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On Cauchy's Interlacing Theorem and the Stability of a Class of Linear Discrete Aggregation Models Under Eventual Linear Output Feedback Controls

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Abstract: This paper links the celebrated Cauchy's interlacing theorem of eigenvalues for partitioned updated sequences of Hermitian matrices with stability and convergence problems and results of related sequences of matrices. The results are also applied to sequences of factorizations of semidefinite matrices with their complex conjugates ones to obtain sufficiency-type stability results for the factors in those factorizations. Some extensions are given for parallel characterizations of convergent sequences of matrices. In both cases, the updated information has a Hermitian structure, in particular, a symmetric structure occurs if the involved vector and matrices are complex. These results rely on the relation of stable matrices and convergent matrices (those ones being intuitively stable in a discrete context). An epidemic model involving a clustering structure is discussed in light of the given results. Finally, an application is given for a discrete-time aggregation dynamic system where an aggregated subsystem is incorporated into the whole system at each iteration step. The whole aggregation system and the sequence of aggregated subsystems are assumed to be controlled via linear-output feedback. The characterization of the aggregation dynamic system linked to the updating dynamics through the iteration procedure implies that such a system is, generally, time-varying.

Keywords: Aggregation dynamic system; Discrete system; Epidemic model; Cauchy's interlacing theorem; Output-feedback control; Stability; Antistable/Stable matrix

1. Introduction

Stability and convergence properties are very important topics when dealing with both continuous- and discrete-time controlled dynamic systems. In this context, one of the most important design tools is the closed-loop stabilization of control systems via the appropriate incorporation of stabilizing controllers; see, for instance, [1–4] and references therein. In particular, in [1], and in some references therein, the robust stable adaptive control of tandem of master-slave robotic manipulators using a multi-estimation scheme is discussed. There are several questions of interest in the analysis, such as the fact that the dynamics may be time-varying and imperfectly known, and the fact that a parallel multi-estimation with eventual switching through time is incorporated into the adaptive controller to improve the transient behavior. The speed estimation and stable control of an induction motor based on the use of artificial neural networks is analyzed in [2]. Strategies of decentralized control, including several applications and stabilization tools, are given in [3,4]. In particular, decentralized control is useful when the various subsystems which are integrated in a whole integrated system are located in separate areas, or when the amount of information needed presents difficulties with regards to obtaining completely optimal suitable performance. Thus, the individual controllers associated with the various subsystems get local information about the corresponding subsystems, and eventually some extra partial information about the remaining ones to achieve stabilization, provided that the neglected coupling dynamics are weak enough. Stabilizing decentralized control designs are described in [3] for networked composite systems. Some technical aspects and the results of non-negative matrices of usefulness to describe the properties and behavior of positive dynamic systems, the robustness of matrices against numerical parameterization perturbations of their entries, and the properties of linear dynamic systems are discussed in [5–8].

This paper focuses on the study of sequences of Hermitian matrices of increasing order which are built via block partition aggregation at each iteration, in such a way that both the current iteration and the next one are Hermitian matrices. The basic mathematical tool is the use of the interlacing Cauchy's theorem of the matrix eigenvalues of the matrices of the sequence, which orders the sequences of the eigenvalues as the iteration progresses [8]. Our main objective is to adapt the interlacing theorem in order to use it to derive stability or convergence conditions of the sequence of matrices, and to use the results for the stability of a large-scale discrete aggregation-type dynamic system [9–14]. The paper is organized as follows. Section 2 is devoted to investigating the properties of boundedness and convergence of the sequences of the determinants and the sequences of eigenvalues as the iteration progresses by aggregation of the updated information while maintaining a Hermitian structure. In the particular case when the matrices describing the problem are real, the updated information has a symmetrical structure. The results are used, in particular, to give stability or anti-stability (in the sense that all the matrix eigenvalues of the matrices of the iterative sequence are unstable) conditions to the matrices used in the standard factorization of Hermitian positive definite matrices. Section 3 extends some of the above results to the convergence of sequences of partitioned Hermitian matrices constructed by aggregation of the updated information. Note that the concept of the convergence of matrices is a discrete counterpart of the matrix stability property in the continuous-time domain, since matrices are stability matrices if all their eigenvalues are in the open complex left-half plane. The basic idea that complex square matrices are convergent if their eigenvalues are within the open unit circle centered at zero is taken into account. An example is discussed concerning a SIR epidemic model with contagions between populations of adjacent clusters in Section 4. Section 5 is devoted to developing an application for the stability of an aggregation discrete-time dynamic controlled system whose order increases by successive incorporations of new subsystems as the iteration index progresses, and whose structure keeps a symmetry. Finally, some conclusions are presented at the end the paper. The relevant mathematical proofs are given in the appendix in order to facilitate a direct reading of the manuscript. The system is assumed to be parameterized by real parameters and controlled by linear output-feedback control laws; it is also assumed that the former whole aggregation system and each new aggregated subsystem at each iteration might eventually be coupled.

Notation and Mathematical Symbols

If M is a square Hermitian matrix, then $M \succ 0$ denotes that it is positive definite and $M \underline{\succ} 0$ denotes it is positive semidefinite. Also, $M \prec 0$ denotes, that it is negative definite.

$$\mathbf{Z}_{0+} = \{z \in \mathbf{Z} : z \geq 0\};\ \mathbf{Z}_{+} = \{z \in \mathbf{Z} : z > 0\},$$

$$\mathbf{R}_{0+} = \{z \in \mathbf{R} : z \geq 0\};\ \mathbf{R}_{+} = \{z \in \mathbf{R} : z > 0\},$$

$$\mathbf{C}_{0+} = \{z \in \mathbf{C} : \operatorname{Re} z \geq 0\};\ \mathbf{C}_{+} = \{z \in \mathbf{C} : \operatorname{Re} z > 0\},$$

$$\overline{n} = \{1, 2, \ldots, n\}$$

I is an identity matrix specified by I_n if it denotes the n-th identity matrix, $A \succ 0$ denotes that the square matrix A is positive definite (positive semidefinite), $A \prec 0$ denotes that the square matrix A is negative definite (respectively, negative semidefinite), $A \succ B$, $A \underline{\succ} B$, $A \prec B$, $A \underline{\prec} B$ denote, respectively, that $A - B \succ 0$, $A - B \underline{\succ} 0$, $A - B \prec 0$ and $A - B \underline{\prec} 0$, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote, respectively, the minimum and maximum eigenvalue of a square real symmetric matrix M, $r(M)$ is the spectral radius of any square complex matrix M, $sp(M)$ is the set of eigenvalues of the Hermitian matrix M.

If such a set is ordered with respect to the partial order relation " $\leq$ " then the ordered spectrum is denoted by $sp_{\leq}(M)$. The superscripts * and T stand, respectively, for complex conjugates or transposes of any vector or matrix, $A \otimes B$ is the Kronecker product of the matrices A, if $A \in \mathbf{C}^{n \times m}$ then its vectorization is a vector $vec(A) \in \mathbf{C}^{n \times m}$ whose components are all the rows of A written in column in its order and respecting the order of its respective entries, $A^{\dagger}$ is the Moore-Penrose pseudoinverse of the matrix A.

2. Technical Results on Partitioned Hermitian Matrices, Cauchy's Interlacing Theorem and Stability

The subsequent result relies on the conditions for the non-singularity of a partitioned Hermitian matrix of order $(n+1)$ which is built by aggregation from a principal Hemitian sub-matrix of order n. Mathematical proof is given in Appendix A.

Lemma 1. *Consider the partitioned matrix* $M\prime = \begin{bmatrix} M & m \\ m^* & d+\widetilde{d} \end{bmatrix} \in \mathbf{C}^{(n+1)\times(n+1)}$ *for any* $n \in \mathbf{Z}_+$, *where* $M \in \mathbf{C}^{n \times n}$ *is Hermitian,* $m \in \mathbf{C}^n$ *and* $d, \widetilde{d} \in \mathbf{R}$. *Then,*

(i) $M\prime$ *is non-singular if and only if* $\det\begin{bmatrix} M & m \\ m^* & \widetilde{d} \end{bmatrix} \neq -d\det M$, *equivalently, if and only if det*$\begin{bmatrix} M & m \\ m^* & 0 \end{bmatrix} \neq -det\begin{bmatrix} M & 0 \\ m^* & d+\widetilde{d} \end{bmatrix}$.

(ii) *Assume that* $M > 0$ *and* $d > 0$. *Then,* $M\prime > 0$ *if and only if* $\det\begin{bmatrix} M & m \\ m^* & \widetilde{d} \end{bmatrix} > -\det\begin{bmatrix} M & 0 \\ m^* & d \end{bmatrix}$. *If* $M > 0$ *and* $d \geq 0$ *then* $M\prime \underline{>} 0$ *if* $\det\begin{bmatrix} M & m \\ m^* & \widetilde{d} \end{bmatrix} \geq 0$. *If* $M < 0$ *and* $d \leq 0$ *then* $M\prime \underline{<} 0$ *if det*$\begin{bmatrix} M & m \\ m^* & \widetilde{d} \end{bmatrix} \leq 0$.

The subsequent result relies on some conditions which guarantee the boundedness of the determinant and eigenvalues of a recursive sequence of Hermitian matrices which were obtained and supported by Lemma 1 and Cauchy's interlacing theorem.

Lemma 2. *Consider the recursive sequence of Hermitian matrices* $\left\{M^{(n)}\right\}_{n=n_0}^{\infty}$ *for a given initial* $M^{(n_0)} \in \mathbf{C}^{n_0 \times n_0}$ *for some given arbitrary* $n_0 \in \mathbf{Z}_+$, *where* $M^{(n)} \in \mathbf{C}^{n \times n}$; $\forall n (\geq n_0) \in \mathbf{Z}_+$, *defined by* $M^{(n+1)} = \begin{bmatrix} M^{(n)} & m^{(n)} \\ m^{(n)*} & d^{(n)}+\widetilde{d}^{(n)} \end{bmatrix}$; $\forall n(\geq n_0) \in \mathbf{Z}_+$ *and assume that there is a real sequence* $\left\{\varepsilon^{(n)}\right\}_{n=n_0}^{\infty} \subset [0, 1)$ *such that* $\frac{1}{k_M^{(n)} k_{\widetilde{d}}^{(n)}} \sqrt{m^{(n)*} m^{(n)}} \leq \varepsilon^{(n)}$, *equivalently* $k_M^{(n)} \geq \frac{1}{k_{\widetilde{d}}^{(n)} \varepsilon^{(n)}} \sqrt{m^{(n)*} m^{(n)}}$; $\forall n(\geq n_0) \in \mathbf{Z}_+$, *where* $k_M^{(n)} = \left|\lambda_{min}\left(M^{(n)}\right)\right| \leq K_M^{(n)} = \left|\lambda_{max}\left(M^{(n)}\right)\right|$; $\left|d^{(n)}\right| \in \left[k_d^{(n)}, K_d^{(n)}\right]$; $\left|\widetilde{d}^{(n)}\right| \in \left[k_{\widetilde{d}}^{(n)}, K_{\widetilde{d}}^{(n)}\right]$; $\forall n(\geq n_0) \in \mathbf{Z}_+$ *with* $K_d^{(n)} \geq K_{\widetilde{d}}^{(n)}$ *and* $k_d^{(n)} \geq k_{\widetilde{d}}^{(n)}$; $\forall n(\geq n_0) \in \mathbf{Z}_+$. *Then, the following properties hold:*

(i) $\limsup_{n\to\infty}\left(\left|\det M^{(n+1)}\right| - \left|\det M^{(n)}\right|\right) \leq 0$, $\left|det M^{(n+1)}\right| \leq \left|det M^{(n)}\right|$ *for any given* $n(\geq n_0) \in \mathbf{Z}_+$ *if* $d^{(n)}$, $\widetilde{d}^{(n)}$ *and* $m^{(n)}$ *satisfy the constraint* $K_{\widetilde{d}}^{(n)} \leq \frac{1-K_d^{(n)}}{1+(2^{n+1}-1)\varepsilon^{(n)}}$; $\forall n(\geq n_0) \in \mathbf{Z}_+$ *with* $K_d^{(n)} \leq 1$, *which becomes* $\left|\widetilde{d}^{(n)}\right| \leq \frac{1-\left|d^{(n)}\right|}{1+(2^{n+1}-1)\varepsilon^{(n)}}$; $\forall n(\geq n_0) \in \mathbf{Z}_+$ *if* $\left|d^{(n)}\right| = K_d^{(n)} = k_d^{(n)} \leq 1$ *and* $\left|\widetilde{d}^{(n)}\right| = K_{\widetilde{d}}^{(n)} = k_{\widetilde{d}}^{(n)}$; $\forall n(\geq n_0) \in \mathbf{Z}_+$.

(ii) *Assume that* $sp_{\leq}\left(M^{(n)}\right)=\left\{\lambda_1^{(n)},\lambda_2^{(n)},\ldots,\lambda_n^{(n)}\right\}$ *and that* $K_{\tilde{d}}^{(n)}\leq\frac{1-K_d^{(n)}}{1+(2^{n+1}-1)\varepsilon^{(n)}}$ *with* $K_d^{(n)}\leq 1$, $\forall n(\geq n_0)\in\mathbf{Z}_+$, *for some given* $n(\geq n_0)\in\mathbf{Z}_+$. *Then, the following relations hold:*

$$\left|\lambda_{n+1}^{(n+1)}\right|\leq\left|\frac{\prod_{i=1}^{n}\lambda_i^{(n)}}{\prod_{i=1}^{n}\lambda_i^{(n+1)}}\right|=\frac{\left|\det M^{(n)}\right|}{\left|\prod_{i=1}^{n}\lambda_i^{(n+1)}\right|}\tag{1}$$

If, furthermore, $M^{(n)}\succeq 0$ *and* $M^{(n+1)}\succeq 0$ *for some* $n(\geq n_0)\in\mathbf{Z}_+$ then

$$\lambda_n^{(n)}\leq\lambda_{n+1}^{(n+1)}\leq\frac{\prod_{i=1}^{n}\lambda_i^{(n)}}{\prod_{i=1}^{n}\lambda_i^{(n+1)}};\ 1\leq\frac{\lambda_{n+1}^{(n+1)}}{\lambda_n^{(n)}}\leq\frac{\prod_{i=1}^{n-1}\lambda_i^{(n)}}{\prod_{i=1}^{n}\lambda_i^{(n+1)}};\ \frac{\prod_{i=2}^{n+1}\lambda_i^{(n+1)}}{\prod_{i=2}^{n}\lambda_i^{(n)}}\leq\frac{\lambda_1^{(n)}}{\lambda_1^{(n+1)}}\leq 1\tag{2}$$

(iii) *Assume that the constraints of Property (ii) hold with* $M^{(n_0)}\succeq 0$; $\forall n(\geq n_0)\in\mathbf{Z}_+$ *and, furthermore,* $\limsup_{n\to\infty}\left(d^{(n)}+\tilde{d}^{(n)}\right)\leq 1$ *and* $m^{(n)}=\min\left(o\left(\|M^{(n)}\|\right),o\left(\left|\tilde{d}^{(n)}\right|\right)\right)$, *which is guaranteed if* $m^{(n)}=min\left(o\left(K_M^{(n)}\right),o\left(K_{\tilde{d}}^{(n)}\right)\right)$. Then, $M^{(n)}\succeq 0$ and $M^{(n+1)}\succeq 0$; $\forall n(\geq n_0)\in\mathbf{Z}_+$, $\left\{\det M^{(n)}\right\}_{n=n_0}^{\infty}$ is bounded and the sequence $\left\{sp\left(M^{(n)}\right)\right\}_{n=n_0}^{\infty}$ is bounded, if *det* $M^{(n_0)}$ is finite, and then $\limsup_{n\to\infty}\left(\det M^{(n+1)}-\det M^{(n)}\right)\leq 0$.

Remark 1. *Concerning Lemma 2 (i), we can focus on the following particular cases of interest A:*

(a) $m^{(n)}=0$ *and* $\left|\det M^{(n+1)}\right|\leq\left|\det M^{(n)}\right|$ *fails for all* $n(\geq\overline{n}_0)\in\mathbf{Z}_+$ *and some* $\overline{n}_0(\geq n_0)\in\mathbf{Z}_+$. *Then,* $|\det M^{(n+1)}|=|\det M^{(n)}||d^{(n)}+\tilde{d}^{(n)}|>|\det M^{(n)}|$ *so that* $K_d^{(n)}+K_{\tilde{d}}^{(n)}\geq\left|d^{(n)}+\tilde{d}^{(n)}\right|>1$ *and* $\frac{1-K_d^{(n)}}{1+(2^{n+1}-1)\varepsilon^{(n)}}\geq K_{\tilde{d}}^{(n)}>1-K_d^{(n)}$ *so that* $1\leq 1+\left(2^{n+1}-1\right)\varepsilon^{(n)}<1$, *a contradiction. Thus, one has* $\left|\det M^{(n+1)}\right|\leq\left|\det M^{(n)}\right|$ *for any* $n(\geq\overline{n}_0)\in\mathbf{Z}_+$ *such that* $m^{(n)}=0$ *and also* $\limsup_{n\to\infty}\left(\left|detM^{(n+1)}\right|-\left|detM^{(n)}\right|\right)\leq 0$.

(b) $d^{(n)}+\tilde{d}^{(n)}=0$ *and* $\left|\det M^{(n+1)}\right|\leq\left|\det M^{(n)}\right|$ *fails for all* $n(\geq\overline{n}_0)\in\mathbf{Z}_+$ *and some* $\overline{n}_0(\geq n_0)\in\mathbf{Z}_+$. *Note from the definition of the recursive sequence* $\left\{M^{(n)}\right\}_{n=n_0}^{\infty}$ *that*

$$\begin{aligned}\det M^{(n+1)}&=\det\begin{bmatrix}M^{(n)}&0\\m^{(n)*}&d^{(n)}\end{bmatrix}+\det\begin{bmatrix}M^{(n)}&m^{(n)}\\m^{(n)*}&\tilde{d}^{(n)}\end{bmatrix}\\&=d^{(n)}\det M^{(n)}+\det\left(\begin{bmatrix}M^{(n)}&0\\0&\tilde{d}^{(n)}\end{bmatrix}\left(I_{n+1}+\begin{bmatrix}M^{(n)^{-1}}&0\\0&1/\tilde{d}^{(n)}\end{bmatrix}\begin{bmatrix}0&m^{(n)}\\m^{(n)*}&0\end{bmatrix}\right)\right)\\&=-\tilde{d}^{(n)}\det M^{(n)}\left(1-\det\left(I_{n+1}+\begin{bmatrix}M^{(n)^{-1}}&0\\0&-1/d^{(n)}\end{bmatrix}\begin{bmatrix}0&m^{(n)}\\m^{(n)*}&0\end{bmatrix}\right)\right);\ \forall n(\geq\overline{n}_0)\in\mathbf{Z}_+\end{aligned}\tag{3}$$

Since $\frac{1}{k_M^{(n)}k_{\tilde{d}}^{(n)}}\sqrt{m^{(n)*}m^{(n)}}\leq\varepsilon^{(n)}$ *and* $\left\{\varepsilon^{(n)}\right\}_{n=n_0}^{\infty}\to 0$ *then*

$$\begin{aligned}&\limsup_{n\to\infty}\left(\left|\det M^{(n+1)}/\det M^{(n)}\right|-\left|\tilde{d}^{(n)}\right|\left|1-\det\left(I_{n+1}+\begin{bmatrix}0&M^{(n)^{-1}}m^{(n)}\\-\left(1/d^{(n)}\right)m^{(n)*}&0\end{bmatrix}\right)\right|\right)\\&=\limsup_{n\to\infty}\left|\det M^{(n+1)}/\det M^{(n)}\right|=0\end{aligned}$$

and $\limsup_{n\to\infty}\left(\left|\det M^{(n+1)}\right|-\left|\det M^{(n)}\right|\right)\le 0$ *since, otherwise, if* $\limsup_{n\to\infty}\left(\left|\det M^{(n+1)}\right|-\left|\det M^{(n)}\right|\right)>0$ *then* $\limsup_{n\to\infty}\left|\det M^{(n+1)}/\det M^{(n)}\right|\in(-\infty,-1)\cup(1,+\infty)$, *a contradiction to* $\limsup_{n\to\infty}\left|\det M^{(n+1)}/\det M^{(n)}\right|=0$. *Note that if* $\widetilde{d}^{(n)}=0$ *then* $\left|d^{(n)}\right|\le 1$ *under the given constraints so that if* $\left\{m^{(n)}\right\}_{n=n_0}^{\infty}\to 0$;

(c) $\forall n(\ge \overline{n}_0)\in\mathbf{Z}_+$ *and some* $\overline{n}_0\ge n_0$ *then* $\limsup_{n\to\infty}\left(\left|\det M^{(n+1)}\right|-\left|\det M^{(n)}\right|\right)\le 0$.

Now, one gets from Lemma 2 [(ii), (iii)] the subsequent dual result concerning the recursion obtained from the inverse of $M^{(n)}$. The use of this result will make it possible to give sufficiency-type conditions regarding the non-singularity of the recursive calculation for any positive integer n, and also as n tends to infinity.

Lemma 3. *For some given arbitrary* $n_0\in\mathbf{Z}_+$ *and all* $n(\ge n_0)\in\mathbf{Z}_+$, *define:*

$$\overline{M}^{(n)}=M^{(n)^{-1}}\left(I_n+m^{(n)}\left(d^{(n)}+\widetilde{d}^{(n)}-m^{(n)*}M^{(n)^{-1}}m^{(n)}\right)^{-1}m^{(n)*}M^{(n)^{-1}}\right)\tag{4}$$

$$\overline{m}^{(n)}=-M^{(n)^{-1}}m^{(n)}\left(d^{(n)}+\widetilde{d}^{(n)}-m^{(n)*}M^{(n)^{-1}}m^{(n)}\right)^{-1}\tag{5}$$

$$\overline{d}^{(n)}+\overline{\overline{d}}^{(n)}=\left(d^{(n)}+\widetilde{d}^{(n)}-m^{(n)*}M^{(n)^{-1}}m^{(n)}\right)^{-1}\tag{6}$$

and assume that:

(1) *there is a real sequence* $\left\{\overline{\varepsilon}^{(n)}\right\}_{n=n_0}^{\infty}\subset[0,1)$ *such that* $\frac{1}{\overline{k}_M^{(n)}(n)\overline{k}_d^{(n)}}\sqrt{\overline{m}^{(n)*}\overline{m}^{(n)}}\le\overline{\varepsilon}^{(n)};\forall n(\ge n_0)\in\mathbf{Z}_+$, *where*

$$\overline{k}_M^{(n)}=\left|\lambda_{\min}\left(\overline{M}^{(n)}\right)\right|\le\overline{K}_M^{(n)}=\left|\lambda_{\max}\left(\overline{M}^{(n)}\right)\right|;\ \overline{d}^{(n)}\in\left[\overline{k}_d^{(n)},\ \overline{K}_d^{(n)}\right];$$
$$\overline{\overline{d}}^{(n)}\in\left[\overline{k}_{\widetilde{d}}^{(n)},\ \overline{K}_{\widetilde{d}}^{(n)}\right];\ \forall n(\ge n_0)\in\mathbf{Z}_+,$$

$$\widetilde{d}^{(n)}\ne m^{(n)*}M^{(n)^{-1}}m^{(n)}-d^{(n)}$$

(2) $\overline{M}^{(n_0)}\underline{>}0$, $\limsup_{n\to\infty}\left(\overline{d}^{(n)}+\overline{\overline{d}}^{(n)}\right)\le 1$ *and* $\overline{m}^{(n)}=\min\left(o\left(\|\overline{M}^{(n)}\|\right),o\left(\overline{\overline{d}}^{(n)}\right)\right)$, *which is guaranteed if* $\overline{m}^{(n)}=\min\left(o\left(\overline{K}_M^{(n)}\right),o\left(\overline{K}_{\widetilde{d}}^{(n)}\right)\right)$. *Then,* $\overline{M}^{(n)}\underline{>}0$ *and* $\overline{M}^{(n+1)}\underline{>}0;\forall n(\ge n_0)\in\mathbf{Z}_+$, $\left\{\det\overline{M}^{(n)}\right\}_{n=n_0}^{\infty}$ *is bounded, the sequence* $\left\{sp\left(\overline{M}^{(n)}\right)\right\}_{n=n_0}^{\infty}$ *is bounded, if* $\det\overline{M}^{(n_0)}$ *is finite, and then* $\limsup_{n\to\infty}\left(\det\left|\overline{M}^{(n+1)}\right|-\det\left|\overline{M}^{(n)}\right|\right)\le 0$.

One gets by combining Lemma 2 and Lemma 3 the two subsequent direct results:

Lemma 4. *Assume that* $M^{(n_0)}\succ 0$ *for some given arbitrary* $n_0\in\mathbf{Z}_+$ *and assume also that the conditions of Lemma 2 (iii) and Lemma 3 hold. Then,* $M^{(n+1)}\succ 0;\ \forall n(\ge n_0)\in\mathbf{Z}_+$.

Lemma 5. *Assume that, for some finite* $n_0\in\mathbf{Z}_+$, $A^{(n_0)}\in\mathbf{C}^{n_0\times n_0}$ *is a stability matrix and construct a sequence* $\left\{M^{(n)}\right\}_{n=n_0}^{\infty}$ *according to the recursive rule:*

$$M^{(n+1)}=A^{(n+1)*}A^{(n+1)}=\begin{bmatrix}M^{(n)} & m^{(n)}\\ m^{(n)*} & d^{(n)}+\widetilde{d}^{(n)}\end{bmatrix}=\begin{bmatrix}A^{(n)*}A^{(n)} & m^{(n)}\\ m^{(n)*} & d^{(n)}+\widetilde{d}^{(n)}\end{bmatrix};\ \forall n\ge n_0$$

with initial condition $M^{(n_0)}=A^{(n_0)*}A^{(n_0)}\succ 0$. *Assume also that* $\left\{M^{(n)}\right\}_{n=n_0}^{\infty}$ *and the sequence of its inverses satisfy the constraints of Lemma 2 [(ii),(iii)] and Lemma 3.*

Then, $\left\{A^{(n)}\right\}_{n=n_0}^{\infty}$ *is a sequence of stability matrices.*

The above result can be directly extended for the case when $A^{(n_0)} \in \mathbf{C}^{n_0 \times n_0}$ is antistable, that is, when all its eigenvalues have positive real parts and $M^{(n_0)} \succ 0$. Then, by using similar arguments, as in the proof of Lemma 5 based on the continuity of the matrix eigenvalues with respect to its entries and supported by Lemmas 2,3, according to Cauchy's interlacing theorem, one concludes that $\left\{A^{(n)}\right\}_{n=n_0}^{\infty}$ consists of antistable members.

Lemma 6. *Lemma 5 holds "mutatis-mutandis" if* $A^{(n_0)} \in \mathbf{C}^{n_0 \times n_0}$ *is antistable.*

3. Some Extended Results Related to Sequences of Convergent Matrices

In order to be able to adapt the above results to discrete dynamic systems, the well-known result that that the stability domain of a convergent matrix (i.e., a "stable" discrete matrix) is the open unit circle of the complex plane centered at zero has to be taken into account. Note that, in particular, $A \in \mathbf{C}^{n \times n}$ is convergent if $sp(A) \subset C_1 = \{z \in \mathbf{C} : |z| < 1\}$ so that $\{A^m\}_0^{\infty} \to 0$ as $m \to \infty$. It turns out that convergent matrices describe the stability property in the discrete sense. In other words, the solution of the discrete difference vector equation $z_{m+1} = Az_m$, where $A \in \mathbf{C}^{n \times n}$, converges to $0 \in \mathbf{R}^n$ for any given $z_0 \in \mathbf{C}^n$ if and only if $A \in \mathbf{C}^{n \times n}$ is convergent. The relevant results of Section 2 can be extended to this situation as follows, provided that $A \in \mathbf{C}^{n \times n}$ is also Hermitian. Consider the following cases:

Case a: $sp(A) \subset C_{1+} = \{z \in \mathbf{C} : 0 < z < 1\} \subset C_1$ so that $A(\in \mathbf{C}^{n \times n}) \succ 0$. Then, it is convergent if and only if $(I_n - A) \succ 0$. The proof is direct since if $(I_n - A) \succ 0$ then $x^*Ax < x^*x$ for any $x(\neq 0) \in \mathbf{C}^n$. Thus, by taking any $\lambda(\neq 0) \in sp(A)$ of eigenvector $x \in \mathbf{C}^n$, one determines that $\lambda < 1$ if $\lambda \neq 0$ and $\lambda = 0$ directly fulfills the constraint. This proves the sufficiency part. The "only if part" follows, since if $(I_n - A) \succ 0$ fails, there is $\lambda(\neq 0) \in sp(A)$ of eigenvector $x \in \mathbf{C}^n$ such that $x^*Ax \geq x^*x$ then $\lambda \geq 1$ and A is not convergent.
Case b: $sp(A) \subset C_{1-} = \{z \in \mathbf{C} : -1 < z < 0\} \subset C_1$ so that $A\,(\in \mathbf{C}^{n \times n}) \prec 0$. Then, it is convergent if and only if $(I_n + A) \succ 0$. The proof is direct, since if $(I_n + A) \succ 0$, then $x^*Ax > -x^*x$ for any $x(\neq 0) \in \mathbf{C}^n$. Thus, by taking any $\lambda(\neq 0) \in sp(A)$ of eigenvector $x \in \mathbf{C}^n$, one determines that $0 > \lambda > -1$. The remainder of the proof follows Case a closely.
Case c: $sp(A) \subset C_1$ so that $A^2(\in \mathbf{C}^{n \times n}) \succ 0$ so that $sp\left(A^2\right) \subset C_{1+}$. Then, it is convergent if and only if $\left(I_n - A^2\right) \succ 0$ according to ***Case a*** by replacing $A \to A^2$. Note that ***Case c*** is included ***Case a*** and ***Case b***.

Now, for ***Case a***, replace $M^{(n+1)}$, defined in Lemma 2, by $I_{n+1} - M^{(n+1)} = \begin{bmatrix} I_n - M^{(n)} & -m^{(n)} \\ -m^{(n)*} & 1 - \left(d^{(n)} + \widetilde{d}^{(n)}\right) \end{bmatrix}$ and it has to be guaranteed that if $M^{(n)}$ is Hermitian, then $\left(I_n - M^{(n)}\right)$ is also Hermitan, and $\left(I_n - M^{(n_0)}\right) \succ 0$ for some $n_0 \in \mathbf{Z}_+$ then $\left(I_{n+1} - M^{(n+1)}\right) \succ 0; \forall n \geq n_0$.

For ***Case b***, replace $M^{(n+1)} \to \left(I_{n+1} + M^{(n+1)}\right) = \begin{bmatrix} I_n + M^{(n)} & m^{(n)} \\ m^{(n)*} & 1 + d^{(n)} + \widetilde{d}^{(n)} \end{bmatrix}$ and it has to be guaranteed that if $M^{(n)}$ is Hermitian, then $\left(I_n + M^{(n)}\right)$ is also Hermitan, and $\left(I_n + M^{(n_0)}\right) \succ 0$ for some $n_0 \in \mathbf{Z}_+$ then $\left(I_{n+1} + M^{(n+1)}\right) \succ 0; \forall n \geq n_0$.

For ***Case c***, note that $\left(I_n - M^{(n)^2}\right) = \left(I_n + M^{(n)}\right)\left(I_n - M^{(n)}\right)$ so that

$$\left(I_n - M^{(n)^2}\right) = \begin{bmatrix} I_n + M^{(n)} & m^{(n)} \\ m^{(n)*} & 1 + d^{(n)} + \widetilde{d}^{(n)} \end{bmatrix} \begin{bmatrix} I_n - M^{(n)} & -m^{(n)} \\ -m^{(n)*} & 1 - \left(d^{(n)} + \widetilde{d}^{(n)}\right) \end{bmatrix}$$

then, replace

$$M^{(n+1)} \to \left(I_{n+1} - M^{(n+1)^2}\right) = \begin{bmatrix} I_n - M^{(n)2} & -\left(\left(d^{(n)} + \widetilde{d}^{(n)}\right)I_n + M^{(n)}\right)m^{(n)} \\ -m^{(n)*}\left(\left(d^{(n)} + \widetilde{d}^{(n)}\right)I_n + M^{(n)*}\right) & 1 - \left(d^{(n)} + \widetilde{d}^{(n)}\right)^2 - m^{(n)*}m^{(n)} \end{bmatrix}$$

and it has to be guaranteed that if $M^{(n)^2}$ is Hermitian and $\left(I_n - M^{(n_0)^2}\right) \succ 0$ for some $n_0 \in \mathbf{Z}_+$ then $\left(I_{n+1} - M^{(n+1)^2}\right) \succ 0$; $\forall n \geq n_0$. Since $A \succ 0$ is Hermitian, it is of the form $A = E^*E$ for some full rank n-matrix E. Then, $A^2 = (E^*E)^2 = E^*EE^*E = A^*A$. If

$A \prec 0$ then $(-A) \succ 0$ and $(-A)^2 = (F^*F)^2 = F^*FF^*F = (-A^*)(-A) = A^*A$ for some full rank n-matrix F. Then, Cases a and b can be dealt with using ***Case c*** by replacing $A^2 \to A^*A$.

By taking advantage from the fact that a complex square matrix A is convergent (i.e., stable in the discrete sense) if and only if the Hermitian matrix A^*A is convergent, we now build a sequence $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ of Hermitian matrices as follows, in order to discuss the convergence of its members, provided that $M^{(n_0)}$ is convergent for some given $n_0 \in \mathbf{Z}_{0+}$ or, with no loss in generality, provided that $M^{(0)}$ is convergent. Then,

$$M^{(n+1)} = \begin{bmatrix} M^{(n)} & \lambda^{(n)}m^{(n)} \\ \lambda^{(n)}m^{(n)*} & \delta^{(n)} \end{bmatrix} = \begin{bmatrix} M^{(n)} & 0 \\ 0 & \delta^{(n)} \end{bmatrix} + \begin{bmatrix} 0 & \lambda^{(n)}m^{(n)} \\ \lambda^{(n)}m^{(n)*} & 0 \end{bmatrix}; \ \forall n \in \mathbf{Z}_{0+} \tag{7}$$

with $\left\{\lambda^{(n)}\right\}_{n=0}^{\infty} \subset [0,\ 1)$.Then,

$$\begin{aligned} \left|\lambda_{\max}\begin{bmatrix} M^{(n)} & \lambda^{(n+1)}m^{(n)} \\ \lambda^{(n)}m^{(n)*} & \delta^{(n)} \end{bmatrix}\right| &\leq \left|\lambda_{\max}\begin{bmatrix} M^{(n)} & 0 \\ 0 & \delta^{(n)} \end{bmatrix}\right| + \left|\lambda_{\max}\begin{bmatrix} 0 & \lambda^{(n)}m^{(n)} \\ \lambda^{(n)}m^{(n)*} & 0 \end{bmatrix}\right| \\ &= \max\left(\left|\lambda_{\max}\left(M^{(n)}\right)\right|,\ \left|\delta^{(n)}\right|\right) + \lambda^{(n)}\sqrt{m^{(n)*}m^{(n)}} \\ &\leq \max\left(1 - \varepsilon^{(n)},\ \left|\delta^{(n)}\right|\right) + \lambda^{(n)}\sqrt{m^{(n)*}m^{(n)}} < 1 - \varepsilon^{(n)};\ \forall n \in \mathbf{Z}_{0+} \end{aligned} \tag{8}$$

which holds if

$$m^{(n)*}m^{(n)} < \frac{1}{\lambda^{(n)2}}\left(1 - \varepsilon^{(n)} - \max\left(1 - \varepsilon^{(n)},\ \left|\delta^{(n)}\right|\right)\right)^2;\ \forall n \in \mathbf{Z}_{0+} \tag{9}$$

or, $\|m^{(n)}\|_2 < \frac{1}{\lambda^{(n)2}}\left(1 - \varepsilon^{(n)} - \max\left(1 - \varepsilon^{(n)},\ \left|\delta^{(n)}\right|\right)\right)$; $\forall n \in \mathbf{Z}_{0+}$, provided that $\varepsilon^{(n+1)} < \varepsilon^{(n)}$; $\forall n \in \mathbf{Z}_{0+}$, that is $\left\{\varepsilon^{(n)}\right\}_{n=0}^{\infty} \subset [0,\ 1)$ is strictly decreasing, so $\left\{\varepsilon^{(n)}\right\}_{n=0}^{\infty} \to 0$, and $\left|\delta^{(n)}\right| < 1 - \varepsilon^{(n+1)}$; $\forall n \in \mathbf{Z}_{0+}$.

Now, assume that the iterations to build $\left\{M^{(n+1)}\right\}_{n=0}^{\infty}$ do not add a new row and column to obtain $M^{(n+1)}$ from $M^{(n)}$ via the contribution of the members of an updating sequence $\left\{\overline{M}^{(n)}\right\}_{n=0}^{\infty}$; $\forall n \in \mathbf{Z}_{0+}$ but a set of the, in general. Then, one may get that:

$$M^{(n+1)} = \begin{bmatrix} M^{(n)} & \lambda^{(n)}\overline{M}^{(n)} \\ \lambda^{(n)}\overline{M}^{(n)*} & \Delta^{(n)} \end{bmatrix} = \begin{bmatrix} M^{(n)} & 0 \\ 0 & \Delta^{(n)} \end{bmatrix} + \begin{bmatrix} 0 & \lambda^{(n)}\overline{M}^{(n)} \\ \lambda^{(n)}\overline{M}^{(n)*} & 0 \end{bmatrix};\ \forall n \in \mathbf{Z}_{0+} \tag{10}$$

so that

$$\begin{aligned} \left|\lambda_{\max}M^{(n+1)}\right| = \left|\lambda_{\max}\begin{bmatrix} M^{(n)} & \lambda^{(n)}\overline{M}^{(n)} \\ \lambda^{(n)}\overline{M}^{(n)*} & \Delta^{(n)} \end{bmatrix}\right| &\leq \left|\lambda_{\max}\begin{bmatrix} M^{(n)} & 0 \\ 0 & \Delta^{(n)} \end{bmatrix}\right| + \left|\lambda_{\max}\begin{bmatrix} 0 & \lambda^{(n)}\overline{M}^{(n)} \\ \lambda^{(n)}\overline{M}^{(n)*} & 0 \end{bmatrix}\right| \\ &= \max\left(\left|\lambda_{\max}\left(M^{(n)}\right)\right|,\ \left|\lambda_{\max}\left(\Delta^{(n)}\right)\right|\right) + \lambda^{(n)}\lambda_{\max}^{1/2}\left(\overline{M}^{(n)*}\overline{M}^{(n)}\right) \\ &\leq \max\left(1 - \varepsilon^{(n)},\ \left|\lambda_{\max}\left(\Delta^{(n)}\right)\right|\right) + \lambda^{(n)}\lambda_{\max}^{1/2}\left(\overline{M}^{(n)*}\overline{M}^{(n)}\right) < 1 - \varepsilon^{(n+1)};\forall n \in \mathbf{Z}_{0+} \end{aligned} \tag{11}$$

which holds by complete induction if $M^{(0)}$ is convergent and

$$\|\overline{M}^{(n)}\|_2^2 = \lambda_{\max}\left(\overline{M}^{(n)*}\overline{M}^{(n)}\right) < \frac{1}{\lambda^{(n)2}}\left(1-\varepsilon^{(n+1)} - \max\left(1-\varepsilon^{(n)}, \left|\lambda_{\max}\left(\Delta^{(n)}\right)\right|\right)\right)^2; \quad \forall n \in \mathbf{Z}_{0+} \tag{12}$$

or, $\|\overline{M}^{(n)}\|_2 < \frac{1}{\lambda^{(n)}}\left(1-\varepsilon^{(n+1)} - \max\left(1-\varepsilon^{(n)}, \left|\lambda_{\max}\left(\Delta^{(n)}\right)\right|\right)\right)$; $\forall n \in \mathbf{Z}_{0+}$, provided that $\varepsilon^{(n+1)} < \varepsilon^{(n)}$, that is $\left\{\varepsilon^{(n)}\right\}_{n=0}^{\infty} \subset [0,1)$ is strictly decreasing, so $\left\{\varepsilon^{(n)}\right\}_{n=0}^{\infty} \to 0$, and $\left|\lambda_{\max}\left(\Delta^{(n)}\right)\right| < 1-\varepsilon^{(n+1)}$; $\forall n \in \mathbf{Z}_{0+}$. This implies that $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ is convergent.

In the particular case that for some $\lambda \in [0,1)$, $\lambda^{(n)} = \lambda^n$; $\forall n \in \mathbf{Z}_{0+}$, such a λ is a forgetting factor of the iteration.

We now consider the matrix factorization $M^{(n)} = A^{(n)*}A^{(n)}$; $\forall n \in \mathbf{Z}_{0+}$. By construction, $M^{(n)}$ is Hemitian (then square), even if $A^{(n)}$ is not square; $\forall n \in \mathbf{Z}_{0+}$. In the case when $A^{(n)}$ is not square, and since its order strictly increases as n increases, it is possible to consider the convergence of the sequence $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ (without invoking the values of its eigenvalues) as the following property $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ is asymptotically convergent if $\lim_{m\to\infty}\left\{\left|\,\|A^{(n+\xi)}\| - \|A^{(n)}\|\,\right|^m\right\}_{n=0}^{\infty} = 0$ for any given $\xi \in \mathbf{Z}_{+}$. $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ is convergent if $\lim_{m\to\infty}\left\{A^{(n)^m}\right\}_{n=0}^{\infty} = 0$. The following related results are direct of simple proofs given in Appendix A:

Lemma 7. *If $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ is convergent then it is asymptotically convergent. The inverse is, in general, not true.*

Lemma 8. *Assume that $M^{(n)} = A^{(n)*}A^{(n)}$; $\forall n \in \mathbf{Z}_{0+}$ is a complex square matrix of any arbitrary order. Then:*

(i) *If $\|A^{(n)}\|_2 < 1$; $\forall n \in \mathbf{Z}_{0+}$ then $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ and $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ are convergent sequences.*

(ii) *If $\|M^{(n)}\|_2 < 1$; $\forall n \in \mathbf{Z}_{0+}$ then $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ and $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ are convergent sequences.*

(iii) *For any $n \in \mathbf{Z}_{0+}$, $\|A^{(n)}\|_2 < 1$ if and only if $\|M^{(n)}\|_2 < 1$. $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ is convergent if and only if $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ is convergent.*

4. Example of SIR-Type Epidemic Models of Inter-Community Clusters

The stability of the equilibrium points of the epidemic models is an interesting topic which is of great relevance to healthcare management. See, for instance, [15–19]. Now, we discuss an epidemic based-model related to stabilization under the given framework of the Cauchy's interlacing theorem.

Example 1. *Consider the subsequent continuous-time linearized epidemic model with Q community clusters:*

$$\begin{aligned}\dot{S}_{i+1} &= \nu_i(1-\mu_i)I_i - \beta_{i+1}S_{i+1}\\ \dot{I}_{i+1} &= \beta_{i+1}S_{i+1} - \nu_{i+1}I_{i+1}\\ \dot{R}_{i+1} &= \nu_{i+1}\mu_{i+1}I_i\end{aligned} \tag{13}$$

for $i = 0, 1, \ldots, Q-1$ with $S_i(0) = S_{i0} \geq 0$, $I_i(0) = I_{i0} \geq 0$, $R_i(0) = R_{i0} \geq 0$; $\forall i \in \overline{Q}$ are the initial conditions of the susceptible, infectious and recovered subpopulations, respectively, and $I_0(0) = 0$, $\nu_0 = 1$. In this model, the infectious subpopulation I_i of a community $i \in \overline{Q} = \{1, 2, \ldots, Q\}$ may infect the population of the neighboring community $(i+1)$. The parameterization is as follows: $\beta_{(.)}$ are the disease transmission rates, $\nu_{(.)}$ are the removal rates and $\mu_{(.)}$ are the separation constants which bifurcate the disease rate between the local community and the total community. Note that the assumption $\mu_0 = 1$ implies that the first cluster is not affected by contagions from any other cluster, [15]. A simple analysis of the trajectory solution of the first cluster shows that

$$S_1(t) = e^{-\beta_1 t}S_{10} \to 0 \ \text{ as } t \to \infty$$

at an exponential rate, irrespective of the initial conditions, and is definitively bounded,

$$I_1(t) = e^{-\nu_1 t}I_{10} + \int_0^t e^{-\nu_1(t-\tau)}S_1(\tau)d\tau = e^{-\nu_1 t}\left(I_{10} + S_{10}\int_0^t e^{(\nu_1-\beta_1)\tau}d\tau\right)$$
$$= \left(e^{-\nu_1 t}I_{10} + \frac{e^{-\beta_1 t}-e^{-\nu_1 t}}{\nu_1-\beta_1}S_{10}\right) \to 0 \text{ as } t \to \infty$$

if $\nu_1 \neq \beta_1$. *If* $\nu_1 = \beta_1$ *then* $I_1(t) = e^{-\nu_1 t}(I_{10} + S_{10}t) \to 0$ *as* $t \to \infty$. *In both cases, the convergence is of exponential order, irrespective of the initial conditions, and is definitively bounded, and*

$$R_1(t) = R_{10} + \nu_1\mu_1\int_0^t I_i(\tau)d\tau = R_{10} + \nu_1\mu_1\int_0^t\left(e^{-\nu_1\tau}I_{10} + \frac{e^{-\beta_1\tau}-e^{-\nu_1\tau}}{\nu_1-\beta_1}S_{10}\right)d\tau$$
$$= R_{10} + \nu_1\mu_1\int_0^t\left(e^{-\nu_1\tau}I_{10} + \frac{e^{-\beta_1\tau}-e^{-\nu_1\tau}}{\nu_1-\beta_1}S_{10}\right)d\tau$$
$$= R_{10} + \nu_1\mu_1 I_{10}\frac{1-e^{-\nu_1 t}}{\nu_1} + \frac{\nu_1\mu_1 S_{10}}{\nu_1-\beta_1}\left(e^{-\beta_1 t} - e^{-\nu_1 t}\right) \to R_{10} + \mu_1 I_{10} + \frac{\nu_1\mu_1 S_{10}}{\nu_1-\beta_1}\left(e^{-\beta_1 t} - e^{-\nu_1 t}\right)$$
$$\text{as } t \to \infty$$

if $\nu_1 \neq \beta_1$ *with solution which is definitively bounded, and, if* $\nu_1 = \beta_1$ *then*

$$R_1(t) = R_{10} + \nu_1\mu_1\int_0^t e^{-\nu_1\tau}(I_{10} + S_{10}\tau)d\tau$$
$$= R_{10} + \mu_1\left[\left(1 - e^{-\nu_1 t}\right)I_{10} + \left(\frac{1}{\nu_1}\left(1 - e^{-\upsilon_1 t}\right) - te^{-\nu_1 t}\right)S_{10}\right] \to R_{10} + \mu_1\left[I_{10} + \frac{S_{10}}{\nu_1}\right] \text{ as}$$
$$t \to \infty$$

with a solution which is definitively bounded. As a result, the total subpopulation at the first cluster is also definitively bounded, and it converges asymptotically to the limit value of the recovered subpopulation. The solution trajectory is also definitively nonnegative since the matrix of dynamics $\begin{bmatrix} -\beta_1 & 0 & 0 \\ \beta_1 & -\nu_1 & 0 \\ 0 & \nu_1\mu_1 & 0 \end{bmatrix}$ *is a Metzler matrix and the initial conditions are non-negative. Interpretation shows that the total equilibrium subpopulation is that of the disease-free equilibrium which only has a recovered subpopulation. It can be surprising at a first glance to see that the usual nonlinear term* $\beta_1 S_1(t)I_1(t)$ *is the susceptible and infectious subpopulations evolutions of the corresponding SIR Kermack-Mcendrick model counterpart is replaced by a linear term. However, for stability purposes, there is no substantial distinct qualitative behavior between both models, since in this case,* $S_1(t) = e^{-\beta_1\int_0^t I_1(\tau)d\tau}S_{10}$ *is strictly decreasing for* $t \geq 0$ *and* $I_1(t) = e^{\int_0^t(\beta_1 S_1(\tau)-\nu_1)d\tau}I_{10}$ *is also strictly decreasing for* $t \geq 0$ *provided that* $S_{10} < \frac{\nu_1}{\beta_1}$. *Now describe the whole model (14) of Nclusters in a more compact way through the individual states* $x_i = (S_i\,,\, I_i\,,R_i)^T$; $i \in \overline{N}$ *and associated matrices of self-dynamics for each* $i \in \overline{N}$ *and coupled dynamics with the respective preceding cluster* $(i-1) \in \overline{N}$:

$$A_i = \begin{bmatrix} -\beta_i & 0 & 0 \\ \beta_i & -\nu_i & 0 \\ 0 & \nu_i\mu_i & 0 \end{bmatrix}, \forall i \in \overline{N};\ A_{i,i-1} = \begin{bmatrix} 0 & \nu_{i-1}(1-\mu_{i-1}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\forall i(\geq 2) \in \overline{N};\ A_{0,-1} = 0 \tag{14}$$

so that (13) is equivalently described as

$$\dot{x}_i(t) = A_i x_i(t) + A_{i,i-1}x_{i-1}(t),\ x_i(0) = x_{i0}(\geq 0);\ \forall i \in \overline{N} \tag{15}$$

with $x_0(t) \equiv 0$ *for* $t \geq 0$, *and compactly, as follows:*

$$\dot{x}(t) = Ax(t),\ x(0) = x_0 \tag{16}$$

where $x(t) = \left(x_1^T(t), x_2^T(t), \cdots, x_N^T(t)\right)^T$ *and*

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ A_{21} & A_2 & 0 & \cdots & 0 \\ 0 & A_{32} & A_3 0 \cdots & \vdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & \\ 0 & 0 \cdots & \cdots 0 & A_{N,N-1} & A_N \end{bmatrix} \tag{17}$$

Thus, system (15), like (16), (17), can be interpreted as an aggregation model given by starting with the first cluster and successively incorporating the dynamics of the remaining clusters. Now, define the symmetric matrix $M = M^{(N)} = A^T A$*. Then, define:*

$$M^{(1)} = A_1^T A_1$$

$$M^{(2)} = \begin{bmatrix} A_1^T & A_{21}^T \\ 0 & A_2^T \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} = \begin{bmatrix} A_1^T A_1 + A_{21}^T A_{21} & A_{21}^T A_2 \\ A_2^T A_{21} & A_2^T A_2 \end{bmatrix} = \begin{bmatrix} M^{(1)} & A_{21}^T A_2 \\ A_2^T A_{21} & A_2^T A_2 \end{bmatrix} + \widetilde{M}^{(2)}$$

$$M^{(3)} = \begin{bmatrix} A_1^T & A_{21}^T & 0 \\ 0 & A_2^T & A_{32}^T \\ 0 & 0 & A_3^T \end{bmatrix}\begin{bmatrix} A_1 & 0 & 0 \\ A_{21} & A_2 & 0 \\ 0 & A_{32} & A_3 \end{bmatrix} = \begin{bmatrix} M^{(2)} & & 0 \\ & & A_{32}^T A_3 \\ 0 & A_3^T A_{32} & A_3^T A_3 \end{bmatrix} + \widetilde{M}^{(3)} \tag{18}$$

$$M^{(N)} = \begin{bmatrix} M^{(N-1)} & m^{(N-1)} \\ m^{(N-1)^T} & A_3^T A_3 \end{bmatrix} + \widetilde{M}^{(N)}$$

$$\widetilde{M}^{(2)} = \begin{bmatrix} A_{21}^T A_{21} & 0 \\ 0 & 0 \end{bmatrix}, \widetilde{M}^{(3)} = \begin{bmatrix} A_{21}^T A_{21} & 0 & 0 \\ 0 & A_{32}^T A_{32} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{M}^{(N)} = \begin{bmatrix} A_{21}^T A_{21} & 0 & \cdots & 0 \\ 0 & A_{32}^T A_{32} & 0 \cdots & 0 \\ 0 & \cdots & \ddots & \\ 0 & \cdots & \cdots & A_{N,N-1}^T A_{N,N-1} \\ & & & & 0 \end{bmatrix}$$

if $N \geq 2$ *and* $\widetilde{M}^{(1)} = 0$*. By inspection of (18), one concludes that* $\|\widetilde{M}^{(n)}\| = 0\left(\sum_{n=2}^{n} \sup^2 \|m^{(n)}\|\right)$ *for* $i = 2, 3, \ldots, N$ *which concludes that, if* $\left\{A_{n-1,n}\right\}_{n=2}^{N} \to 0$ *as* $N \to \infty$ *in such a way that* $\left\{\|m^{(j)}\|\right\}_{j=2}^{N} \to 0$ *as* $N \to \infty$*, for instance, if the convergence is at exponential rate, then* $\lim_{N\to\infty} \sum_{n=2}^{N} \sup^2 \|m^{(n)}\| \leq K_m < +\infty$*. Furthermore, if* A_1 *is a stability matrix of absolute stability abscissa which are sufficiently larger than* K_m*, then the dynamic system (16),(17) is globally asymptotically stable according to Lemma 5. In particular, note that if there is any pair of stable complex conjugate eigenvalues* $s_{1,2} = \mu \pm i\nu$ $(\mu < 0, \nu > 0)$ *for the first cluster, then there is a submatrix of* $M^{(1)}$*,*

$$M^{s(1)} = A_1^{sT} A_1^s = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}\begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix} = \begin{bmatrix} \mu^2 + \nu^2 & 0 \\ 0 & \mu^2 + \nu^2 \end{bmatrix} = \begin{bmatrix} \lambda_1^{(1)} & 0 \\ 0 & \lambda_2^{(1)} \end{bmatrix}$$

in the real canonical form. Since $\mu^{(1)} = \mu < 0$ *then* $\lambda_2^{(1)} = \lambda_2^{(1)} = \mu^2 + \nu^2 > 0$*. The maximum and minimum corresponding eigenvalues of* $M^{s(2)}$ *are no less than* $\lambda_2^{(1)}$ *and no larger than* $\lambda_1^{(1)}$*, respectively, from Cauchy's* *interlacing theorem. Since the eigenvalues are continuous functions of the matrix entries, and since $M^{s(n)}$ is positive definite any critical eigenvalue of a member $M^{s(j)}$ of the sequence, $\left\{M^{s(n)}\right\}_{n=2}^{\infty}$ implies a lot of stability of the corresponding A_j^s. This is avoided if $\lim_{N\to\infty}\sum_{n=2}^{N}\sup^2\|m^{(n)}\|\leq K_m<+\infty$, implying also that $\left\{m^{(n)}\right\}_2^N\to 0$ as $N\to\infty$, and the sequence of separation constants $\{\mu_n\}_{n=2}^N\to 1$ as $N\to\infty$ if K_m is small enough related to μ. The physical interpretation relies on the fact that the contagion link from a cluster to the next one is weakened sufficiently quickly as the cluster index increases, due to the fact of the numbers of the infected subpopulations are rapidly decreasing as the cluster index increases at a sufficiently large rate.*

5. Dynamic Linear Discrete Aggregation Model with Output Delay and Linear Feedback Control

In this section, the convergence results of Section 3 are applied to a dynamic discrete system which is built by the aggregation of discrete dynamic subsystems subject to linear output feedback control. Since we are dealing with a physical system, it turns out that the formalism of Section 2 can be developed by invoking conditions related to real symmetric systems, rather than to complex Hermitian ones, when necessary. It would suffice to describe the state by expressing the matrix of dynamics in the real canonical form and to transform the control and output matrices by the appropriate similarity matrix. The necessary mathematical proof is given in Appendix A.

Consider the aggregation linear discrete dynamic system subject to r point delays under linear output-feedback:

$$x^{0(n+1)}=A^{0(n)}x^{(n)}+\hat{A}^{0(n)}\hat{x}^{(n)}+\sum_{j=1}^{r}B_j^{(n)}y^{(n-j)}+B_0^{(n)}u^{(n)} \tag{19}$$

$$y^{(n)}=C^{(n)}x^{(n)} \tag{20}$$

$$u^{(n)}=\sum_{j=0}^{r_n}K_j^{(n)}y^{(n-j)}=\sum_{j=0}^{r_n}K_j^{(n)}Cx^{(n-j)} \tag{21}$$

$\forall n\in\mathbf{Z}_{0+}$, with initial conditions $x^{0(0)}=x_0$, where $\{n_i\}_{i=0}^{\infty}$ is a sequence of positive integer numbers, $x^{(n)}\in\mathbf{R}^{\sum_{i=0}^{n}n_i}$ is the "a priori" vector state at the n-th iteration, $\hat{x}^{(n)}\in\mathbf{R}^{n_n}$ is the aggregated "a priori" new substate at the n-th iteration (that is basically, the new information needed to update the state vector and its dimension) and $x^{0(n+1)}\in\mathbf{R}^{\sum_{i=0}^{n}n_i}$ is the "a priori" whole state at the $(n+1)$-th iteration. Also, $x^{0(n)}\in\mathbf{R}^{\sum_{i=0}^{n-1}n_i}$, $u^{(n)}\in\mathbf{R}^{\sum_{i=0}^{n}m_i}$ and $y^{(n)}\in\mathbf{R}^{\sum_{i=0}^{n}p_i}$ are, respectively, the "a priori" input and measurable output vectors at the n-th iteration and $r\subset\mathbf{Z}_+$ is a sequence of delays influencing the global dynamics. The sequences of matrices of dynamics $\left\{A^{0(n)}\right\}_{n=0}^{\infty}$ and $\left\{\hat{A}^{0(n)}\right\}_{n=0}^{\infty}$, control $\left\{B_0^{(n)}\right\}_{n=0}^{\infty}$, output-state coupling $\left\{B_j^{(n)}\right\}_{n=0}^{\infty}$ for $j\in\bar{r}$ are of members $A^{0(n)}\in\mathbf{R}^{(\sum_{i=0}^{n}n_i)\times(\sum_{i=0}^{n}n_i)}$, $\hat{A}^{(n)}\in\mathbf{R}^{(\sum_{i=0}^{n}n_i)\times n_n}$, and $B_0^{(n)}\in\mathbf{R}^{(\sum_{i=0}^{n}n_i)\times(\sum_{i=0}^{n}m_i)}$ and $B_j^{(n)}\in\mathbf{R}^{(\sum_{i=0}^{n}n_i)\times(\sum_{i=0}^{n-j}p_i)}$ for $j\in\bar{r}$ and the output matrix $C^{(n)}\in\mathbf{R}^{(\sum_{i=0}^{n}p_i)\times(\sum_{i=0}^{n}n_i)}$. The sequences of matrices $\left\{K_j^{(n)}\right\}_{n=0}^{\infty}$, with $K_j^{(n)}\in\mathbf{R}^{(\sum_{i=0}^{n}m_i)\times(\sum_{i=0}^{n}p_i)}$ for $j\in\bar{r}\cup\{0\}$, are the output-feedback control gains which generate the control law sequence $\left\{u^{(n)}\right\}_{n=0}^{\infty}$.

The dynamics of the new dynamics at the $(n+1)$-th iteration aggregated to the former global aggregation system of state $x^{(n)}$ obtained at the n-th iteration, are assumed to be described by:

$$\hat{x}^{(n+1)}=\hat{A}^{(n+1)}x^{(n)}+\left(\hat{D}^{(n)}+\widetilde{\hat{D}}^{(n)}\right)\hat{x}^{(n)}+\sum_{j=1}^{r}\left(\hat{B}_j^{a(n)}y^{(n-j)}+\hat{B}_j^{(n)}\hat{y}^{(n-j)}\right)+\hat{B}_0^{(n)}\hat{u}^{(n)} \tag{22}$$

$$\hat{y}^{(n)}=\hat{C}^{(n)}\hat{x}^{(n)} \tag{23}$$

$$\hat{u}^{(n)}=\sum_{j=0}^{r}\left(\hat{K}_j^{(n)}\hat{y}^{(n-j)}+\hat{K}_j^{a(n)}y^{(n-j)}\right)=\sum_{j=0}^{r}\left(\hat{K}_j^{(n)}\hat{C}^{(n)}\hat{x}^{(n-j)}+\hat{K}_j^{a(n)}C^{(n)}x^{(n-j)}\right) \tag{24}$$

$\forall n \in \mathbf{Z}_{0+}$, where $\hat{x}^{(n+1)} \in \mathbf{R}^{n_{n+1}}$ is the "a posteriori" state of the aggregated subsystem at the $(n+1)$-th iteration whose "a priori" value is $\hat{x}^{(n)} \in \mathbf{R}^{n_n}$, $\hat{y}^{(n)} \in \mathbf{R}^{p_n}$, $\hat{u}^{(n)} \in \mathbf{R}^{m_n}$, $\hat{A}^{(n+1)} \in \mathbf{R}^{n_{n+1}\times(\sum_{i=0}^{n} n_i)}$, $\hat{B}_0^{(n)} \in \mathbf{R}^{n_{n+1}\times m_n}$, $\hat{B}_j^{(n)} \in \mathbf{R}^{n_{n+1}\times p_{n-j}}$, $\hat{B}_j^{a(n)} \in \mathbf{R}^{n_{n+1}\times(\sum_{i=0}^{n-j} p_i)}$ for $j \in \overline{r}$; $\hat{C}^{(n-j)} \in \mathbf{R}^{p_{n-j}\times n_{n-j}}$, $\hat{K}_j^{(n)} \in \mathbf{R}^{m_{n+1}\times p_{n-j}}$, $\hat{K}_j^{a(n)} \in \mathbf{R}^{m_{n+1}\times(\sum_{i=0}^{n-j} p_i)}$ for $j \in \overline{r} \cup \{0\}$, and $\hat{D}^{(n)}$, $\widetilde{\hat{D}}^{(n)} \in \mathbf{R}^{n_{n+1}\times n_n}$ and $\hat{B}_0^{(n)} \in \mathbf{R}^{n_{n+1}\times m_{n+1}}$, $\hat{B}_j^{(n)} \in \mathbf{R}^{n_{n+1}\times p_{n-j}}$ for $j \in \overline{r}$; $\forall n \in \mathbf{Z}_{0+}$.

Note that the aggregated subsystem (22)–(24) is coupled to the former global state $x^{(n)}$ describing the total system's dynamics prior to the aggregation action. It can be seen that the coupling terms do not necessary demonstrate infinite memory requirements as n tends to infinity, since the matrices $A^{(n+1)}$, $\hat{B}_j^{a(n)}$ and $\hat{K}_j^{a(n)}$ can contain nonzero columns associated with the most recent state/output data related to the previous aggregation system; see, for instance, [12]. Note also that, due to the coupling between the a priori whole state at the n-th iterations with the a priori new aggregated substate, it can happen that the a posteriori vector after the new aggregated substate has a higher dimension than its a priori version. The various dynamics, control and output matrices have the appropriate orders.

After incorporating the control law, we can write this whole system of extended states $x^{(n)} = \left(x^{0(n)^T}, \hat{x}^{(n)^T}\right)^T \in \mathbf{R}^{\sum_{i=0}^{n} n_i}$; $\forall n \in \mathbf{Z}_{0+}$ in a compact way:

$$x^{(n+1)} = \begin{bmatrix} x^{0(n+1)} \\ \hat{x}^{(n+1)} \end{bmatrix} = \begin{bmatrix} A^{0(n)} + B_0^{(n)} K_0^{(n)} C^{(n)} & \hat{A}^{0(n)} \\ \hat{A}^{(n+1)} + \hat{B}_0^{(n)} \hat{K}_0^{a(n)} C^{(n)} & \hat{D}^{(n)} + \widetilde{\hat{D}}^{(n)} + \hat{B}_0^{(n)} \hat{K}_0^{(n)} \hat{C}^{(n)} \end{bmatrix} \begin{bmatrix} x^{(n)} \\ \hat{x}^{(n)} \end{bmatrix}$$
$$+ \sum_{j=1}^{r} \begin{bmatrix} \left(B_j^{(n)} + B_0^{(n)} K_j^{(n)}\right) C^{(n)} & 0 \\ \left(\hat{B}_j^{a(n)} + \hat{B}_0^{(n)} \hat{K}_j^{a(n)}\right) C^{(n-j)} & \left(\hat{B}_j^{(n)} + \hat{B}_0^{(n)} \hat{K}_j^{(n)}\right) \hat{C}^{(n-j)} \end{bmatrix} x^{(n-j)}; \ \forall n \in \mathbf{Z}_{0+} \tag{25}$$

so that $x^{(n)}, x^{0(n+1)} \in \mathbf{R}^{\sum_{i=0}^{n} n_i}$ and $\hat{x}^{(n)} \in \mathbf{R}^{n_n}$ and $\hat{x}^{(n+1)} \in \mathbf{R}^{n_{n+1}}$ imply that $x^{(n+1)} \in \mathbf{R}^{\sum_{i=0}^{n+1} n_i}$, $x^{(n+1)} = \left(x^{0(n+1)^T}, \hat{x}^{(n+1)^T}\right)^T \in \mathbf{R}^{\sum_{i=0}^{n+1} n_i}$.

In order to construct a state vector which includes delayed dynamics, we now define the modified extended state $\overline{x}^{(n)}$ defined by $\overline{x}^{(n)} = \left(x^{(n)^T}, x^{(n-1)^T}, \ldots, x^{(n-r)^T}\right)^T \in \mathbf{R}^{\sum_{j=n-r}^{n} \sum_{i=0}^{j} n_i}$; $\forall n \in \mathbf{Z}_{0+}$. Thus, one determines from (25) that:

$$\overline{x}^{(n+1)} = \overline{A}^{(n)} \overline{x}^{(n)}; \ \forall n \in \mathbf{Z}_{0+} \tag{26}$$

where

$$\overline{A}^{(n)} = \begin{bmatrix} A^{0(n)} + B_0^{0(n)} K_0^{(n)} C^{(n)} & \hat{A}^{0(n)} & \left(B_1^{(n)} + B_0^{(n)} K_1^{(n)}\right) C^{(n-1)} & 0 & \cdots & \left(B_r^{(n)} + B_0^{(n)} K_r^{(n)}\right) C^{(n-r)} & 0 \\ \hat{A}^{(n+1)} + \hat{B}_0^{(n)} \hat{K}_0^{a(n)} C^{(n)} & D^{(n)} + \widetilde{D}^{(n)} + \hat{B}_0^{(n)} \hat{K}_0^{(n)} \hat{C}^{(n)} & \left(\hat{B}_1^{a(n)} + \hat{B}_0^{(n)} \hat{K}_1^{a(n)}\right) C^{(n-1)} & \left(\hat{B}_1^{(n)} + \hat{B}_0^{(n)} \hat{K}_1^{(n)}\right) \hat{C}^{(n-1)} & & \left(\hat{B}_r^{a(n)} + \hat{B}_0^{(n)} \hat{K}_r^{a(n)}\right) C^{(n-r)} & \left(\hat{B}_r^{(n)} + \hat{B}_0^{(n)} \hat{K}_r^{(n)}\right) \hat{C}^{(n-r)} \\ & I_{\sum_{i=0}^{n-1} n_i} & & \cdots & \cdots & & \cdots \\ & \cdots & & \ddots & \cdots & & \cdots \\ & I_{\sum_{i=0}^{n-r} n_i} & & \cdots & 0 & & 0 \end{bmatrix} \tag{27}$$

$\forall n \in \mathbf{Z}_{0+}$, with $\overline{A}^{(n)} \in \mathbf{R}^{(\sum_{j=n-r}^{n+1} \sum_{i=0}^{j} n_i)\times(\sum_{j=n-r}^{n} \sum_{i=0}^{j} n_i)}$. Now, consider the symmetric matrices:

$$\overline{M}^{(n)} = \overline{A}^{(n)^T} \overline{A}^{(n)} = \begin{bmatrix} \overline{A}^{(n)^T} \overline{A}^{(n)} & \overline{A}^{(n)^T} \overline{B}^{(n)} \\ \overline{B}^{(n)^T} \overline{A}^{(n)} & \overline{B}^{(n)^T} \overline{B}^{(n)} \end{bmatrix}$$

$$M^{(n)} = \begin{bmatrix} A^{0(n+1)^T} A^{0(n+1)} & A^{0(n+1)^T} B^{0(n+1)} \\ B^{0(n+1)^T} A^{0(n+1)} & B^{0(n+1)^T} B^{0(n+1)} \end{bmatrix}$$
$$\in \mathbf{R}^{(\sum_{i=0}^{n+1} n_i + \sum_{j=n-r}^{n} (\sum_{i=0}^{j} n_i + n_j)) \times (\sum_{i=0}^{n+1} n_i + \sum_{j=n-r}^{n} (\sum_{i=0}^{j} n_i + n_j))}; \ \forall n \in \mathbf{Z}_{0+} \tag{28}$$

where the relations between the a priori dynamics of the new iteration after the aggregation of a new substate to the whole dynamics with the a posteriori dynamics of the former iteration are given by:

$$A^{0(n+1)} = \begin{bmatrix} A^{(n)} & 0_{(\Sigma_{i=0}^{n+1} n_i)\times n_{n+1}} \end{bmatrix}$$
$$= \begin{bmatrix} A^{0(n)} + B_0^{(n)} K_0^{(n)} C^{(n)} & \hat{A}^{0(n)} & 0 \\ \hat{A}^{(n+1)} + \hat{B}_0^{(n)} \hat{K}_0^{a(n)} C^{(n)} & \hat{D}^{(n)} + \widetilde{\hat{D}}^{(n)} + \hat{B}_0^{(n)} \hat{K}_0^{(n)} \hat{C}^{(n)} & 0 \end{bmatrix} \in \mathbf{R}^{(\Sigma_{i=0}^{n+1} n_i)\times(\Sigma_{i=0}^{n+1} n_i)} \tag{29}$$

$$B^{0(n+1)} = B^{(n)} = \begin{bmatrix} \left(B_1^{(n)} + B_0^{(n)} K_1^{(n)}\right) C^{(n-1)} & 0 & \cdots & \left(B_j^{(n)} + B_0^{(n)} K_r^{(n)}\right) C^{(n-r)} & 0 \\ \left(\hat{B}_1^{a(n)} + \hat{B}_0^{(n)} \hat{K}_1^{a(n)}\right) C^{(n-1)} & \left(\hat{B}_1^{(n)} + \hat{B}_0^{(n)} \hat{K}_1^{(n)}\right) \hat{C}^{(n-1)} & & \left(\hat{B}_r^{a(n)} + \hat{B}_0^{(n)} \hat{K}_r^{a(n)}\right) C^{(n-r)} & \left(\hat{B}_j^{(n)} + \hat{B}_0^{(n)} \hat{K}_r^{(n)}\right) \hat{C}^{(n-r)} \end{bmatrix}$$
$$\in \mathbf{R}^{(\Sigma_{i=0}^{n+1} n_i)\times(\Sigma_{j=n-r}^{n-1} (\Sigma_{i=0}^{j} n_i + n_j))} \tag{30}$$

$\forall n \in \mathbf{Z}_{0+}$. which are built in order to complete a square a priori matrix of dynamics of the $(n+1)$- the aggregated system which was obtained after the aggregation of the $(n+1)$-th subsystem.

The stability of the aggregation dynamic system (19) to (24) under discrete delays is now discussed via the modified extended system (26), subject to (27), which can be obtained via Lemmas 7,8 from the convergence of the symmetric matrix (28), subject to (29),(30). The following result holds:

Theorem 1. *The following properties hold:*

(i) $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ *and* $\left\{\overline{A}^{(n)}\right\}_{n=0}^{\infty}$ *are convergent, and also asymptotically convergent, if and only if* $\lim_{m\to\infty} \overline{A}^{(n)^m} = 0$; $\forall n \in \mathbf{Z}_{0+}$.

(ii) $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ *and* $\left\{\overline{A}^{(n)}\right\}_{n=0}^{\infty}$ *are asymptotically convergent if and only if* $\lim_{m\to\infty} \left\{\left| \|\overline{A}^{(n+\xi)}\| - \|\overline{A}^{(n)}\| \right|^m \right\}_{n=0}^{\infty} = 0$ *for any given* $n \in \mathbf{Z}_{0+}$, $\xi \in \mathbf{Z}_{+}$.

(iii) If $\left\{\overline{A}^{(n)}\right\}_{n=0}^{\infty}$ *(and then* $\left\{M^{(n)}\right\}_{n=0}^{\infty}$*) is convergent, then the state of the modified extended system, (26), converges asymptotically to zero, i.e.* $\overline{x}^{(n+m)} \to 0$ *and also* $x^{(n+m)} \to 0$ *as* $m \to \infty$ *for any given initial condition* $x^{(0)}$ *and any* $n \in \mathbf{Z}_{0+}$ *so that the aggregation system is globally asymptotically stable.*

(iv) If $\left\{\overline{A}^{(n)}\right\}_{n=0}^{\infty}$ *(and then* $\left\{M^{(n)}\right\}_{n=0}^{\infty}$*) is asymptotically convergent then* $\|\overline{x}^{(n+m+\xi)} - \overline{x}^{(n+m)}\| \to 0$ *as* $m \to \infty$ *for any given* $n \in \mathbf{Z}_{0+}$, $\xi \in \mathbf{Z}_{+}$ *and any given initial condition* $x^{(0)}$ *and also* $\|x^{(n+m+\xi)} - x^{(n+m)}\| \to 0$ *as* $m \to \infty$ *for any given initial condition* $x^{(0)}$ *and any given* $n \in \mathbf{Z}_{0+}$, $\xi \in \mathbf{Z}_{+}$ *so that the incremental aggregation system is globally asymptotically stable.*

(v) Assume that $M^{(0)}$ *is convergent and that* $\left|\lambda_{max}\left(B^{(n)^T} B^{(n)}\right)\right| < min\left(1 - \varepsilon^{(n+1)}, \varepsilon^{(n)} - \varepsilon^{(n+1)}\right)$; $\forall n \in \mathbf{Z}_{0+}$ *for some strictly decreasing real sequence* $\left\{\varepsilon^{(n)}\right\}_{n=0}^{\infty} \subset [0, 1)$. *Then,* $\|M^{(n)}\|_2 < 1$; $\forall n \in \mathbf{Z}_{0+}$ *and* $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ *and* $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ *are convergent sequences.*

It is of interest to now discuss how the stability properties of the aggregation system of Theorem 1 can be guaranteed or addressed by the synthesis of the basic controller (21) on the current aggregated system, and how its updated rule (24) can be applied to the new aggregated subsystem to generate the aggregated system for the next iteration step. This discussion invokes conditions to guarantee that the equation of dimensionally compatible real matrices

$$BKC = A_m - A \tag{31}$$

is solvable in K for a given quadruple (A, B, C, A_m) with A and A_m being square, A_m being convergent (basically stable in the discrete context) and defining the closed-loop system dynamics after linear output-feedback control $u = Ky = KCx$ via the linear stabilizing controller of gain K; A, B and C are the open-loop dynamics (i.e., the one being got for $K = 0$) and B and C are the control and output matrices. Equation (31) is written in equivalent vector form for the unknown K as follows:

$$\left(B \otimes C^T\right) vec\,(K) = vec(A_m - A) \tag{32}$$

It turns out that (31) is solvable in K if and only if (32) is solvable in $vec\,(K)$, that is, if $rank\left(B \otimes C^T\right) = rank\left[B \otimes C^T, vec(A_m - A)\right]$ according to the Rouché-Froebenius theorem for solvability of linear systems of algebraic equations. Note that if A_m satisfies the constraint $A_m = EA$, for some square matrix E of the same order as A, then $\|A_m\|_2 < 1$ (so that A_m is convergent) if $\|E\|_2 < 1/\|A\|_2$. In particular, if $A_m = \rho A$ with $\rho \in \mathbf{R}$ then A_m is convergent if $|\rho| < 1/\|A\|_2$. A preliminary technical result concerning the solvability if the concerned algebraic system (31), or equivalently (32), is (either indeterminate or determinate) compatible to be then used follows:

Lemma 9. *Assume that $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{m\times n}$ and $C \in \mathbf{R}^{p\times n}$. Then, the following properties hold:*

(i) *linear output-feedback controller exists which stabilizes the closed-loop matrix of dynamics $A_m = EA$ for some E with $\|E\|_2 < 1/\|A\|_2$, [5,7], which satisfies the rank constraint:*

$$rank\left(B \otimes C^T\right) = rank\left[B \otimes C^T, \left(I \otimes A^T\right) vec(E) - vec(A)\right] \tag{33}$$

If (33) holds, then the set of stabilizing linear-output feedback controllers of gains K which solve (32), equivalently (31), which is a compatible algebraic linear system, for $A_m = EA$, are given by

$$vec\,(K) = \left(B \otimes C^T\right)^{\dagger} \left[\left(I \otimes A^T\right) vec(E) - vec(A)\right] + \left[I - \left(B \otimes C^T\right)^{\dagger} \left(B \otimes C^T\right)\right] k_w \tag{34}$$

with k_w being any arbitrary real vector of the same dimension as $vec\,(K)$. Assume that $\left(B \otimes C^T\right)^{\dagger} = \left(B \otimes C^T\right)^{-1}$ (a necessary condition being $\min(m,p) \geq n$). Then (32) for $A_m = EA$ is a compatible determinate, and the unique solution to (33) is

$$vec\,(K) = \left(B \otimes C^T\right)^{\dagger} \left[\left(I \otimes A^T\right) vec(E) - vec(A)\right] \tag{35}$$

If $A_m = \rho A$ with $|\rho| < 1/\|A\|_2$ then (33), (34) and (35) become, in particular,

$$rank\left(B \otimes C^T\right) = rank\left[B \otimes C^T, (\rho - 1) vec(A)\right] \tag{36}$$

$$vec\,(K) = (\rho - 1)\left(B \otimes C^T\right)^{\dagger} vec(A) + \left[I - \left(B \otimes C^T\right)^{\dagger} \left(B \otimes C^T\right)\right] k_w \tag{37}$$

and

$$vec\,(K) = \left(B \otimes C^T\right)^{\dagger} \left(B \otimes C^T\right)^{\dagger} vec(A) \tag{38}$$

(ii) *Assume that $A_m = EA$ and*

$$rank\left[B \otimes C^T, \left(I \otimes A^T\right) vec(E) - vec(A)\right] = rank\left(B \otimes C^T\right) + 1 \tag{39}$$

Then, (32), equivalently (31), is an algebraically incompatible system of equations, and

$$vec\,(K) = \left(B \otimes C^T\right)^{\dagger} \left[\left(I \otimes A^T\right) vec(E) - vec(A)\right], \text{ i.e.} \tag{40}$$

i.e., Equation (34) for $k_w = 0$, is the best least-squares approximated solution to (32) in the sense that the corresponding controller gain minimizes the norm error $\|BKC + A - A_m\|_2^2$. If (39) holds for any A_m of the form $A_m = EA$ then there is no solution to (31) in K; only best approximation solutions exist.

Particular cases of interest which are well-known from basic Control Theory (see e.g., [13]) are:

(1) $p = n$, $C \in \mathbf{R}^{n\times n}$ is non-singular and (A, B) is stabilizable, i.e., any unstable or critically unstable mode of the open-loop dynamics can be closed-loop stabilized under linear state feed-back control. Thus, $rank\,[\lambda I_n - A\,,\,B] = n$; $\forall\lambda \in \mathbf{C}$ with $|\lambda| \geq 1$ (discrete form of Popov-Belevitch-Hautus stabilizability test [6,13,14]). Then (31) becomes $BK = (A_m - A)C^{-1}$, which is solvable in K, and there is always an output-feedback stabilizing linear controller generating a stabilizing controller of gain K, generating a control $u = Ky = KCx$, such that the closed-loop dynamics is defined by a convergent matrix $A_m = A + BKC$.

(2) In Case 1, $C = I_n$. Then, the control law is a linear state-feedback control, and a state-feedback stabilizing linear controller generating a control $u = Kx$ exists, leading to closed-loop dynamics defined by the convergent matrix $A_m = A + BK$.

Lemma 9 is useful to guarantee the relevant results of Theorem 1 in terms of the controller gains choices under certain algebraic solvability conditions. This feature is addressed in the subsequent result:

Theorem 2. *Assume that:*

(1) $rank\left(B_0^{(n)} \otimes C^{(n)^T}\right) = rank\left(B_0^{(n)} \otimes C^{(n)^T}, A_f^{(n)} - A^{0(n)}\right)$ so that $A^{0(n)} + B_0^{(n)}K_0^{(n)}C^{(n)} = A_f^{(n)}$ is solvable in $K_0^{(n)}$ for some convergent matrix $A_f^{(n)}$ of appropriate order; $\forall n \in \mathbf{Z}_{0+}$, $rank\left(\hat{B}_0^{(n)} \otimes C^{(n)^T}\right) = rank\left(\hat{B}_0^{(n)} \otimes C^{(n)^T}, \hat{A}_f^{a(n+1)} - \hat{A}^{(n+1)}\right)$ so that $\hat{A}^{(n+1)} + \hat{B}_0^{(n)}\hat{K}_0^{(n)}C^{(n)} = \hat{A}_f^{a(n+1)}$ is solvable in $\hat{K}_0^{a(n)}$ for some matrix $\hat{A}_f^{a(n+1)}$ of appropriate order; $\forall n \in \mathbf{Z}_{0+}$, $rank\left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n)^T}\right) = rank\left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n)^T}, \hat{A}_f^{(n+1)} - \hat{D}^{(n)} - \widetilde{\hat{D}}^{(n)}\right)$ so that $\hat{A}_f^{(n)} = \hat{D}^{(n)} + \widetilde{\hat{D}}^{(n)} + \hat{B}_0^{(n)}\hat{K}_0^{(n)}\hat{C}^{(n)}$ is solvable in $\hat{K}_0^{(n)}$ for some matrix $\hat{A}_f^{(n)}$ of appropriate order; $\forall n \in \mathbf{Z}_{0+}$,

(2) and that subsequent rank conditions hold:

$$\begin{gathered}
rank\left(B_0^{(n)} \otimes C^{(n-i)^T}\right) = rank\left(B_0^{(n)} \otimes C^{(n-i)^T}, B_i^{(n)}C^{(n-i)}\right) \\
rank\left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n-i)^T}\right) = rank\left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n-i)^T}, \hat{B}_i^{(n)}\hat{C}^{(n-i)}\right) \\
rank\left(\hat{B}_0^{(n)} \otimes C^{(n-i)^T}\right) = rank\left(\hat{B}_0^{(n)} \otimes C^{(n-i)^T}, \hat{B}_i^{a(n)}C^{(n-i)}\right)
\end{gathered}$$

$\forall i \in \bar{r}$ so that the following matrix equations are solvable in the delayed controller gains $K_i^{(n)}$, $\hat{K}_i^{(n)}$ and $\hat{K}_i^{a(n)}$:

$$\begin{gathered}
B_0^{(n)}K_i^{(n)}C^{(n-i)} = -B_i^{(n)}C^{(n-i)}; \hat{B}_0^{(n)}\hat{K}_i^{(n)}\hat{C}^{(n-i)} = -\hat{B}_i^{(n)}\hat{C}^{(n-i)}; \\
\hat{B}_0^{(n)}\hat{K}_i^{a(n)}C^{(n-i)} = -\hat{B}_i^{(n)}C^{(n-i)}; \forall i \in \bar{r}; \forall n \in \mathbf{Z}_{0+}.
\end{gathered}$$

Then, the matrix equations

$$\begin{gathered}
A^{0(n)} + B_0^{(n)}K_0^{(n)}C^{(n)} = A_f^{(n)},\ \hat{A}^{(n+1)} + \hat{B}_0^{(n)}\hat{K}_0^{a(n)}C^{(n)} = \hat{A}_f^{a(n+1)}, \\
\hat{A}_f^{(n)} = \hat{D}^{(n)} + \widetilde{\hat{D}}^{(n)} + \hat{B}_0^{(n)}\hat{K}_0^{(n)}\hat{C}^{(n)},\ B_0^{(n)}K_i^{(n)}C^{(n-i)} = -B_i^{(n)}C^{(n-i)}; \\
\hat{B}_0^{(n)}\hat{K}_i^{(n)}\hat{C}^{(n-i)} = -\hat{B}_i^{(n)}\hat{C}^{(n-i)};\ \forall i \in \bar{r};\ \forall n \in \mathbf{Z}_{0+})
\end{gathered} \tag{41}$$

are solvable in the controller gains $K_0^{(n)}$, $K_i^{(n)}$ and $\hat{K}_i^{(n)}$; $\forall i \in \overline{r}$; $\forall n \in \mathbf{Z}_{0+}$ leading to the solutions

$$
\begin{aligned}
vec\left(K_0^{(n)}\right) &= \left(B_0^{(n)} \otimes C^{(n)^T}\right)^{\dagger} vec\left(A_f^{(n)} - A^{0(n)}\right) + \left(I - \left(B_0^{(n)} \otimes C^{(n)^T}\right)^{\dagger}\left(B_0^{(n)} \otimes C^{(n)^T}\right)\right) vec\left(K_0^{v(n)}\right) \\
vec\left(\hat{K}_0^{a(n)}\right) &= \left(\hat{B}_0^{(n)} \otimes C^{(n)^T}\right)^{\dagger} vec\left(\hat{A}_f^{a(n+1)} - \hat{A}^{(n+1)}\right) + \left(I - \left(\hat{B}_0^{(n)} \otimes C^{(n)^T}\right)^{\dagger}\left(\hat{B}_0^{(n)} \otimes C^{(n)^T}\right)\right) vec\left(\hat{K}_0^{av(n)}\right) \\
vec\left(\hat{K}_0^{(n)}\right) &= \left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n)^T}\right)^{\dagger} vec\left(\hat{A}_f^{(n+1)} - \hat{D}^{(n)} - \widetilde{\hat{D}}^{(n)}\right) + \left(I - \left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n)^T}\right)^{\dagger}\left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n)^T}\right)\right) vec\left(\hat{K}_0^{v(n)}\right) \\
vec\left(K_i^{(n)}\right) &= -\left(B_0^{(n)} \otimes C^{(n-i)^T}\right)^{\dagger} vec\left(B_i^{(n)} C^{(n-i)}\right) + \left(I - \left(B_0^{(n)} \otimes C^{(n-i)^T}\right)^{\dagger}\left(B_0^{(n)} \otimes C^{(n-i)^T}\right)\right) vec\left(K_i^{v(n)}\right) \\
vec\left(\hat{K}_i^{(n)}\right) &= -\left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n-i)^T}\right)^{\dagger} vec\left(\hat{B}_i^{(n)} \hat{C}^{(n-i)}\right) + \left(I - \left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n-i)^T}\right)^{\dagger}\left(\hat{B}_0^{(n)} \otimes \hat{C}^{(n-i)^T}\right)\right) vec\left(\hat{K}_i^{v(n)}\right)
\end{aligned}
\quad (42)
$$

$$
vec\left(\hat{K}_i^{a(n)}\right) = -\left(\hat{B}_0^{(n)} \otimes C^{(n-i)^T}\right)^{\dagger} vec\left(\hat{B}_i^{a(n)} C^{(n-i)}\right) + \left(I - \left(\hat{B}_0^{(n)} \otimes C^{(n-i)^T}\right)^{\dagger}\left(\hat{B}_0^{(n)} \otimes C^{(n-i)^T}\right)\right) vec\left(\hat{K}_i^{av(n)}\right)
$$

with $K_0^{v(n)}$, $K_0^{av(n)}$, $\hat{K}_0^{v(n)}$, $K_i^{v(n)}$, $\hat{K}_i^{v(n)}$ and $\hat{K}_i^{av(n)}$; $\forall i \in \overline{r}$; $\forall n \in \mathbf{Z}_{0+}$ being arbitrary matrices of appropriate orders for the corresponding equation (above) in each case whose equivalent vector expressions are denoted by vec (.).

It should be pointed out that it can be of interest to apply the results on interlacing Cauchy's theorem and some of its extensions (see e.g., [20–22]) to the stability of aggregation models based on dynamic systems formulated via differential, difference or hybrid differential/difference equations.

6. Conclusions

This paper relies on partitioned Hermitian matrices and Cauchy's interlacing theorem and the associated stability results. Based on the fact that convergent matrices are a discrete counterpart of stability matrices, the results presented above are then extended to sequences of convergent matrices. Then, an example of a SIR-type epidemic model continuous-time consisting of intercommunity clusters is discussed relative to the previously given stability theoretic results under the proposed framework based on Cauchy's interlacing theorem, and which may be of interest for healthcare management. Later, a dynamic linear discrete aggregation model is discussed, which involves output delay and linear output feedback, and which can be also be reformulated for linear-state feedback by identifying state and output, that is, by taking the output matrix equal to the identity, provided that the state is available for measurement. The studied aggregation model is built through the successive incorporation of discrete subsystems with particular coupled dynamics. Stabilizing decentralized controllers are proposed and discussed for this type of aggregation model.

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Appendix A Mathematical Proofs

Proof of Lemma 1. From the given assumptions, M is Hermitian, and note that

$$
\det M\prime = \det\begin{bmatrix} M & 0 \\ m^* & d \end{bmatrix} + \det\begin{bmatrix} M & m \\ m^* & \widetilde{d} \end{bmatrix} = \det\begin{bmatrix} M & 0 \\ m^* & \widetilde{d} \end{bmatrix} + \det\begin{bmatrix} M & m \\ m^* & d \end{bmatrix} = \det\begin{bmatrix} M & 0 \\ m^* & d+\widetilde{d} \end{bmatrix} + \det\begin{bmatrix} M & m \\ m^* & 0 \end{bmatrix}.
$$

Thus, Properties (i) to (iii) follow directly from the above relations by noting, furthermore, that $\det\begin{bmatrix} M & 0 \\ m^* & c \end{bmatrix} = c\det M$. □

Proof of Lemma 2. First, note that

$$k_d^{(n)} - k_{\widetilde{d}}^{(n)} \le \| d^{(n)} | - | \widetilde{d}^{(n)} \| \le | d^{(n)} + \widetilde{d}^{(n)} | \le K_d^{(n)} + K_{\widetilde{d}}^{(n)};\ \forall n(\ge n_0) \in \mathbf{Z}_+.$$

Thus,

$$\begin{gathered} \det M^{(n+1)} = \det\begin{bmatrix} M^{(n)} & 0 \\ m^{(n)*} & d^{(n)} \end{bmatrix} + \det\begin{bmatrix} M^{(n)} & m^{(n)} \\ m^{(n)*} & \widetilde{d}^{(n)} \end{bmatrix} \\ = d^{(n)}\det M^{(n)} + \det\left(\begin{bmatrix} M^{(n)} & 0 \\ 0 & \widetilde{d}^{(n)} \end{bmatrix}\left(I_{n+1} + \begin{bmatrix} {M^{(n)}}^{-1} & 0 \\ 0 & 1/\widetilde{d}^{(n)} \end{bmatrix}\begin{bmatrix} 0 & m^{(n)} \\ m^{(n)*} & 0 \end{bmatrix}\right)\right) \\ = d^{(n)}\det M^{(n)} + \widetilde{d}^{(n)}\det M^{(n)}\det\left(I_{n+1} + \begin{bmatrix} {M^{(n)}}^{-1} & 0 \\ 0 & 1/\widetilde{d}^{(n)} \end{bmatrix}\begin{bmatrix} 0 & m^{(n)} \\ m^{(n)*} & 0 \end{bmatrix}\right); \\ \forall n(\ge n_0) \in \mathbf{Z}_+ \end{gathered} \tag{A1}$$

if $\left\{{M^{(n)}}^{-1}\right\}_{n=n_0}^{\infty}$ exists so that $\det M^{(n)} \ne 0$, that is, if $0 < k_M^{(n)} = \left|\lambda_{\min}\left(M^{(n)}\right)\right| \le K_M^{(n)} = \left|\lambda_{\max}\left(M^{(n)}\right)\right| < +\infty$; $\forall n(\ge n_0) \in \mathbf{Z}_+$. Thus,

$$\begin{gathered} \left|\det M^{(n+1)}\right| \le \left|d^{(n)}\det M^{(n)}\right| + \left|\widetilde{d}^{(n)}\det M^{(n)}\right|\left|\det\left(I_{n+1} + \begin{bmatrix} {M^{(n)}}^{-1} & 0 \\ 0 & 1/\widetilde{d}^{(n)} \end{bmatrix}\begin{bmatrix} 0 & m^{(n)} \\ m^{(n)*} & 0 \end{bmatrix}\right)\right| \\ \le nK_d^{(n)}K_M^{(n)} + nK_{\widetilde{d}}^{(n)}K_M^{(n)}\left(1 + \frac{1}{k_M^{(n)}k_{\widetilde{d}}^{(n)}}\sqrt{m^{(n)*}m^{(n)}}\right)^{n+1} \\ \le nK_d^{(n)}K_M^{(n)} + nK_{\widetilde{d}}^{(n)}K_M^{(n)}\left(1+\varepsilon^{(n)}\right)^{n+1} \le nK_d^{(n)}K_M^{(n)} + nK_{\widetilde{d}}^{(n)}K_M^{(n)}\left(1 + \left(2^{n+1}-1\right)\varepsilon^{(n)}\right); \\ \forall n(\ge n_0) \in \mathbf{Z}_+ \end{gathered} \tag{A2}$$

since $\lambda_{\max}\left(m^{(n)*}m^{(n)}\right) = m^{(n)*}m^{(n)}$; $\forall n(\ge n_0) \in \mathbf{Z}_+$. Since it is assumed the existence of a real sequence $\left\{\varepsilon^{(n)}\right\}_{n=n_0}^{\infty} \subset [0, 1)$ such that $\frac{1}{k_M^{(n)}k_{\widetilde{d}}^{(n)}}\sqrt{m^{(n)*}m^{(n)}} \le \varepsilon^{(n)}$; $\forall n(\ge n_0) \in \mathbf{Z}_+$ and since

$$\begin{gathered} \left(1+\varepsilon^{(n)}\right)^{n+1} = \sum_{i=0}^{n+1}\binom{n+1}{i}{\varepsilon^{(n)}}^{i} = 1 + \sum_{i=1}^{n+1}\binom{n+1}{i}{\varepsilon^{(n)}}^{i} \le 1 + \left(2^{n+1}-1\right)\varepsilon^{(n)}; \\ \forall n(\ge n_0) \in \mathbf{Z}_+ \end{gathered} \tag{A3}$$

thus, it follows that $\left|\det M^{(n+1)}\right| \le (n+1)K_M^{(n+1)} \le nK_M^{(n)}$ holds for any given $n(\ge n_0) \in \mathbf{Z}_+$ if

$$nK_d^{(n)}K_M^{(n)} + nK_{\widetilde{d}}^{(n)}K_M^{(n)}\left(1 + \left(2^{n+1}-1\right)\varepsilon^{(n)}\right) \le nK_M^{(n)}$$

for any given $n(\ge n_0) \in \mathbf{Z}_+$, that is, if

$$nK_M^{(n)}\left(K_d^{(n)} - 1\right) + nK_{\widetilde{d}}^{(n)}K_M^{(n)}\left(1 + \left(2^{n+1}-1\right)\varepsilon^{(n)}\right) \le 0$$

for such $n \in \mathbf{Z}_+$, equivalently, if $nK_{\widetilde{d}}^{(n)}\left(1+\left(2^{n+1}-1\right)\varepsilon^{(n)}\right) \leq n\left(1-K_d^{(n)}\right)$, equivalently if $K_d^{(n)} \leq 1$, and furthermore, and accordingly to the restriction on the sequence $\left\{\varepsilon^{(n)}\right\}_{n=n_0}^{\infty}$, if

$$K_{\widetilde{d}}^{(n)} \leq \frac{\left(1-K_d^{(n)}\right)}{1+(2^{n+1}-1)\varepsilon^{(n)}} \leq \frac{\left(1-K_d^{(n)}\right)k_M^{(n)}k_d^{(n)}}{k_M^{(n)}k_d^{(n)}+(2^{n+1}-1)\sqrt{\lambda_{\max}\left(m^{(n)*}m^{(n)}\right)}} \tag{A4}$$

for such an $n \in \mathbf{Z}_{0+}$. In particular, if $\left|d^{(n)}\right| = K_d^{(n)} = k_d^{(n)} \leq 1$ and $\left|\widetilde{d}^{(n)}\right| = K_{\widetilde{d}}^{(n)} = k_{\widetilde{d}}^{(n)}$ for some given $n(\geq n_0) \in \mathbf{Z}_{0+}$, then (A4) holds if

$$\left|\widetilde{d}^{(n)}\right| \leq \frac{\left(1-\left|d^{(n)}\right|\right)}{1+(2^{n+1}-1)\varepsilon^{(n)}} \leq \frac{\left(1-\left|d^{(n)}\right|\right)k_M^{(n)}\left|d^{(n)}\right|}{k_M^{(n)}\left|d^{(n)}\right|+(2^{n+1}-1)\sqrt{\lambda_{\max}\left(m^{(n)*}m^{(n)}\right)}} \tag{A5}$$

and Property (i) has been proved. On the other hand, since Property (ii) assumes that the constraints of Property (i) hold the constraint (1) is a direct consequence of Property (i). Also, if $M^{(n)}\succeq 0$ and $M^{(n+1)}\succeq 0$ then the constraints (2) follow directly from (1) and Cauchy's interlacing theorem of the eigenvalues which are real non-negative. Property (ii) has been proved. Property (iii) follows since $\limsup_{n\to\infty}\left(\det\left|M^{(n+1)}\right| - \det\left|M^{(n)}\right|\right) \leq 0$ directly from (A1) and the given assumptions. □

Proof of Lemma 3. By the inversion of a block partitioned matrix, one gets:

$$M^{(n+1)^{-1}} = \begin{bmatrix} M^{(n)} & m^{(n)} \\ m^{(n)*} & d^{(n)}+\widetilde{d}^{(n)} \end{bmatrix}^{-1} = \begin{bmatrix} \overline{M}^{(n)} & \overline{m}^{(n)} \\ \overline{m}^{(n)*} & \overline{d}^{(n)}+\overline{\widetilde{d}}^{(n)} \end{bmatrix} \tag{A6}$$

where

$$\overline{M}^{(n)} = M^{(n)^{-1}}\left(I_n + m^{(n)}\left(d^{(n)}+\widetilde{d}^{(n)} - m^{(n)*}M^{(n)^{-1}}m^{(n)}\right)^{-1}m^{(n)*}M^{(n)^{-1}}\right) \tag{A7}$$

$$\overline{m}^{(n)} = -M^{(n)^{-1}}m^{(n)}\left(d^{(n)}+\widetilde{d}^{(n)} - m^{(n)*}M^{(n)^{-1}}m^{(n)}\right)^{-1} \tag{A8}$$

$$\overline{d}^{(n)}+\overline{\widetilde{d}}^{(n)} = \left(d^{(n)}+\widetilde{d}^{(n)} - m^{(n)*}M^{(n)^{-1}}m^{(n)}\right)^{-1}$$

since $M^{(n)}$ is non-singular and $\widetilde{d}^{(n)} \neq m^{(n)*}M^{(n)^{-1}}m^{(n)} - d^{(n)}$ (i.e., $\left(d^{(n)}+\widetilde{d}^{(n)} - m^{(n)*}M^{(n)^{-1}}m^{(n)}\right)$ is non-singular). The proof follows directly from Lemma 2 [(ii)-(iii)] by replacing $M^{(n+1)} \to \overline{M}^{(n+1)} = \overline{M}^{(n+1)^{-1}}$; $\forall n(\geq n_0) \in \mathbf{Z}_+$. □

Proof of Lemma 4. Since $M^{(n_0)} \succ 0$ then its inverse is also positive definite and then both of them fulfil the positive semi-definiteness constraint of Lemma 2 [(ii),(iii)] and Lemma 3. The proof follows directly since $\sup_{n\geq n_0}\max\left(\left|\det M^{(n)}\right|, \left|\det M^{(n)^{-1}}\right|\right) < +\infty$ and $\limsup_{n\to\infty}\left(\sup\max\left(\left|\det M^{(n)}\right|, \left|\det M^{(n)^{-1}}\right|\right)\right) < +\infty$ then $M^{(n)} \succ 0$ for $n \geq n_0$ if $M^{(n_0)} \succ 0$, $\limsup_{n\to\infty} M^{(n)} \succ 0$ and $\liminf_{n\to\infty} M^{(n)} \succ 0$. □

Proof of Lemma 5. Note that $M^{(n_0)} = A^{(n_0)*}A^{(n_0)} \succ 0$ since $A^{(n_0)}$ is a stability matrix. From Lemma 2 and Lemma 3, the sequence $\left\{M^{(n)}\right\}_{n=n_0}^{\infty}$ consists of positive definite members. Thus, the singular values of the elements of the sequence $\left\{A^{(n)}\right\}_{n=n_0}^{\infty}$ are positive and bounded. Since $A^{(n_0)}$ is stable and since the eigenvalues of any square matrix are continuous functions of its entries there is no zero eigenvalue in any member of the sequence $\left\{A^{(n)}\right\}_{n=n_0}^{\infty}$. Any member of this sequence has no eigenvalues at the imaginary complex axis other than zero (i.e., any nonzero critically stable eigenvalues) since then the corresponding $M^{(n)} = A^{(n)*}A^{(n)}$ is not positive definite contradicting the given assumption. As a result, no member of the sequence $\left\{A^{(n)}\right\}_{n=n_0}^{\infty}$ has a critically stable eigenvalue (i.e., located on the imaginary complex axis) or unstable eigenvalue (i.e., located on the complex open right half plane).

$M^{(n_0)} \succ 0$ for some given arbitrary $n_0 \in \mathbf{Z}_+$ and assume also that the conditions of Lemma 2 (iii) and Lemma 3 hold. Then, $M^{(n+1)} \succ 0$; $\forall n(\geq n_0) \in \mathbf{Z}_+$. □

Proof of Lemma 7. It is direct since $\lim_{m\to\infty}\left\{A^{(n)m}\right\}_{n=0}^{\infty} = 0$ implies that $\lim_{m\to\infty}\left\{\left|\|A^{(n+\xi)}\| - \|A^{(n)}\|\right|^m\right\}_{n=0}^{\infty} = 0$ for any given $\xi \in \mathbf{Z}_+$. □

Proof of Lemma 8. If for any given $n \in \mathbf{Z}_{0+}$, $\|A^{(n)}\|_2 < 1$ then $\lim_{m\to\infty} A^{(n)m} = 0$ for any $n \in \mathbf{Z}_{0+}$, then $\lim_{m\to\infty}\left\{A^{(n)m}\right\}_{n=0}^{\infty} = 0$, $\|M^{(n)}\|_2^2 = \|A^{(n)*}A^{(n)}\|_2^2 \leq \|A^{(n)}\|_2^4 < 1$ and then $\|M^{(n)}\|_2 < 1$, $\lim_{m\to\infty} M^{(n)m} = 0$ and $\lim_{m\to\infty}\left\{M^{(n)m}\right\}_{n=0}^{\infty} = 0$ for any given $n \in \mathbf{Z}_{0+}$. Thus, if $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ is convergent then $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ is convergent, hence Property (i) holds. On the other hand, since $M^{(n)}$ is semidefinite positive Hermitian by construction; $\forall n \in \mathbf{Z}_{0+}$ then the condition $\|M^{(n)}\|_2 < 1$; $\forall n \in \mathbf{Z}_{0+}$ leads to the convergence of $\left\{M^{(n)}\right\}_{n=1}^{\infty}$, and if $\|M^{(n)}\|_2 < 1$ for some $n \in \mathbf{Z}_{0+}$ then

$$r^2\left(M^{(n)}\right) = \lambda_{\max}\left(M^{(n)*}M^{(n)}\right) = \lambda_{\max}^2\left(A^{(n)*}A^{(n)}\right) = \lambda_{\max}^2\left(M^{(n)}\right) = \|M^{(n)}\|_2^2 < 1$$

Then, $\|A^{(n)}\|_2 = \lambda_{\max}^{1/2}\left(A^{(n)*}A^{(n)}\right) = \|M^{(n)}\|_2^{1/2} < 1$. Thus, if the above holds for any $n \in \mathbf{Z}_{0+}$, one concludes that $\left\{A^{(n)}\right\}_{n=0}^{\infty}$ is convergent if $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ is convergent. Hence, Property (ii) follows. Property (iii) is a combination of the other two properties since for any $n \in \mathbf{Z}_{0+}$, $\|M^{(n)}\|_2 < 1$, $\|A^{(n)}\|_2 < 1$ implies that $\|M^{(n)}\|_2 < 1$ and $\|M^{(n)}\|_2 < 1$ implies that $\|A^{(n)}\|_2 < 1$. □

Proof of Theorem 1. Properties (i),(ii) follows from Lemma 7and Lemma 8 by taking into account the factorization (28). Property (iii) follows from Property (i) and (20) since

$$\|\overline{x}^{(n+m)}\| \leq \left(\prod_{i=0}^{m-1} \|\overline{A}^{(n-i)}\|\right)\|\overline{x}^{(n)}\| \leq \max_{0\leq i\leq m-1}\|\overline{A}^{(n-i)}\|^m \|\overline{x}^{(n)}\|;\ \forall n \in \mathbf{Z}_{0+}, \forall m \in \mathbf{Z}_+ \tag{A9}$$

and then $\overline{x}^{(n+m)} \to 0$ as $m \to \infty$ for any $n \in \mathbf{Z}_{0+}$. Property (iv) is proved in the same way as Property (iii) via Property (ii). The proof of Property (v) is made by comparing (28) with (10)–(12) by replacing $\lambda^{(n)}\overline{M}^{(n)} \to A^{(n)^T}B^{(n)}$ and $\Delta^{(n)} \to B^{(n)^T}B^{(n)}$. One gets, via complete induction, that if $M^{(0)}$ is convergent and, furthermore,

$$\lambda_{\max}\left(B^{(n)^T}A^{(n)}A^{(n)^T}B^{(n)}\right) < \left(1 - \varepsilon^{(n+1)} - \max\left(1 - \varepsilon^{(n)}, \left|\lambda_{\max}\left(B^{(n)^T}B^{(n)}\right)\right|\right)\right)^2;\quad \forall n \in \mathbf{Z}_{0+} \tag{A10}$$

Then, $\left\{M^{(n)}\right\}_{n=0}^{\infty}$ is convergent, since $M^{(0)}$ is convergent, provided that $\varepsilon^{(n+1)} < \varepsilon^{(n)}$, that is $\left\{\varepsilon^{(n)}\right\}_{n=0}^{\infty} \subset [0, 1)$ is strictly decreasing, and

$$\left|\lambda_{\max}\left(B^{(n)^T}B^{(n)}\right)\right| < 1 - \varepsilon^{(n+1)};\ \forall n \in \mathbf{Z}_{0+} \tag{A11}$$

The constrains (A10), (A11) are jointly fulfilled if $\left|\lambda_{\max}\left(B^{(n)^T}B^{(n)}\right)\right| < \min\left(1 - \varepsilon^{(n+1)}, \varepsilon^{(n)} - \varepsilon^{(n+1)}\right)$; $\forall n \in \mathbf{Z}_{0+}$.□

Proof of Lemma 9. Since $A_m = EA = IEA$ then $vec\,(EA) = vec(IEA) = \left(I \otimes A^T\right) vec(E)$, and (32) becomes for I being the identity matrix of the same order as E and A:

$$\left(B \otimes C^T\right) vec\,(K) = vec((E-I)A) = \left(I \otimes A^T\right) vec(E) - vec(A) \tag{A12}$$

whose solutions are given by (34) if (33) holds and the whole set of solutions reduces to (35) if the solution is unique. The corresponding Equations (36)–(38) are got for the particular case when $A_m = \rho A$ with $|\rho| < 1/\|A\|_2$. Property (i) has been proved. Property (ii) follows since the least-squares best approximation to the corresponding incompatible algebraic system (31), or (32), is (40), that is, (34) for $k_w = 0$, [13,14]. □

Proof of Theorem 2. The solvability of (41) in the form (42) follows from the Rouché-Froebenius rank conditions from the algebraic compatibility under Assumptions 1,2. By defining

$$A^{0(n+1)} = \left[A^{(n)} \;\; 0_{(\Sigma_{i=0}^{n+1} n_i)\times n_{n+1}}\right];\; B^{0(n+1)} = B^{(n)}$$

in order to complete a square "a priori" matrix of dynamics of the $(n+1)$-the aggregated system obtained after the aggregation of the $(n+1)$-th subsystem, note that

$$A^{0(n+1)} = \begin{bmatrix} A^{0(n)} + B_0^{(n)} K_0^{(n)} C^{(n)} & \hat{A}^{0(n)} & 0 \\ \hat{A}^{(n+1)} + \hat{B}_0^{(n)} \hat{K}_0^{a(n)} C^{(n)} & \hat{D}^{(n)} + \widetilde{\hat{D}}^{(n)} + \hat{B}_0^{(n)} \hat{K}_0^{(n)} \hat{C}^{(n)} & 0 \end{bmatrix} \in \mathbf{R}^{(\Sigma_{i=0}^{n+1} n_i)\times(\Sigma_{i=0}^{n+1} n_i)} \tag{A13}$$

$$B^{0(n+1)} = \begin{bmatrix} \left(B_1^{(n)} + B_0^{(n)} K_1^{(n)}\right) C^{(n-1)} & 0 & \cdots & \left(B_j^{(n)} + B_0^{(n)} K_r^{(n)}\right) C^{(n-r)} & 0 \\ \left(\hat{B}_1^{a(n)} + \hat{B}_0^{(n)} \hat{K}_1^{a(n)}\right) C^{(n-1)} & \left(\hat{B}_1^{(n)} + \hat{B}_0^{(n)} \hat{K}_1^{(n)}\right) \hat{C}^{(n-1)} & & \left(\hat{B}_r^{a(n)} + \hat{B}_0^{(n)} \hat{K}_r^{a(n)}\right) C^{(n-r)} & \left(\hat{B}_j^{(n)} + \hat{B}_0^{(n)} \hat{K}_r^{(n)}\right) \hat{C}^{(n-r)} \end{bmatrix} \in \mathbf{R}^{(\Sigma_{i=0}^{n+1} n_i)\times(\Sigma_{j=n-r}^{n-1} (\Sigma_{i=0}^{j+1} n_i))} \tag{A14}$$

$\forall n \in \mathbf{Z}_{0+}$. From (41), with corresponding associated controller explicit solutions (42), one gets that the $(n+1)$-th aggregated delay-free dynamics is described by the matrix:

$$A^{0(n+1)} = \left[A^{(n)} \;\; 0_{(\Sigma_{i=0}^{n+1} n_i)\times n_{n+1}}\right] = \begin{bmatrix} A_f^{(n)} & \hat{A}^{0(n)} & 0 \\ \hat{A}_f^{a(n+1)} & \hat{A}_f^{(n)} & 0 \end{bmatrix} \in \mathbf{R}^{(\Sigma_{i=0}^{n+1} n_i)\times(\Sigma_{i=0}^{n+1} n_i)};\; \forall n \in \mathbf{Z}_{0+} \tag{A15}$$

Having in mind (27), construct

$$M^{(n+1)} = \overline{A}^{(n)} \overline{A}^{(n)T} = \begin{bmatrix} A_f^{(n)} & \hat{A}^{0(n)} & 0 & \overline{B}_1^{0(n)} \\ \hat{A}_f^{a(n+1)} & \hat{A}_f^{(n)} & 0 & \overline{B}_2^{0(n)} \\ i\left(\overline{B}^{0(n)}\right) I & & 0 & \end{bmatrix} \begin{bmatrix} A_f^{(n)} & \hat{A}^{0(n)} & 0 & \overline{B}_1^{0(n)} \\ \hat{A}_f^{a(n+1)} & \hat{A}_f^{(n)} & 0 & \overline{B}_2^{0(n)} \\ i\left(\overline{B}^{0(n)}\right) I & 0 & & \end{bmatrix}^T$$

$$= \begin{bmatrix} A_f^{(n)} A_f^{(n)^T} + \hat{A}^{0(n)} \hat{A}^{0(n)T} + \overline{B}_1^{0(n)} \overline{B}_1^{0(n)^T} & A_f^{(n)} \hat{A}_f^{a(n+1)^T} + \hat{A}^{0(n)} \hat{A}_f^{(n)^T} + \overline{B}_1^{0(n)} \overline{B}_2^{0(n)^T} & \begin{bmatrix} A_f^{(n)} & \hat{A}^{0(n)} \\ \hat{A}_f^{a(n+1)} & \hat{A}_f^{(n)} \end{bmatrix} \\ \begin{bmatrix} A_f^{(n)} & \hat{A}^{0(n)} \\ \hat{A}_f^{a(n+1)} & \hat{A}_f^{(n)} \end{bmatrix}^T & & i\left(\overline{B}^{0(n)}\right) I \end{bmatrix}$$

$$= \begin{bmatrix} A_f^{(n)} A_f^{(n)^T} + \hat{A}^{0(n)} \hat{A}^{0(n)T} + \overline{B}_1^{0(n)} \overline{B}_1^{0(n)^T} & A_f^{(n)} \hat{A}_f^{a(n+1)^T} + \hat{A}^{0(n)} \hat{A}_f^{(n)^T} + \overline{B}_1^{0(n)} \overline{B}_2^{0(n)^T} & \begin{bmatrix} A_f^{(n)} \\ \hat{A}_f^{a(n+1)} \end{bmatrix} \begin{bmatrix} \hat{A}^{0(n)} \\ \hat{A}_f^{(n)} \end{bmatrix} \\ \begin{bmatrix} A_f^{(n)^T} & \hat{A}_f^{a(n+1)^T} \end{bmatrix} \begin{bmatrix} \hat{A}_f^{0(n)^T} & \hat{A}_f^{(n)^T} \end{bmatrix} & & i\left(\overline{B}^{0(n)}\right) \delta^{(n)} I \end{bmatrix}$$

$$+ i\left(\overline{B}^{0(n)}\right) \begin{bmatrix} 0 & 0 \\ 0 & \left(1-\delta^{(n)}\right) I \end{bmatrix}$$

where

$$\overline{B}^{0(n)} = \begin{bmatrix} \overline{B}_1^{0(n)} \\ \overline{B}_2^{0(n)} \end{bmatrix} = \begin{bmatrix} \left(B_1^{(n)} + B_0^{(n)} K_1^{(n)}\right) C^{(n-1)} & 0 & \cdots & \left(B_r^{(n)} + B_0^{(n)} K_r^{(n)}\right) C^{(n-r)} & 0 \\ \left(\hat{B}_1^{a(n)} + \hat{B}_0^{(n)} \hat{K}_1^{a(n)}\right) C^{(n-1)} & \left(\hat{B}_1^{(n)} + \hat{B}_0^{(n)} \hat{K}_1^{(n)}\right) \hat{C}^{(n-1)} & & \left(\hat{B}_r^{a(n)} + \hat{B}_0^{(n)} \hat{K}_r^{a(n)}\right) C^{(n-r)} & \left(\hat{B}_r^{(n)} + \hat{B}_0^{(n)} \hat{K}_r^{(n)}\right) \hat{C}^{(n-r)} \end{bmatrix} \tag{A16}$$

and

(a) $i\left(\overline{B}^{0(n)}\right)$ is a binary indicator function defined as $i\left(\overline{B}^{0(n)}\right) = 1$ if $\left(\overline{B}^{0(n)}\right) \neq 0$ and $i\left(\overline{B}^{0(n)}\right) = 0$ if $\left(\overline{B}^{0(n)}\right) = 0$. The reason of the use of this indicator is that, in fact, if the delayed dynamics is zero then the dimension of the extended state, so that of $\overline{A}^{(n)}$, decreases since the resulting block identity matrices are removed,

(b) $\left\{\delta^{(n)}\right\}_{n=0}^{\infty} \subset [0, 1]$ is a design sequence which satisfies $\left\{\delta^{(n)}\right\}_{n=0}^{\infty} \to 0$.

Note that the fact that $I = i\left(\overline{B}^{0(n)}\right) I = i\left(\overline{B}^{0(n)}\right) \delta^{(n)} I + i\left(\overline{B}^{0(n)}\right)\left(1 - \delta^{(n)}\right) I$ justifies (A16). □

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