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Geometric Properties of Certain Classes of Analytic Functions Associated with a q-Integral Operator

Shahid Mahmood [1], Nusrat Raza [2,*], Eman S. A. AbuJarad [3], Gautam Srivastava [4,5], H. M. Srivastava [6,7] and Sarfraz Nawaz Malik [8]

1. Department of Mechanical Engineering, Sarhad University of Science & Information Technology, Ring Road, Peshawar 25000, Pakistan; shahidmahmood757@gmail.com
2. Mathematics Section, Women's College, Aligarh Muslim University, Aligarh 202001, Uttar Pradesh, India
3. Department of Mathematics, Aligarh Muslim University, Aligarh 202001, Uttar Pradesh, India; emanjarad2@gmail.com
4. Department of Mathematics and Computer Science, Brandon University, 270 18th Street, Brandon, MB R7A 6A9, Canada; srivastavag@brandonu.ca
5. Research Center for Interneural Computing, China Medical University, Taichung 40402, Taiwan, Republic of China
6. Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
7. Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China
8. Department of Mathematics, COMSATS University Islamabad, Wah Campus 47040, Pakistan; snmalik110@yahoo.com

* Correspondence: nraza.maths@gmail.com

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Abstract: This article presents certain families of analytic functions regarding q-starlikeness and q-convexity of complex order γ $(\gamma \in \mathbb{C}\setminus\{0\})$. This introduced a q-integral operator and certain subclasses of the newly introduced classes are defined by using this q-integral operator. Coefficient bounds for these subclasses are obtained. Furthermore, the (δ, q)-neighborhood of analytic functions are introduced and the inclusion relations between the (δ, q)-neighborhood and these subclasses of analytic functions are established. Moreover, the generalized hyper-Bessel function is defined, and application of main results are discussed.

Keywords: Geometric Function Theory; q-integral operator; q-starlike functions of complex order; q-convex functions of complex order; (δ, q)-neighborhood

MSC: 30C15; 30C45

1. Introduction

Recently, many researchers have focused on the study of q-calculus keeping in view its wide applications in many areas of mathematics, e.g., in the q-fractional calculus, q-integral calculus, q-transform analysis and others (see, for example, [1,2]). Jackson [3] was the first to introduce and develop the q-derivative and q-integral. Purohit [4] was the first one to introduce and analyze a class in open unit disk and he used a certain operator of fractional q-derivative. His remarkable contribution was to give q-extension of a number of results that were already known in analytic function theory. Later, the q-operator was studied by Mohammed and Darus regarding its geometric properties on certain analytic functions, see [5]. A very significant usage of the q-calculus in the context of Geometric Function Theory was basically furnished and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [6] pp. 347 et seq.; see also [7]). Earlier, a class of q-starlike functions were introduced by Ismail et al. [8]. These are the generalized form of the known starlike functions by using the q-derivatives. Sahoo and Sharma [9] obtained many results of q-close-to-convex functions. Also, some recent results and investigations associated with the q-derivatives operator have been in [6,10–13].

It is worth mentioning here that the ordinary calculus is a limiting case of the quantum calculus. Now, we recall some basic concepts and definitions related to q-derivative, to be used in this work. For more details, see References [3,14–16].

The quantum derivative (named as q-derivative) of function f is defined as:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \neq 0;\ 0 < q < 1).$$

We note that $D_q f(z) \longrightarrow f'(z)$ as $q \longrightarrow 1-$ and $D_q f(0) = f'(0)$, where f' is the ordinary derivative of f.

In particular, q-derivative of $h(z) = z^n$ is as follows :

$$D_q h(z) = [n]_q z^{n-1}, \tag{1}$$

where $[n]_q$ denotes q-number which is given as:

$$[n]_q = \frac{1-q^n}{1-q} \qquad (0 < q < 1). \tag{2}$$

Since we see that $[n]_q \longrightarrow n$ as $q \longrightarrow 1-$, therefore, in view of Equation (1), $D_q h(z) \longrightarrow h'(z)$ as $q \longrightarrow 1-$, where h' represents ordinary derivative of h.

The q-gamma function Γ_q is defined as:

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+t}} \qquad (t > 0;\ 0 < q < 1), \tag{3}$$

which has the following properties:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \tag{4}$$

and

$$\Gamma_q(t+1) = [t]_q!\,, \tag{5}$$

where $t \in \mathbb{N}$ and $[.]_q!$ denotes the q-factorial and defined as:

$$[t]_q! = \begin{cases} [t]_q [t-1]_q \dots [2]_q [1]_q, & t = 1, 2, 3, \dots; \\ 1, & t = 0. \end{cases} \tag{6}$$

Also, the q-beta function B_q is defined as:

$$B_q(t,s) = \int_0^1 x^{t-1} (1-qx)_q^{s-1} d_q x \qquad (t, s > 0;\ 0 < q < 1), \tag{7}$$

which has the following property:

$$B_q(t,s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}, \tag{8}$$

where Γ_q is given by Equation (3).

Furthermore, q-binomial coefficients are defined as [17]:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_n![n-k]_q!}, \tag{9}$$

where $[.]_q!$ is given by Equation (6).

We consider the class $\mathcal{A}$ comprising the functions that are analytic in open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and are of the form given as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{10}$$

Using Equation (1), the q-derivative of f, defined by Equation (10) is as follows:

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \qquad (z \in \mathbb{U};\ 0 < q < 1), \tag{11}$$

where $[n]_q$ is given by Equation (2).

The two important subsets of the class $\mathcal{A}$ are the families $\mathcal{S}^*$ consisting of those functions that are starlike with reference to origin and $\mathcal{C}$ which is the collection of convex functions. A function f is from S^*if for each point $x \in f(\mathbb{U})$ the linear segment between 0 and x is contained in $f(\mathbb{U})$. Also, a function $f \in \mathcal{C}$ if the image $f(\mathbb{U})$ is a convex subset of complex plane $\mathbb{C}$, i.e., $f(\mathbb{U})$ must have every line segment that joins its any two points.

Nasr and Aouf [18] defined the class of those functions which are starlike and are of complex order γ $(\gamma \in \mathbb{C}\setminus \{0\})$, denoted by $\mathcal{S}^*(\gamma)$ and Wiatrowski [19] gave the class of similar type convex functions i.e., of complex order γ $(\gamma \in \mathbb{C}\setminus \{0\})$, denoted by $\mathcal{C}(\gamma)$ as:

$$\mathcal{S}^*(\gamma) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0 \ \ (z \in \mathbb{U}; \gamma \in \mathbb{C}\setminus \{0\}) \right\} \tag{12}$$

and

$$\mathcal{C}(\gamma) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right) > 0 \ \ (z \in \mathbb{U}; \gamma \in \mathbb{C}\setminus \{0\}) \right\}, \tag{13}$$

respectively.

From Equations (12) and (13), it is clear that $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ are subclasses of the class $\mathcal{A}$.

The class denoted by ${\mathcal{S}^*}_q(\mu)$ of such q-starlike functions that are of order μ is defined as:

$${\mathcal{S}^*}_q(\mu) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zD_q f(z)}{f(z)}\right) > \mu \ \ (z \in \mathbb{U}; 0 \leq \mu < 1) \right\}. \tag{14}$$

Also, the class $\mathcal{C}_q(\mu)$ of q-convex functions of order μ is defined as:

$$\mathcal{C}_q(\mu) = \left\{ f \in \mathcal{A} : \Re\left(\frac{D_q(zD_q f(z))}{D_q f(z)}\right) > \mu \ \ (z \in \mathbb{U}; 0 \leq \mu < 1) \right\}. \tag{15}$$

For more detail, see [20]. From Equations (14) and (15), it is clear that $\mathcal{S}_q^*(\mu)$ and $\mathcal{C}_q(\mu)$ are subclasses of the class $\mathcal{A}$.

Next, we recall that the δ-neighborhood of the function $f(z) \in \mathcal{A}$ is defined as [21]:

$$\mathcal{N}_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \,\middle|\, \sum_{n=2}^{\infty} n\,|a_n - b_n| \leq \delta \right\} \qquad (\delta \geq 0). \tag{16}$$

In particular, the δ-neighborhood of the identity function $p(z) = z$ is defined as [21]:

$$\mathcal{N}_{\delta}(p) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \middle| \sum_{n=2}^{\infty} n\,|b_n| \leq \delta \right\} \qquad (\delta \geq 0). \tag{17}$$

Finally, we recall that the Jung-Kim-Srivastava integral operator $\mathcal{Q}_{\beta}^{\alpha} : \mathcal{A} \to \mathcal{A}$ are defined as [22]:

$$\begin{aligned} \mathcal{Q}_{\beta}^{\alpha} f(z) =& \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^{\beta}} \int_0^z t^{\beta-1} \left(1 - \frac{t}{z}\right)^{\alpha-1} f(t)dt \\ =& z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \qquad (\beta > -1;\ \alpha > 0;\ f \in \mathcal{A}). \end{aligned} \tag{18}$$

The Bessel functions are associated with a wide range of problems in important areas of mathematical physics and Engineering. These functions appear in the solutions of heat transfer and other problems in cylindrical and spherical coordinates. Rainville [23] discussed the properties of the Bessel function.

The generalized Bessel functions $w_{\nu,b,d}(z)$ are defined as [24]:

$$w_{\nu,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n!\Gamma\left(\nu + n + \dfrac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu}, \tag{19}$$

where $\nu, b, d, z \in \mathbb{C}$.

Orhan, Deniz and Srivastava [25] defined the function $\varphi_{\nu,b,d}(z) : \mathbb{U} \to \mathbb{C}$ as:

$$\varphi_{\nu,b,d}(z) = 2^{\nu}\Gamma\left(\nu + \frac{b+1}{2}\right) z^{-\frac{\nu}{2}} w_{\nu,b,d}(\sqrt{z}), \tag{20}$$

by using the Generalized Bessel function $w_{\nu,b,d}(z)$, given by Equation (12).

The power series representation for the function $\varphi_{\nu,b,d}(z)$ is as follows [25]:

$$\varphi_{\nu,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d/4)^n}{(c)_n n!} z^n, \tag{21}$$

where $c = \nu + \dfrac{b+1}{2} > 0, \nu, b, d \in \mathbb{R}$ and $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

The hyper-Bessel function is defined as [26]:

$$J_{\alpha_d}(z) = \sum_{n=0}^{\infty} \frac{(z/d+1)^{\alpha_1+\ldots\alpha_d}}{\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_d+1)} {}_0F_d \left(-, (\alpha_d+1); -\left(\frac{z}{d+1}\right)^{d+1} \right), \tag{22}$$

where the hypergeometric function ${}_pF_q$ is defined by:

$${}_pF_q\left((\beta_p);(\eta_q);x\right) = \sum_{n=0}^{\infty} \frac{(\beta_1)_n(\beta_2)_n \ldots (\beta_p)_n}{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n} \frac{x^n}{n!}, \tag{23}$$

using above Equation (23) in Equation (22), then the function $J_{\alpha_d}(z)$ has the following power series:

$$J_{\alpha_d}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\alpha_1+n+1)\Gamma(\alpha_2+n+1)\ldots\Gamma(\alpha_d+n+1)} \left(\frac{z}{d+1}\right)^{n(d+1)+\alpha_1+\ldots\alpha_d}, \tag{24}$$

By choosing $d = 1$ and putting $\alpha_1 = \nu$, we get the classical Bessel function

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} z^{2n+\nu}. \tag{25}$$

In the next section, we introduce the classes of q-starlike functions that are of complex order γ $(\gamma \in \mathbb{C}\setminus\{0\})$ and similarly, q-convex functions that are of complex order γ $(\gamma \in \mathbb{C}\setminus\{0\})$, which are denoted by $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$, respectively. Also, we define a q-integral operator and define the subclasses $\mathcal{S}_q(\alpha,\beta,\gamma)$ and $\mathcal{C}_q(\alpha,\beta,\gamma)$ of the class $\mathcal{A}$ by using this q-integral operator. Then, we find the coefficient bounds for these subclasses.

First, we define the q-starlike function of complex order γ $(\gamma \in \mathbb{C}\setminus\{0\})$, denoted by $\mathcal{S}_q^*(\gamma)$ and the q-convex function of complex order γ $(\gamma \in \mathbb{C}\setminus\{0\})$, denoted by $\mathcal{C}_q(\gamma)$ by taking the q-derivative in place of ordinary derivatives in Equations (12) and (13), respectively.

The respective definitions of the classes $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$ are as follows:

Definition 1. *The function $f \in \mathcal{A}$ will belong to the class $\mathcal{S}_q^*(\gamma)$ if it satisfies the following inequality:*

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{zD_q f(z)}{f(z)} - 1\right)\right) > 0 \qquad (\gamma \in \mathbb{C}\setminus\{0\},\ 0 < q < 1). \tag{26}$$

Definition 2. *The function $f \in \mathcal{A}$ will belong to the class $\mathcal{C}_q(\gamma)$ if it satisfies the following inequality:*

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right)\right) > 0 \qquad (\gamma \in \mathbb{C}\setminus\{0\}, 0 < q < 1). \tag{27}$$

Remark 1. *(i) If $\gamma \in \mathbb{R}$ and $\gamma = 1 - \mu$ $(0 \le \mu < 1)$, then the subclasses $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$ give the sub classes $\mathcal{S}_q^*(\mu)$ and $\mathcal{C}_q(\mu)$, respectively.*

(ii) Using the fact that $\lim_{q\to 1-} D_q f(z) = f'(z)$, we get that $\lim_{q\to 1-} \mathcal{S}_q^*(\gamma) = \mathcal{S}^*(\gamma)$ and $\lim_{q\to 1-} \mathcal{C}_q(\gamma) = \mathcal{C}(\gamma)$.

Now, we introduce the q-integral operator $\chi^{\alpha}_{\beta,q}$ as:

$$\chi^{\alpha}_{\beta,q} f(z) = \binom{\alpha+\beta}{\beta}_q \frac{[\alpha]_q}{z^\beta} \int_0^z t^{\beta-1}\left(1 - \frac{qt}{z}\right)_q^{\alpha-1} f(t)d_q t$$
$$(\alpha > 0;\ \beta > -1;\ 0 < q < 1;\ |z| < 1;\ f \in \mathcal{A}). \tag{28}$$

It is clear that $\chi^{\alpha}_{\beta,q} f(z)$ is analytic in open disc $\mathbb{U}$.

Using Equations (4), (5) and (7)–(9), we get the following power series for the function $\chi^{\alpha}_{\beta,q} f$ in $\mathbb{U}$:

$$\chi^{\alpha}_{\beta,q} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_n z^n \qquad (\alpha > 0;\ \beta > -1;\ 0 < q < 1;\ f \in \mathcal{A}). \tag{29}$$

Remark 2. *For $q \longrightarrow 1-$, Equation (29), gives the Jung-Kim-Srivastava integral operator $\mathcal{Q}^{\alpha}_{\beta}$, given by Equation (18).*

Remark 3. *Taking $\alpha = 1$ in Equation (28) and using Equations (4), (5) and (9), we get the q-Bernardi integral operator, defined as [27]:*

$$\mathcal{F}(z) = \frac{[1+\beta]_q}{z^\beta} \int_0^z t^{\beta-1} f(t) d_q t \qquad \beta = 1, 2, 3, \ldots$$

Next, in view of the Definitions 1 and 2 and the fact that $\Re(z) < |z|$, we introduce the subclasses $\mathcal{S}_q(\alpha,\beta,\gamma)$ and $\mathcal{C}_q(\alpha,\beta,\gamma)$ of the classes $\mathcal{S}_q^*(\gamma)$ and $\mathcal{C}_q(\gamma)$, respectively, by using the operator $\chi_{\beta,q}^{\alpha}$, as:

Definition 3. *The function $f \in \mathcal{A}$ will belong to $\mathcal{S}_q(\alpha,\beta,\gamma)$ if it satisfies the following inequality:*

$$\left|\frac{zD_q(\chi_{\beta,q}^{\alpha}f(z))}{\chi_{\beta,q}^{\alpha}f(z)} - 1\right| < |\gamma|, \tag{30}$$

where $\alpha > 0;\ \beta > -1;\ 0 < q < 1;\ \gamma \in \mathbb{C}\setminus\{0\}$.

Definition 4. *The function $f \in \mathcal{A}$ will belong to $\mathcal{C}_q(\alpha,\beta,\gamma)$ if it satisfies the following inequality:*

$$\left|\frac{D_q\left(zD_q\chi_{\beta,q}^{\alpha}f(z)\right)}{D_q\chi_{\beta,q}^{\alpha}f(z)}\right| < |\gamma|, \tag{31}$$

where $\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C}\setminus\{0\}$.

Now, we establish the following result, which gives the coefficient bound for the subclass $\mathcal{S}_q(\alpha,\beta,\gamma)$:

Lemma 1. *If f is an analytic function such that it belongs to the class $\mathcal{S}_q(\alpha,\beta,\gamma)$, then*

$$\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}\left([n]_q - |\gamma| - 1\right)a_n < |\gamma| \qquad (\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C}\setminus\{0\}), \tag{32}$$

where Γ_q and $[n]_q$ are given by Equations (3) and (2), respectively.

Proof. Let $f \in \mathcal{A}$, then using Equations (11) and (29), we have

$$\left|\frac{zD_q(\chi_{\beta,q}^{\alpha}f(z))}{\chi_{\beta,q}^{\alpha}f(z)} - 1\right| = \left|\frac{z + \sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_q a_n z^n}{z + \sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}a_n z^n} - 1\right|. \tag{33}$$

If $f \in \mathcal{S}_q(\alpha,\beta,\gamma)$, then in view of Definition 3 and Equation (33), we have

$$\left|\frac{z + \sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_q a_n z^n}{z + \sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}a_n z^n} - 1\right| < |\gamma|,$$

which, on simplifying, gives

$$\left|\frac{\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}\left([n]_q - 1\right)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}a_n z^{n-1}}\right| < |\gamma|. \tag{34}$$

Now, using the fact that $\Re(z) < |z|$ in the Inequality (34), we get

$$\Re\left(\frac{\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}\left([n]_q-1\right)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}a_nz^{n-1}}\right) < |\gamma|. \tag{35}$$

Since $\chi^{\alpha}_{\beta,q}f(z)$ is analytic in $\mathbb{U}$, therefore taking limit $z \to 1$—through real axis, Inequality (35), gives the Assertion (32). □

Also, we establish the following result, which gives the coefficient bound for the subclass $\mathcal{C}_q(\alpha, \beta, \gamma)$:

Lemma 2. *If f is an analytic function such that it belongs to the class $\mathcal{C}_q(\alpha, \beta, \gamma)$ and $|\gamma| \geq 1$ then*

$$\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q\left([n]_q-|\gamma|\right))a_n < |\gamma|-1 \quad (\alpha>0;\ \beta>-1;\ 0<q<1; \gamma\in\mathbb{C}\setminus\{0\}), \tag{36}$$

where Γ_q and $[n]_q$ are given by Equations (3) and (2), respectively.

Proof. Let $f \in \mathcal{A}$, then using Equations (11) and (29), we get

$$\left|\frac{D_q\left(zD_q\chi^{\alpha}_{\beta,q}f(z)\right)}{D_q\chi^{\alpha}_{\beta,q}f(z)}\right| = \left|\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q)^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_qa_nz^{n-1}}\right|. \tag{37}$$

If $f \in \mathcal{C}_q(\alpha, \beta, \gamma)$, then in view of Definition 4 and Equation (37), we have

$$\left|\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q)^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_qa_nz^{n-1}}\right| < |\gamma| \tag{38}$$

Now, using the fact that $\Re(z) < |z|$ in Inequality (38), we get

$$\Re\left(\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q)^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}[n]_qa_nz^{n-1}}\right) < |\gamma| \tag{39}$$

Since $\chi^{\alpha}_{\beta,q}f(z)$ is analytic in $\mathbb{U}$, therefore taking limit $z \to 1-$ through real axis, Inequality (39) gives the Assertion (36). □

In the next section, we define (δ, q)-neighborhood of the function $f \in \mathcal{A}$ and establish the inclusion relations of the subclasses $\mathcal{S}_q(\alpha, \beta, \gamma)$ and $\mathcal{C}_q(\alpha, \beta, \gamma)$ with the (δ, q)-neighborhood of the identity function $p(z) = z$.

2. The Classes $\mathcal{N}_{\delta,q}(f)$ and $\mathcal{N}_{\delta,q}(p)$

In view of Equation (16), we define the (δ, q)-neighborhood of the function $f \in \mathcal{A}$ as:

$$\mathcal{N}_{\delta,q}(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \,\middle|\, \sum_{n=2}^{\infty}[n]_q\,|a_n - b_n| \leq \delta \right\} \qquad (\delta \geq 0,\ 0 < q < 1), \tag{40}$$

where $[n]_q$ is given by Equation (2).

In particular, the (δ, q)-neighborhood of the identity function $p(z) = z$, defined as:

$$\mathcal{N}_{\delta,q}(p) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \,\middle|\, \sum_{n=2}^{\infty} [n]_q \, |b_n| \leq \delta \right\} \qquad (\delta \geq 0,\ 0 < q < 1). \tag{41}$$

Since $[n]_q$ approaches n as q approaches $1-$, therefore, from Equations (16) and (40), we note that $\lim_{q \to 1-} \mathcal{N}_{\delta,q}(f) = \mathcal{N}_{\delta}(f)$, where $\mathcal{N}_{\delta}(f)$ is defined by Equation (16). In particular, $\lim_{q \to 1-} \mathcal{N}_{\delta,q}(p) = \mathcal{N}_{\delta}(p)$.

Now, we establish the following inclusion relation between the class $\mathcal{S}_q(\alpha, \beta, \gamma)$ and (δ, q)-neighborhood $\mathcal{N}_{\delta,q}(p)$ of identity function p for the specified range of values of δ:

Theorem 1. *If* $-1 < \beta \leq 0$, $|\gamma| \leq [n]_q - 1 \ \ (n = 2, 3, \ldots)$ *and*

$$\delta \geq \frac{|\gamma| [2]_q \Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)}{([2]_q - |\gamma| - 1) \Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}, \tag{42}$$

then

$$\mathcal{S}_q(\alpha, \beta, \gamma) \subset \mathcal{N}_{\delta,q}(p) \qquad (\gamma \in \mathbb{C} \setminus \{0\}\,;\ \alpha > 0;\ 0 < q < 1). \tag{43}$$

Proof. Let $f \in \mathcal{S}_q(\alpha, \beta, \gamma)$, then, in view of Lemma 1, Inequality (32) holds. Since for $\alpha > 0$, $-1 < \beta \leq 0$, the sequence $\left\{ \dfrac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \right\}_{n=2}^{\infty}$ is non-decreasing, therefore, we have

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \left([2]_q - |\gamma| - 1 \right) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta + n) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + n) \Gamma_q(\beta + 1)} \left([n]_q - |\gamma| - 1 \right) a_n,$$

which in view of Inequality (32), gives

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \left([2]_q - |\gamma| - 1 \right) \sum_{n=2}^{\infty} a_n < |\gamma|, \tag{44}$$

or, equivalently,

$$\sum_{n=2}^{\infty} a_n < \frac{|\gamma| \Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)}{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1) \left([2]_q - |\gamma| - 1 \right)}. \tag{45}$$

Again, using the fact that the sequence $\left\{ \dfrac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \right\}_{n=2}^{\infty}$ is non-decreasing for $\alpha > 0$ and $-1 < \beta \leq 0$, Inequality (32), gives

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} ([n]_q - |\gamma| - 1) a_n < |\gamma|,$$

or, equivalently,

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} [n]_q a_n < |\gamma| + \frac{(1 + |\gamma|) \Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} a_n, \tag{46}$$

which on using the Inequality (45), gives

$$\frac{\Gamma_q(\beta + 2) \Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\alpha + \beta + 2) \Gamma_q(\beta + 1)} \sum_{n=2}^{\infty} [n]_q a_n < |\gamma| + \frac{(1 + |\gamma|) |\gamma|}{[2]_q - |\gamma| - 1},$$

or, equivalently,

$$\sum_{n=2}^{\infty}[n]_q\, a_n < \frac{|\gamma|[2]_q\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|-1)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}. \tag{47}$$

Now, if we take $\delta \geq \dfrac{|\gamma|[2]_q\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|-1)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}$, then in view of Equation (41) and Inequality (47), we obtain that $f(z) \in \mathcal{N}_{\delta,q}(p)$, which proves the inclusion Relation (43). $\square$

Next, we establish the following inclusion relation between the class $\mathcal{C}_q(\alpha,\beta,\gamma)$ and (δ,q)-neighborhood $\mathcal{N}_{\delta,q}(p)$ of identity function p for the specified range of values of δ:

Theorem 2. *If* $-1<\beta\leq 0$, $|\gamma|\geq 1$ *and*

$$\delta \geq \frac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|)\,\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}, \tag{48}$$

then

$$\mathcal{C}_q(\alpha,\beta,\gamma)\subset\mathcal{N}_{\delta,q}(p) \qquad (\alpha>0;\ \gamma\in\mathbb{C}\setminus\{0\};\ 0<q<1). \tag{49}$$

Proof. Let $f\in\mathcal{C}_q(\alpha,\beta,\gamma)$, then, in view of Lemma 2, Inequality (36) holds. Since for $\alpha>0$, $-1<\beta\leq 0$, the sequence $\left\{\dfrac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\right\}_{n=2}^{\infty}$ is non-decreasing, therefore we have

$$\frac{\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}\left([2]_q-|\gamma|\right)\sum_{n=2}^{\infty}[n]_q a_n \leq \sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)}([n]_q\left([n]_q-|\gamma|\right))a_n,$$

which, in view of Inequality (36), gives

$$\frac{\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}\left([2]_q-|\gamma|\right)\sum_{n=2}^{\infty}[n]_q a_n < |\gamma|-1, \tag{50}$$

or, equivalently

$$\sum_{n=2}^{\infty}[n]_q a_n < \frac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|)\,\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}. \tag{51}$$

Now, if we take $\delta \geq \dfrac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{([2]_q-|\gamma|)\,\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}$, then in view of Equation (41) and Inequality (51), we obtain that $f(z)\in\mathcal{N}_{\delta,q}(p)$, which proves the inclusion Relation (49). $\square$

3. The Classes $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$

In this section, the classes $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$ are defined. Then, we establish the inclusion relations between the neighborhood of a function belonging to $\mathcal{S}_q(\alpha,\beta,\gamma)$ and $\mathcal{C}_q(\alpha,\beta,\gamma)$ with $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$, respectively. First, we define the class $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ as follows.

Definition 5. *The function* $f\in\mathcal{A}$, *belongs to* $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ $(\alpha>0;\ -1<\beta;\ \gamma\in\mathbb{C}\setminus\{0\};\ 0<q<1;\ 0\leq\eta<1)$ *if there exists a function* $g(z)\in\mathcal{S}_q(\alpha,\beta,\gamma)$ *that satisfies*

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\eta, \tag{52}$$

where

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n. \tag{53}$$

Similarly, we define the class $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ as:

Definition 6. *The function $f \in \mathcal{A}$, belongs to $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$ $(\alpha>0;\ -1<\beta;\ \gamma\in\mathbb{C}\setminus\{0\};\ 0<q<1;\ 0\leq \eta<1)$ if there exists a function g, given by Equation (53), in the class $\mathcal{C}_q(\alpha,\beta,\gamma)$, satisfying the Inequality (52).*

Now, we establish the following inclusion relation between a neighborhood $\mathcal{N}_{\delta,q}(g)$ of any function $g \in \mathcal{S}_q(\alpha,\beta,\gamma)$ and the class $\mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$ for the specified range of values of η:

Theorem 3. *Let the function g, given by Equation (53), belongs to the class $\mathcal{S}_q(\alpha,\beta,\gamma)$ and*

$$\eta < 1 - \frac{\delta\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q-|\gamma|-1\right)}{[2]_q\left(\left([2]_q-|\gamma|-1\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-|\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}, \tag{54}$$

then

$$\mathcal{N}_{\delta,q}(g) \subset \mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma), \tag{55}$$

where $\alpha>0;\ -1<\beta\leq 0;\ \gamma\in\mathbb{C}\setminus\{0\};\ \delta\geq 0;\ 0<q<1;\ 0\leq\eta<1$.

Proof. We assume that $f \in \mathcal{N}_{\delta,q}(g)$, then in view of Relation (40), we have

$$\sum_{n=2}^{\infty}[n]_q\left|a_n-b_n\right| \leq \delta. \tag{56}$$

Since $\left\{[n]_q\right\}_{n=2}^{\infty}$ is non-decreasing sequence, therefore

$$\sum_{n=2}^{\infty}[2]_q\left|a_n-b_n\right| \leq \sum_{n=2}^{\infty}[n]_q\left|a_n-b_n\right|,$$

This implies that

$$[2]_q\sum_{n=2}^{\infty}\left|a_n-b_n\right| \leq \sum_{n=2}^{\infty}[n]_q\left|a_n-b_n\right|,$$

which in view of Inequality (56) gives

$$[2]_q\sum_{n=2}^{\infty}\left|a_n-b_n\right| \leq \delta,$$

or, equivalently

$$\sum_{n=2}^{\infty}\left|a_n-b_n\right| \leq \frac{\delta}{[2]_q} \qquad (0<q<1;\ \delta\geq 0). \tag{57}$$

Since $-1<\beta\leq 0$, therefore, for the function g, given by Equation (53), in the class $\mathcal{S}_q(\alpha,\beta,\gamma)$, using Inequality (45), we get

$$\sum_{n=2}^{\infty}b_n \leq \frac{|\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q+|\gamma|-1\right)}. \tag{58}$$

Using Equations (10), (53) and the fact that $|z|<1$, we get

$$\left|\frac{f(z)}{g(z)}-1\right| = \left|\frac{\sum_{n=2}^{\infty}(a_n-b_n)z^{n-1}}{1+\sum_{n=2}^{\infty}b_nz^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}|b_n|} \leq \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n}, \tag{59}$$

Now, using Inequalities (57) and (58) in Inequality (59), we get

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\delta\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q-|\gamma|-1\right)}{[2]_q\left(\left([2]_q-|\gamma|-1\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-|\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}. \tag{60}$$

If we take $\eta < 1 - \dfrac{\delta\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)\left([2]_q-|\gamma|-1\right)}{[2]_q\left(\left([2]_q-|\gamma|-1\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-|\gamma|\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}$, then in view of Definition 5 and Inequality (60), we obtain that $f \in \mathcal{S}_q^{(\eta)}(\alpha,\beta,\gamma)$, which proves the inclusion Relation (55). □

Next, we establish the following inclusion relation between a neighborhood $\mathcal{N}_{\delta,q}(g)$ of any function $g \in \mathcal{C}_q(\alpha,\beta,\gamma)$ and the class $\mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$ for the specified range of values of η:

Theorem 4. *Let the function g, given by Equation (53), belongs to the class $\mathcal{C}_q(\alpha,\beta,\gamma)$ and*

$$\eta < 1 - \frac{\delta[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{[2]_q\left([2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}, \tag{61}$$

then

$$\mathcal{N}_{\delta,q}(g) \subset \mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma), \tag{62}$$

where $|\gamma| > 1,\ \alpha > 0;\ -1 < \beta \leq 0;\ \gamma \in \mathbb{C}\setminus\{0\};\ 0 < q < 1;\ \delta \geq 0;\ 0 \leq \eta < 1.$

Proof. If we take any $f \in \mathcal{N}_{\delta,q}(g)$, then Inequality (57) holds.

Now, since $-1 < \beta \leq 0$, therefore, for any function g, given by Equation (53), in the class $\mathcal{C}_q(\alpha,\beta,\gamma)$, using Inequality (51) and the fact that the sequence $\{[n]_q\}_{n=2}^{\infty}$ is non-decreasing, we get

$$\sum_{n=2}^{\infty} b_n < \frac{(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)}{[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}. \tag{63}$$

Using Inequalities (57) and (63) in Inequality (59), we get

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\delta[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{[2]_q\left([2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}. \tag{64}$$

If we take $\eta < 1 - \dfrac{\delta[2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)}{[2]_q\left([2]_q\left([2]_q-|\gamma|\right)\Gamma_q(\beta+2)\Gamma_q(\alpha+\beta+1)-(|\gamma|-1)\Gamma_q(\alpha+\beta+2)\Gamma_q(\beta+1)\right)}$, then in view of Definition 6 and Inequality (64), we obtain that $f \in \mathcal{C}_q^{(\eta)}(\alpha,\beta,\gamma)$, which proves the Assertion (61). □

4. Application

First, we define the generalized hyper-Bessel function $w_{c,b,\alpha_d}(z)$ as :

$$w_{c,b,\alpha_d}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\prod_{i=1}^{d}\Gamma\left(\alpha_i+n+\dfrac{b+1}{2}\right)}\left(\frac{z}{d+1}\right)^{n(d+1)+\sum_{i=1}^{d}\alpha_i} \tag{65}$$

where $v, b, d, z \in \mathbb{C}$.

Second, we define the function $\varphi_{\alpha_d,b,c}(z) : \mathbb{U} \to \mathbb{C}$ as:

$$\varphi_{\alpha_d,b,c}(z) = (d+1)^{\sum_{i=1}^{d}\alpha_i} \prod_{i=1}^{d} \Gamma\left(\alpha_i + \frac{b+1}{2}\right) z^{1-\frac{\sum_{i=1}^{d}\alpha_i}{d+1}} w_{\alpha_d,b,c}(z^{1/d+1}), \tag{66}$$

by using Equation (65) in Equation (66), we get

$$\begin{aligned} \varphi_{c,b,\alpha_d}(z) &= \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \prod_{i=1}^{d}\left(\alpha_i + \frac{b+1}{2}\right)_n (d+1)^{n(d+1)}} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{(n-1)! \prod_{i=1}^{d}\left(\alpha_i + \frac{b+1}{2}\right)_{n-1} (d+1)^{(n-1)(d+1)}} z^n \end{aligned} \tag{67}$$

by choosing $d = 1$ and $\alpha_1 = \nu$, then the functions $w_{c,b,\alpha_d}(z)$ and $\varphi_{\alpha_d,b,c}(z)$ are reduce to $w_{\nu,b,d}(z)$ and $\phi_{\nu,b,d}(z)$, respectively.

Third, we applying the introduced function $\varphi_{c,b,\alpha_d}(z)$, given by Equation (67) in the results of Lemma 1 and Lemma 2, we get the conditions for that function $\varphi_{c,b,\alpha_d}(z)$ to be in the classes $\mathcal{S}_q(\alpha,\beta,\gamma)$ and $\mathcal{C}_q(\alpha,\beta,\gamma)$ in the following corollaries, respectively:

Corollary 1. *If $\varphi_{c,b,\alpha_d}(z)$ is an analytic function such that it belongs to the class $\mathcal{S}_q(\alpha,\beta,\gamma)$, then*

$$\sum_{n=2}^{\infty} \frac{(-c)^{n-1}\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{(n-1)! \prod_{i=1}^{d}\left(\alpha_i + \frac{b+1}{2}\right)_{n-1} (d+1)^{(n-1)(d+1)}\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} \times \left([n]_q - |\gamma| - 1\right) < |\gamma| \quad (\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C}\setminus\{0\}),$$

where Γ_q and $[n]_q$ are given by Equations (2) and (3), respectively.

Corollary 2. *If $\varphi_{c,b,\alpha_d}(z)$ is an analytic function such that it belongs to the class $\mathcal{C}_q(\alpha,\beta,\gamma)$ and $|\gamma| \geq 1$ then*

$$\sum_{n=2}^{\infty} \frac{(-c)^{n-1}\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{(n-1)! \prod_{i=1}^{d}\left(\alpha_i + \frac{b+1}{2}\right)_{n-1} (d+1)^{(n-1)(d+1)}\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} \times \left([n]_q\left([n]_q - |\gamma|\right)\right)a_n < |\gamma| - 1 \quad (\alpha > 0;\ \beta > -1;\ 0 < q < 1; \gamma \in \mathbb{C}\setminus\{0\}),$$

where Γ_q and $[n]_q$ are given by Equations (2) and (3), respectively.

5. Discussion of Results and Future Work

The concept of q-derivatives has so far been applied in many areas of not only mathematics but also physics, including fractional calculus and quantum physics. However, research on q-calculus is in connection with function theory and especially geometric properties of analytic functions such as starlikeness and convexity, which is fairly familiar on this topic. Finding sharp coefficient bounds for analytic functions belonging to Classes of starlikeness and convexity defined by q-calculus operators is of particular importance since any information can shed light on the study of the geometric properties of such functions. Our results are applicable by using any analytic functions.

6. Conclusions

In this paper, we have used q-calculus to introduce a new q-integral operator which is a generalization of the known Jung-Kim-Srivastava integral operator. Also, a new subclass involving the q-integral operator introduced has been defined. Some interesting coefficient bounds for these subclasses of analytic functions have been studied. Furthermore, the (δ, q)-neighborhood of analytic functions and the inclusion relation between the (δ, q)-neighborhood and the subclasses involving the q-integral operator have been derived. The ideas of this paper may stimulate further research in this field.

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