^0[a,b]$ for

$$\left\{ x = \frac{1}{\sqrt{(b-a)}} f, y = \frac{1}{\sqrt{b-a}} g, z = \frac{1}{\sqrt{(b-a)}}, u = \frac{1}{\sqrt{b-a}} h \right\},$$

where $f,g,h \in C^0[a,b]$. We remark that $(x,y|z) = det \begin{pmatrix} \langle x,y\rangle & \langle x,z\rangle \\ \langle z,y\rangle & \langle z,z\rangle \end{pmatrix} = T(f,g)$. Consequently, we obtain an inequality in terms of the Chebyshev functional, as follows:

$$0 \leq \frac{T(g,g)}{T(h,h)} \left(\frac{T(f,g)T(g,h)}{T(g,g)} - T(f,h) \right)^2 \leq T(f,f)T(g,g) - [T(f,g)]^2, \tag{31}$$

where $f,g,h \in C^0[a,b], T(g,g), T(h,h) > 0$.

This inequality is an improvement of Inequality (30). Using the inequalities $T(f,f) \leq \frac{1}{4}(\Gamma_1 - \gamma_1)^2$ and $T(g,g) \leq \frac{1}{4}(\Gamma_2 - \gamma_2)^2$, we find a refinement of the Grüss inequality, given by:

$$0 \leq \frac{T(g,g)}{T(h,h)} \left(\frac{T(f,g)T(g,h)}{T(g,g)} - T(f,h) \right)^2 \leq \frac{1}{16}(\Gamma_1 - \gamma_1)^2(\Gamma_2 - \gamma_2)^2 - [T(f,g)]^2, \tag{32}$$

where $f,g,h \in C^0[a,b], T(g,g), T(h,h) > 0$.

5. Conclusions

In this paper, we establish new results related to several inequalities in an inner product space and in a 2-inner product space. Among these inequalities we mention the Cauchy–Schwarz inequality and the Ostrowski inequality. We obtain some inequalities of the Cauchy–Schwarz type and Ostrowski type. By conveniently choosing the inner product space, other interesting inequalities can also be obtained.

In [25] Malčeski and Anevska proved that if $(X, \|.|\|)$ is a 2-normed space and $\{a, b\}$ is a linearly independent subset of X, then the normed space $(X, \|.\|_{a,b,2})$, where $\|x\|_{a,b,2} = (\|x|a\|^2 + \|x|b\|^2)^{\frac{1}{2}}$, is a pre-Hilbert space, endowed with the inner product given by

$$(x, y)_{a,b} = (x, y|a) + (x, y|b).$$

We can demonstrate that $\|x, y\|_{a,b,2} \geq \sqrt{2}\sqrt{|(a, b|x)|}$, for all $x \in X$, from the following:

$$\|x\|_{a,b,2} = ((x, x|a) + (x, x|b))^{\frac{1}{2}} = ((a, a|x) + (b, b|x))^{\frac{1}{2}} \geq$$

$$\sqrt{2}((a, a|x)(b, b|x))^{\frac{1}{4}} \geq \sqrt{2}\sqrt{|(a, b|x)|}.$$

In the same way we proved the inequality:

$$\|x\|_{a,b,2}\|y\|_{a,b,2} \geq 2((x, x|a)(y, y|a)(x, x|b)(y, y|b))^{\frac{1}{4}} \geq 2\sqrt{|(x, y|a)(x, y|b)|}.$$

for any $x, y \in X$.

Another future direction of research can be the study of Ostrowski's inequality for space $(X, \|.\|_{a,b,2})$.

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