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Strong Convergence of a System of Generalized Mixed Equilibrium Problem, Split Variational Inclusion Problem and Fixed Point Problem in Banach Spaces

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Abstract: The purpose of this paper is to introduce a new algorithm to approximate a common solution for a system of generalized mixed equilibrium problems, split variational inclusion problems of a countable family of multivalued maximal monotone operators, and fixed-point problems of a countable family of left Bregman, strongly asymptotically non-expansive mappings in uniformly convex and uniformly smooth Banach spaces. A strong convergence theorem for the above problems are established. As an application, we solve a generalized mixed equilibrium problem, split Hammerstein integral equations, and a fixed-point problem, and provide a numerical example to support better findings of our result.

Keywords: split variational inclusion problem; generalized mixed equilibrium problem; fixed point problem; maximal monotone operator; left Bregman asymptotically nonexpansive mapping; uniformly convex and uniformly smooth Banach space

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1. Introduction and Preliminaries

Let E be a real normed space with dual E^*. A map $B : E \to E^*$ is called:

(i) monotone if, for each $x, y \in E$, $\langle \eta - \nu, x - y \rangle \geq 0$, $\forall\, \eta \in Bx$, $\nu \in By$, where $\langle \cdot, \cdot \rangle$ denotes duality pairing,

(ii) ϵ-inverse strongly monotone if there exists $\epsilon > 0$, such that $\langle Bx - By, x - y \rangle \geq \epsilon ||Bx - By||^2$,

(iii) maximal monotone if B is monotone and the graph of B is not properly contained in the graph of any other monotone operator. We note that B is maximal monotone if, and only if it is monotone, and $R(J + tB) = E^*$ for all $t > 0$, J is the normalized duality map on E and $R(J + tB)$ is the range of $(J + tB)$ (cf. [1]).

Let H_1 and H_2 be Hilbert spaces. For the maximal monotone operators $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$, Moudafi [2] introduced the following split monotone variational inclusion:

$$\begin{aligned} &find\ x^* \in H_1\ such\ that\ 0 \in f(x^*) + B_1(x^*), \\ &y^* = Ax^* \in H_2\ solves\ 0 \in g(y^*) + B_2(y^*), \end{aligned}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are given operators. In 2000, Moudafi [3] proposed the viscosity approximation method, which is formulated by considering the approximate well-posed problem and combining the non-expansive mapping S with a contraction mapping f on a non-empty, closed, and convex subset C of H_1. That is, given an arbitrary x_1 in C, a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n,$$

converges strongly to a point of $F(S)$, the set of fixed point of S, whenever $\{\alpha_n\} \subset (0,1)$ such that $\alpha_n \to 0$ as $n \to \infty$.

In [4,5], the viscosity approximation method for split variational inclusion and the fixed point problem in a Hilbert space was presented as follows:

$$\begin{aligned} u_n &= J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)T^n(u_n), \forall n \geq 1, \end{aligned} \tag{1}$$

where B_1 and B_2 are maximal monotone operators, $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are resolvent mappings of B_1 and B_2, respectively, f is the Meir Keeler function, T a non-expansive mapping, and A^* is the adjoint of A, $\gamma_n, \alpha_n \in (0,1)$ and $\lambda > 0$.

The algorithm introduced by Schopfer et al. [6] involves computations in terms of Bregman distance in the setting of p-uniformly convex and uniformly smooth real Banach spaces. Their iterative algorithm given below converges weakly under some suitable conditions:

$$x_{n+1} = \Pi_C J^{-1}(Jx_n + \gamma A^* J(P_Q - I)Ax_n),\ n \geq 0, \tag{2}$$

where Π_C denotes the Bregman projection and P_C denotes metric projection onto C. However, strong convergence is more useful than the weak convergence in some applications. Recently, strong convergence theorems for the split feasibility problem (SFP) have been established in the setting of p-uniformly convex and uniformly smooth real Banach spaces [7–10].

Suppose that

$$F(x,y) = f(x,y) + g(x,y)$$

where $f, g : C \times C \longrightarrow \mathbb{R}$ are bifunctions on a closed and convex subset C of a Banach space, which satisfy the following special properties $(A_1) - (A_4), (B_1) - (B_3)$ and (C):

$$\begin{cases} (A_1)\ f(x,y)=0, \forall x \in C; \\ (A_2)\ f \text{ is maximal monotone;} \\ (A_3)\ \forall x,y,z \in C \text{ and } t \in [0,1] \text{ we have} \limsup_{n\to 0^+}(f(tz+(1-t)x,y) \leq f(x,y)); \\ (A_4)\ \forall x \in C, \text{the function } y \mapsto f(x,y) \text{is convex and weakly lower semi-continuous;} \\ (B_1)\ g(x,x)=0\ \forall\ x \in C; \\ (B_2)\ g \text{ is maximal monotone, and weakly upper semi-continuous in the first variable;} \\ (B_3)\ g \text{ is convex in the second variable;} \\ (C) \text{ for fixed } \lambda > 0 \text{ and } x \in C, \text{there exists a bounded set } K \subset C \\ \text{and } a\ \in K \text{ such that } f(a,z)+g(z,a)+\frac{1}{\lambda}(a-z,z-x) < 0\ \ \forall x \in C \backslash K. \end{cases} \tag{3}$$

The well-known, generalized mixed equilibrium problem (GMEP) is to find an $x \in C$, such that

$$F(x,y) + \langle Bx, y-x\rangle \geq 0 \quad \forall \quad y \in C,$$

where B is nonlinear mapping.

In 2016, Payvand and Jahedi [11] introduced a new iterative algorithm for finding a common element of the set of solutions of a system of generalized mixed equilibrium problems, the set of common fixed points of a finite family of pseudo contraction mappings, and the set of solutions of the variational inequality for inverse strongly monotone mapping in a real Hilbert space. Their sequence is defined as follows:

$$\begin{cases} g_i(u_{n,i},y) + \langle C_i u_{n,i} + S_{n,i}x_n, y-u_{n,i}\rangle + \theta_i(y) - \theta_i(u_{n,i}) \\ +\frac{1}{r_{n,i}}\langle y-u_{n,i}, u_{n,i}-x_n\rangle \geq 0\ \forall y \in K,\ \forall i \in I, \\ y_n = \alpha_n v_n + (1-\alpha_n(I-f)P_K(\sum_{i=0}^{\infty}\delta_{n,i}u_{n,i} - \lambda_n A\sum_{i=0}^{\infty}\delta_{n,i}u_{n,i}), \\ x_{n+1} = \beta_n x_n + (1+\beta_n)(\gamma_0 + \sum_{j=1}^{\infty}\gamma_j T_j)P_K(y_n - \lambda_n A y_n) n \geq 1, \end{cases} \tag{4}$$

where g_i are bifunctions, S_i are $\epsilon-$ inverse strongly monotone mappings, C_i are monotone and Lipschtz continuous mappings, θ_i are convex and lower semicontinuous functions, A is a $\Phi-$ inverse strongly monotone mapping, and f is an $\iota-$contraction mapping and $\alpha_n, \delta_n, \beta_n, \lambda_n, \gamma_0 \in (0,1)$.

In this paper, inspired by the above cited works, we use a modified version of (1), (2) and (4) to approximate a solution of the problem proposed here. Both the iterative methods and the underlying space used here are improvements and extensions of those employed in [2,6,7,9–11] and the references therein.

Let $p,q \in (1,\infty)$ be conjugate exponents, that is, $\frac{1}{p}+\frac{1}{q}=1$. For each $p>1$, let $g(t)=t^{p-1}$ be a gauge function where $g:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $g(0)=0$ and $\lim_{t\to\infty} g(t)=\infty$. We define the generalized duality map $J_p : E \longrightarrow 2^{E^*}$ by

$$J_{g(t)} = J_p(x) = \{x^* \in E^*; \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = g(\|x\|) = \|x\|^{p-1}\}.$$

In the sequel, $a \vee b$ denotes $\max\{a,b\}$.

Lemma 1 ([12])**.** *In a smooth Banach space E, the Bregman distance $\triangle_p$ of x to y, with respect to the convex continuous function $f: E \to R$, such that $f(x)=\frac{1}{p}\|x\|^p$, is defined by*

$$\triangle_p(x,y) = \frac{1}{q}\|x\|^p - \langle J^p(x), y\rangle + \frac{1}{p}\|y\|^p,$$

for all $x,y \in E$ and $p>1$.

A Banach space E is said to be uniformly convex if, for $x, y \in E$, $0 < \delta_E(\epsilon) \leq 1$, where $\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\|; \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon$, where $0 \leq \epsilon \leq 2\}$.

Definition 1. *A Banach space E is said to be uniformly smooth, if for $x, y \in E$, $\lim_{r\to 0}(\frac{\rho_E(r)}{r}) = 0$ where $\rho_E(r) = \frac{1}{2}\sup\{\|x+y\| + \|x-y\| - 2 : \|x\| = 1, \|y\| \leq r; 0 \leq r < \infty$ and $0 \leq \rho_E(r) < \infty\}$.*
It is shown in [12] that:

1. *ρ_E is continuous, convex, and nondecreasing with $\rho_E(0) = 0$ and $\rho_E(r) \leq r$*
2. *The function $r \mapsto \frac{\rho_E(r)}{r}$ is nondecreasing and fulfils $\frac{\rho_E(r)}{r} > 0$ for all $r > 0$.*

Definition 2 ([13])**.** *Let E be a smooth Banach space. Let $\triangle_p$ be the Bregman distance. A mapping $T : E \longrightarrow E$ is said to be a strongly non-expansive left Bregman with respect to the non-empty fixed point set of T, $F(T)$, if $\triangle_p(T(x), v) \leq \triangle_p(x, v)\ \forall\ x \in E$ and $v \in F(T)$.*

Furthermore, if $\{x_n\} \subset C$ is bounded and $\lim\limits_{n\to\infty}(\triangle_p(x_n, v) - \triangle_p(Tx_n, v)) = 0$, then it follows that $\lim\limits_{n\to\infty} \triangle_p(x_n, Tx_n) = 0$.

Definition 3. *Let E be a smooth Banach space. Let $\triangle_p$ be the Bregman distance. A mapping $T : E \longrightarrow E$ is said to be a strongly asymptotically non-expansive left Bregman with $\{k_n\} \subset [1, \infty)$ if there exists non-negative real sequences $\{k_n\}$ with $\lim_{n\to\infty} k_n = 1$, such that $\triangle_p(T^n(x), T^n(v)) \leq k_n \triangle_p(x, v), \forall (x, v) \in E \times F(T)$.*

Lemma 2 ([14])**.** *Let E be a real uniformly convex Banach space, K a non-empty closed subset of E, and $T : K \to K$ an asymptotically non-expansive mapping. Then, $I - T$ is demi-closed at zero, if $\{x_n\} \subset K$ converges weakly to a point $p \in K$ and $\lim\limits_{n\to\infty}\|Tx_n - x_n\| = 0$, then $p = Tp$.*

Lemma 3 ([12])**.** *In a smooth Banach space E, let $x_n \in E$. Consider the following assertions:*

1. $\lim_{n\to\infty}\|x_n - x\| = 0$
2. $\lim_{n\to\infty}\|x_n\| = \|x\|$ *and* $\lim_{n\to\infty}\langle J^p(x_n), x\rangle = \langle J^p(x), x\rangle$
3. $\lim_{n\to\infty}\triangle_p(x_n, x) = 0$.

The implication (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are valid. If E is also uniformly convex, then the assertions are equivalent.

Lemma 4. *Let E be a smooth Banach space. Let $\triangle_p$ and V_p be the mappings defined by $\triangle_p(x, y) = \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p$ for all $(x, y) \in E \times E$ and $V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p$ for all $(x, x^*) \in E \times E^*$. Then, $\triangle_p(x, y) = V_p(x^*, y)$ for all $x, y \in E$.*

Lemma 5 ([12])**.** *Let E be a reflexive, strictly convex, and smooth Banach space, and J^p be a duality mapping of E. Then, for every closed and convex subset $C \subset E$ and $x \in E$, there exists a unique element $\Pi_C^p(x) \in C$, such that $\triangle_p(x, \Pi_C^p(x)) = \min_{y\in C}\triangle_p(x, y)$; here, $\Pi_C^p(x)$ denotes the Bregman projection of x onto C, with respect to the function $f(x) = \frac{1}{p}\|x\|^p$. Moreover, $x_0 \in C$ is the Bregman projection of x onto C if*

$$\langle J^p(x_0 - x), y - x_0\rangle \geq 0$$

or equivalently

$$\triangle_p(x_0, y) \leq \triangle_p(x, y) - \triangle_p(x, x_0)\ for\ every\ y \in C.$$

Lemma 6 ([15])**.** *In the case of a uniformly convex space, E, with the duality map J^q of E^*, $\forall x^*, y^* \in E^*$ we have*

$$\|x^* - y^*\|^q \leq \|x^*\|^q - q\langle J^q(x^*), y^*\rangle + \bar{\sigma}_q(x^*, y^*), \quad where$$

$$\bar{\sigma_q}(x^*,y^*) = qG_q \int_0^1 \frac{(\|x^*-ty^*\| \vee \|x^*\|)^q}{t} \rho_{E^*}\left(\frac{t\|y^*\|}{2(\|x^*-ty^*\| \vee \|x^*\|)}\right) dt \tag{5}$$

and $G_q = 8 \vee 64cK_q^{-1}$ *with* $c, K_q > 0$.

Lemma 7 ([12]). *Let E be a reflexive, strictly convex, and smooth Banach space. If we write* $\triangle_q^*(x,y) = \frac{1}{p}\|x^*\|^q - \langle J_{E^*}^q x^*, y^*\rangle + \frac{1}{q}\|y^*\|^q$ *for all* $(x^*,y^*) \in E^* \times E^*$ *for the Bregman distance on the dual space* E^* *with respect to the function* $f_q^*(x^*) = \frac{1}{q}\|x^*\|^q$*, then we have* $\triangle_p(x,y) = \triangle_q^*(x^*,y^*)$.

Lemma 8 ([16]). *Let* $\{\alpha_n\}$ *be a sequence of non-negative real numbers, such that* $\alpha_{n+1} \le (1-\beta_n)\alpha_n + \delta_n$*,* $n \ge 0$*, where* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in R, such that*

1. $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$;
2. $\limsup_{n\to\infty} \frac{\delta_n}{\beta_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 9. *Let E be reflexive, smooth, and strictly convex Banach space. Then, for all* $x,y,z \in E$ *and* $x^*, z^* \in E^*$ *the following facts hold:*

1. $\triangle_p(x,y) \ge 0$ *and* $\triangle_p(x,y) = 0$ *iff* $x = y$;
2. $\triangle_p(x,y) = \triangle_p(x,z) + \triangle_p(z,y) + \langle x^* - z^*, z - y\rangle$.

Lemma 10 ([17]). *Let E be a real uniformly convex Banach space. For arbitrary* $r > 1$*, let* $B_r(0) = \{x \in E : \|x\| \le r\}$*. Then, there exists a continuous strictly increasing convex function*

$$g : [0,\infty) \longrightarrow [0,\infty), g(0) = 0$$

such that for every $x, y \in B_r(0)$*,* $f_x \in J_p(x)$*,* $f_y \in J_p(y)$ *and* $\lambda \in [0,1]$*, the following inequalities hold:*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - (\lambda^p(1-\lambda) + (1-\lambda)^p\lambda)g(\|x-y\|)$$

and

$$\langle x - y, f_x - f_y\rangle \ge g(\|x-y\|).$$

Lemma 11 ([18]). *Suppose that* $\sum_{n=1}^{\infty} sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty$*. Then, for each* $y \in C$*,* $\{T_n y\}$ *converges strongly to some point of C. Moreover, let T be a mapping of C onto itself, defined by* $Ty = \lim_{n\to\infty} T_n y$ *for all* $y \in C$*. Then,* $\lim_{n\to\infty} sup\{\|Tz - T_n z\| : z \in C\} = 0$*. Consequently, by Lemma 3,* $\lim_{n\to\infty} sup\{\triangle_p(Tz, T_n z) : z \in C\} = 0$.

Lemma 12 ([19]). *Let E be a reflexive, strictly convex, and smooth Banach space, and C be a non-empty, closed convex subset of E. If* $f, g : C \times C \longrightarrow \mathbb{R}$ *be two bifunctions which satisfy the conditions* $(A_1) - (A_4), (B_1) - (B_3)$*and*$(C)$*, in (3), then for every* $x \in E$ *and* $r > 0$*, there exists a unique point* $z \in C$ *such that* $f(z,y) + g(z,y) + \frac{1}{r}\langle y - z, jz - jx\rangle \ge 0 \,\forall\, y \in C$.

For $f(x) = \frac{1}{p}\|x\|^p$, Reich and Sabach [20] obtained the following technical result:

Lemma 13. *Let E be a reflexive, strictly convex, and smooth Banach space, and C be a non-empty, closed, and convex subset of E. Let* $f, g : C \times C \longrightarrow \mathbb{R}$ *be two bifunctions which satisfy the conditions* $(A_1) -$

$(A_4), (B_1) - (B_3)$and(C), in (3). Then, for every $x \in E$ and $r > 0$, we define a mapping $S_r : E \longrightarrow C$ as follows;

$$S_r(x) = \{z \in C : f(z,y) + g(z,y) + \frac{1}{r}\langle y - z, J_E^p z - J_E^p x\rangle \geq 0 \forall y \in C\}. \tag{6}$$

Then, the following conditions hold:

1. *S_r is single-valued;*
2. *S_r is a Bregman firmly non-expansive-type mapping, that is,*

$$\forall x, y \in E \langle S_r x - S_r y, J_E^p S_r x - J_E^p S_r y\rangle \leq \langle S_r x - S_r y, J_E^p x - J_E^p y\rangle$$

   or equivalently
   $\triangle_p(S_r x, S_r y) + \triangle_p(S_r y, S_r x) + \triangle_p(S_r x, x) + \triangle_p(S_r y, y) \leq \triangle_p(S_r x, y) + \triangle_p(S_r y, x);$
3. *$F(S_r) = MEP(f, g)$, here MEP stands for mixed equilibrium problem;*
4. *$MEP(f, g)$ is closed and convex;*
5. *for all $x \in E$ and for all $v \in F(S_r)$, $\triangle_p(v, S_r x) + \triangle_p(S_r x, x) \leq \triangle_p(v, x)$.*

2. Main Results

Let E_1 and E_2 be uniformly convex and uniformly smooth Banach spaces and E_1^* and E_2^* be their duals, respectively. For $i \in I$, let $U_i : E_1 \to 2^{E_1^*}$ and $T_i : E_2 \to 2^{E_2^*}$, $i \in I$ be multi-valued maximal monotone operators. For $i \in I$, $\delta > 0$, $p, q \in (1, \infty)$ and $K \subset E_1$ closed and convex, let $\Phi_i : K \times K \to \mathbb{R}$, $i \in I$, be bifunctions satisfying $(A1) - (A4)$ in (3), let $B_\delta^{U_i} : E_1 \to E_1$ be resolvent operators defined by $B_\delta^{U_i} = (J_{E_1}^p + \delta U_i)^{-1} J_{E_1}^p$ and $B_\delta^{T_i} : E_2 \to E_2$ be resolvent operators defined by $B_\delta^{T_i} = (J_{E_2}^p + \delta T_i)^{-1} J_{E_2}^p$. Let $A : E_1 \to E_2$ be a bounded and linear operator, A^* denotes the adjoint of A and AK be closed and convex. For each $i \in I$, let $S_i : E_1 \to E_1$ be a uniformly continuous Bregman asymptotically non-expansive operator with the sequences $\{k_{n,i}\} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} k_{n,i} = 1$. Denote by $\Upsilon : E_1^* \to E_1^*$ a firmly non-expansive mapping. Suppose that, for $i \in I$, $\theta_i : K \to R$ are convex and lower semicontinuous functions, $G_i : K \to E_1$ are $\varepsilon-$ inverse strongly monotone mappings and $C_i : K \to E_1$, are monotone and Lipschitz continuous mappings. Let $f : E_1 \to E_1$ be a $\zeta-$contraction mapping, where $\zeta \in (0, 1)$. Suppose that $\Pi_{AK}^p : E_2 \to AK$ is a generalized Bregman projection onto AK. Let $\Omega = \{x^* \in \cap_{i=1}^\infty SOLVIP(U_i); Ax^* \in \cap_{i=1}^\infty SOLVIP(T_i)\}$ be the set of solution of the split variational inclusion problem, $\omega = \{x^* \in \cap_{i=1}^\infty GMEP(G_i, C_i, \theta_i, g_i)\}$ be the solution set of a system of generalized mixed equilibrium problems, and $\Im = \{x^* \in \cap_{i=1}^\infty F(S_i)\}$ be the common fixed-point set of S_i for each $i \in I$. Let the sequence $\{x_n\}$ be defined as follows:

$$\begin{cases} \Phi_i(u_{n,i}, y) + \langle J_{E_1}^p G_{n,i} x_n, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, J_{E_1}^p u_{n,i} - J_{E_1}^p x_n\rangle \geq 0 \forall y \in K, \\ \forall i \in I, \\ x_{n+1} = J_{E_1^*}^q \left(\sum_{i=0}^\infty \alpha_{n,i} B_{\delta_n}^{U_i} \left(J_{E_1}^p x_n - \sum_{i=0}^\infty \beta_{n,i} \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\right)\right), \end{cases} \tag{7}$$

where $\Phi_i(x, y) = g_i(x, y) + \langle J_{E_1}^p C_i x, y - x\rangle + \theta_i(y) - \theta_i(x)$.

We shall strictly employ the above terminology in the sequel.

Lemma 14. *Suppose that $\bar{\sigma}_q$ is the function (5) in Lemma 6 for the characteristic inequality of the uniformly smooth dual E_1^*. For the sequence $\{x_n\} \subset E_1$ defined by (7), let $0 \neq x_n \in E_1$, $0 \neq A$, $0 \neq J_{E_1}^p G_{n,i} x_n \in E_1^*$ and $0 \neq \sum_{i=0}^\infty \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i} \in E_2^*$, $i \in I$. Let , for $\lambda_{n,i} > 0$ and $r_{n,i} > 0, i \in I$ be defined by*

$$\lambda_{n,i} = \frac{1}{\|A\|} \frac{1}{\|\sum_{i=0}^\infty \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\|}, \text{ and} \tag{8}$$

$$r_{n,i} = \frac{1}{\|J_{E_1}^p G_{n,i} x_n\|}, \text{ respectively.} \tag{9}$$

Then for $\mu_{n,i} = \frac{1}{\|x_n\|^{p-1}}$,

$$2^q G_q \|J^p_{E_1} x_n\|^p \rho_{E_1^*}(\mu_{n,i}) \geq \begin{cases} \frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_n, r_{n,i}J^p_{E_1}G_{n,i}x_n) \\ \frac{1}{q}\bar{\sigma}_q\left(J^p_{E_1}x_n, \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* \sum_{i=0}^{\infty}\beta_{n,i}J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\right), \end{cases} \tag{10}$$

where G_q *is the constant defined in Lemma 6 and* $\rho_{E_1^*}$ *is the modulus of smoothness of* E_1^*.

Proof. By Lemma 12, (6) in Lemma 13 and (7), for each $i \in I$, we have that $u_{n,i} = J^q_{E_1^*}(\Upsilon_{r_{n,i}}(J^p_{E_1}x_n - r_{n,i}J^p_{E_1}G_{n,i}x_n))$. By Lemma 6, we get

$$\frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_n, r_{n,i}J^p_{E_1}G_{n,i}x_n) = G_q \int_0^1 \frac{(\|J^p_{E_1}x_n - tr_{n,i}J^p_{E_1}G_{n,i}x_n\| \vee \|J^p_{E_1}x_n\|)^q}{t} \times$$
$$\rho_{E^*}\left(\frac{t\|r_{n,i}J^p_{E_1}G_{n,i}x_n\|}{(\|J^p_{E_1}x_n - tr_{n,i}J^p_{E_1}G_{n,i}x_n\| \vee \|J^p_{E_1}x_n\|)}\right) dt, \tag{11}$$

$$for\ every\ t \in [0,1].$$

However, by (9) and Definition 1(2), we have

$$\rho_{E_1^*}\left(\frac{t\|r_{n,i}J^p_{E_1}G_{n,i}x_n\|}{(\|J^p_{E_1}x_n - tr_{n,i}J^p_{E_1}G_{n,i}x_n\| \vee \|J^p_{E_1}x_n\|)}\right) \leq \rho_{E_1^*}\left(\frac{t\|r_{n,i}J^p_{E_1}G_{n,i}x_n\|}{\|x_n\|^{p-1}}\right)$$
$$= \rho_{E_1^*}(t\mu_{n,i}). \tag{12}$$

Substituting (12) into (11), and using the nondecreasing of function $\rho_{E_1^*}$, we have

$$\frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_n, r_{n,i}J^p_{E_1}G_{n,i}x_n) \leq 2^q G_q \|x_n\|^p \rho_{E_1^*}(\mu_{n,i}). \tag{13}$$

In addition, by Lemma 6, we have

$$\frac{1}{q}\bar{\sigma}_q\left(J^p_{E_1}x_n, \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\right)$$
$$= G_q \int_0^1 \frac{\left(\left\|J^p_{E_1}x_n - \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\right\| \vee \|J^p_{E_1}x_n\|\right)^q}{t} \times$$
$$\rho_{E^*}\left(\frac{t\|\sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\|}{\left(\left\|J^p_{E_1}x_n - \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\right\| \vee \|J^p_{E_1}x_n\|\right)}\right) dt, \tag{14}$$

$$for\ every\ t \in [0,1].$$

However, by (8) and Definition 1(2), we have

$$\rho_{E_1^*}\left(\frac{t\left\|\sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\right\|}{\left(\left\|J^p_{E_1}x_n - t\sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\right\| \vee \|J^p_{E_1}x_n\|\right)}\right)$$
$$\leq \rho_{E_1^*}\left(\frac{t\left\|\sum_{i=0}^{\infty}\beta_{n,i}\lambda_{n,i} A^* J^p_{E_2}(I - \Pi^p_{AK}B^{T_i}_{\delta_n})Au_{n,i}\right\|}{\|x_n\|^{p-1}}\right) = \rho_{E_1^*}(t\mu_{n,i}). \tag{15}$$

Substituting (15) into (14), and using the nondecreasing of function $\rho_{E_1^*}$, we get

$$\frac{1}{q}\bar{\sigma}_q\left(J_{E_1}^p x_n, \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J_{E_2}^p (I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\right) \leq 2^q G_q\|x_n\|^p \rho_{E_1^*}(\mu_{n,i}). \tag{16}$$

By (13) and (16), the result follows. □

Lemma 15. *For the sequence $\{x_n\} \subset E_1$, defined by (7), $i \in I$, let $0 \neq \sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^p(I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i} \in E_2^*$, $0 \neq J_{E_1}^p G_{n,i}x_n \in E_1^*$, and $\lambda_n > 0$ and $r_{n,i} > 0$, $i \in I$, be defined by*

$$\lambda_n = \frac{1}{\|A\|}\frac{1}{\|\sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^p(I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\|} \tag{17}$$

and

$$r_{n,i} = \frac{1}{\|J_{E_1}^p G_{n,i}x_n\|}, \tag{18}$$

where $\iota, \gamma \in (0,1)$ and $\mu_{n,i} = \frac{1}{\|x_n\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\mu_{n,i}) = \frac{\iota}{2^q G_q\|A\|} \times \frac{\|\sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^p(I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\|^p}{\|x_n\|^p\|\sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^p(I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\|^{p-1}}, \tag{19}$$

and

$$\rho_{E_1^*}(\mu_{n,i}) = \frac{\gamma\langle J_{E_1}^p G_{n,i}x_n, x_n - v\rangle}{2^q G_q\|x_n\|^p\|J_{E_1}^p G_{n,i}x_n\|}. \tag{20}$$

Then, for all $v \in \Gamma$, we get

$$\begin{aligned}\triangle_p(x_{n+1}, v) &\leq \triangle_p(x_n, v) \\ &- [1-\iota] \times \frac{\left\langle \sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^p(I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}, \sum_{i=0}^{\infty}\beta_{n,i}(I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\right\rangle}{\|A\|\|\sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^p(I-\Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\|}\end{aligned} \tag{21}$$

and

$$\triangle_p(u_n, v) \leq \triangle_p(x_n, v) - [1-\gamma] \times \frac{\langle J_{E_1}^p G_{n,i}x_n, x_n - v\rangle}{\|J_{E_1}^p G_{n,i}x_n\|}, \text{ respectively}. \tag{22}$$

Proof. By Lemmas 13, 4 and 6, for each $i \in I$, we get that $u_{n,i} = J_{E_1^*}^q(\Upsilon_{r_{n,i}}(J_{E_1}^p x_n - r_{n,i}J_{E_1}^p G_{n,i}x_n))$, and hence it follows that

$$\begin{aligned}\triangle_p(u_{n,i}, v) &\leq V_p(J_{E_1}^p x_n - r_{n,i}J_{E_1}^p G_{n,i}x_n, v) \\ &= -\langle J_{E_1}^p x_n, v\rangle + r_{n,i}\langle J_{E_1}^p G_{n,i}x_n, v\rangle \\ &+ \frac{1}{q}\|J_{E_1}^p x_n - r_{n,i}J_{E_1}^p G_{n,i}x_n\|^q + \frac{1}{p}\|v\|^p.\end{aligned} \tag{23}$$

By Lemmas 6 and 14, we have

$$\begin{aligned} &\frac{1}{q}\|J_{E_1}^p x_n - r_{n,i}J_{E_1}^p G_{n,i}x_n\|^q \\ &\quad \leq \frac{1}{q}\|J_{E_1}^p x_n\|^q - r_{n,i}\langle J_{E_1}^p G_{n,i}x_n, x_n\rangle + 2^q G_q\|J_{E_1}^p x_n\|^p \rho_{E_1^*}(\mu_{n,i}). \end{aligned} \tag{24}$$

Substituting (24) into (23), we have, by Lemma 4

$$\begin{aligned} \triangle_p(u_{n,i}, v) \leq \triangle_p(x_n, v) + 2^q G_q\|J_{E_1}^p x_n\|^p \rho_{E_1^*}(\mu_{n,i}) \\ - r_{n,i}\langle J_{E_1}^p G_{n,i}x_n, x_n - v\rangle \end{aligned} \tag{25}$$

Substituting (18) and (20) into (25), we have

$$\begin{aligned} \triangle_p(u_{n,i}, v) &\leq \triangle_p(x_n, v) + \frac{\gamma\langle J_{E_1}^p G_{n,i}x_n, x_n - v\rangle}{\|J_{E_1}^p G_{n,i}x_n\|} - \frac{\langle J_{E_1}^p G_{n,i}x_n, x_n - v\rangle}{\|J_{E_1}^p G_{n,i}x_n\|} \\ &= \triangle_p(x_n, v) - [1-\gamma] \times \frac{\langle J_{E_1}^p G_{n,i}x_n, x_n - v\rangle}{\|J_{E_1}^p G_{n,i}x_n\|}. \end{aligned}$$

Thus, (22) holds.

Now, for each $i \in I$, let $v = B_\gamma^{U_i} v$ and $Av = B_\gamma^{T_i} Av$. By Lemma 4, we have

$$\begin{aligned} \triangle_p(y_n, v) \leq \frac{1}{q}\left\| J_{E_1}^p u_{n,i} - \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\right\|^q + \frac{1}{p}\|v\|^p \\ - \langle J_{E_1}^p u_{n,i}, v\rangle + \left\langle \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, v\right\rangle, \end{aligned} \tag{26}$$

where,

$$\begin{aligned} &\left\langle \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, v\right\rangle \\ &= -\left\langle \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n J_{E_2}^p (\Pi_{AK}^p B_{\delta_n}^{T_i} - I) A u_{n,i}, (Av - \sum_{i=0}^{\infty}\beta_{n,i} A u_{n,i}) - \sum_{i=0}^{\infty}\beta_{n,i}(\Pi_{AK}^p B_{\delta_n}^{T_i} - I) A u_{n,i}\right\rangle \\ &- \left\langle \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, \sum_{i=0}^{\infty}\beta_{n,i}(I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\right\rangle \\ &+ \left\langle \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, A u_{n,i}\right\rangle. \end{aligned}$$

As AK is closed and convex, by Lemma 5 and the variational inequality for the Bregman projection of zero onto $AK - \sum_{i=0}^{\infty}\beta_{n,i} A u_{n,i}$, we arrive at

$$\left\langle \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n J_{E_2}^p (\Pi_{AK}^p B_{\delta_n}^{T_i} - I) A u_{n,i}, (Av - \sum_{i=0}^{\infty}\beta_{n,i} A u_{n,i}) - \sum_{i=0}^{\infty}\beta_{n,i}(\Pi_{AK}^p B_{\delta_n}^{T_i} - I) A u_{n,i}\right\rangle \geq 0$$

and therefore,

$$\left\langle \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, v \right\rangle$$
$$\leq - \left\langle \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, \sum_{i=0}^{\infty} \beta_{n,i} (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i} \right\rangle$$
$$+ \left\langle \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n J_{E_2}^p (I - \Pi_{\Gamma}^p B_{\delta_n}^{T_i}) A u_{n,i}, A u_{n,i} \right\rangle. \quad (27)$$

By Lemma 6, 14 and (27), we get

$$\triangle_p(y_n, v) \leq \triangle_p(u_{n,i}, v) + 2^p G_p \|J_{E_1}^p u_{n,i}\|^p \rho_{E_1^*}(\tau_{n,i})$$
$$- \left\langle \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, \sum_{i=0}^{\infty} \beta_{n,i} (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i} \right\rangle. \quad (28)$$

Substituting (17) and (19) into (28), we have

$$\triangle_p(y_n, v) \leq \triangle_p(u_{n,i}, v) - [1 - \iota]$$
$$\times \frac{\left\langle \sum_{i=0}^{\infty} \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}, \sum_{i=0}^{\infty} \beta_{n,i} (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i} \right\rangle}{\|A\| \| \sum_{i=0}^{\infty} \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\|}.$$

Thus, (21) holds as desired. □

We now prove our main result.

Theorem 1. *Let $g_i : K \times K \to R$, $i \in I$, be bifunctions satisfying $(A1) - (A4)$ in (3). For $\delta > 0$ and $p, q \in (1, \infty)$, let $(I - \Pi_{AK}^p B_{\delta}^{T_i})$, $i \in I$, be demi-closed at zero. Let $x_1 \in E_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} g_i(u_{n,i}, y) + \langle J_{E_1}^p C_i u_{n,i} + J_{E_1}^p G_{n,i} x_n, y - u_{n,i} \rangle + \theta_i(y) - \theta_i(u_{n,i}) \\ + \frac{1}{r_{n,i}} \langle y - u_{n,i}, J_{E_1}^p u_{n,i} - J_{E_1}^p x_n \rangle \geq 0 \ \forall y \in K, \ \forall i \in I, \\ y_n = J_{E_1^*}^q \left(\sum_{i=0}^{\infty} \alpha_{n,i} B_{\delta_n}^{U_i} \left(J_{E_1}^p u_{n,i} - \sum_{i=0}^{\infty} \beta_{n,i}\lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i} \right) \right), \\ x_{n+1} = J_{E_1^*}^q \left(\eta_{n,0} J_{E_1}^p (f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} J_{E_1}^p (S_{n,i}(y_n)) \right) n \geq 1, \end{cases} \quad (29)$$

where $r_{n,i} = \frac{1}{\|J_{E_1}^p G_{n,i} x_n\|}$, $\mu_{n,i} = \frac{1}{\|x_n\|^{p-1}}$ and $\gamma \in (0,1)$ such that $\rho_{E_1^*}(\mu_{n,i}) = \frac{\gamma \langle J_{E_1}^p G_{n,i} x_n, x_n - v \rangle}{2^q G_q \|x_n\|^p \|J_{E_1}^p G_{n,i} x_n\|}$,

$$\lambda_n = \begin{cases} \frac{1}{\|A\|} \frac{1}{\| \sum_{i=0}^{\infty} \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\|}, & u_{n,i} \neq 0 \\ \frac{1}{\|A\|^p} \frac{\| \sum_{i=0}^{\infty} \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\|^{p(p-1)}}{\| \sum_{i=0}^{\infty} \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\|^p}, & u_{n,i} = 0, \end{cases} \quad (30)$$

$\iota \in (0,1)$ and $\tau_{n,i} = \frac{1}{\|u_{n,i}\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\tau_{n,i}) = \frac{\iota}{2^q G_q \|A\|} \times \frac{\| \sum_{i=0}^{\infty} \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\|^p}{\|u_{n,i}\|^p \| \sum_{i=0}^{\infty} \beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i}) A u_{n,i}\|^{p-1}}, \quad (31)$$

with, $\lim_{n\to\infty}\eta_{n,0} = 0$, $\eta_{n,0} \leq \sum_{i=1}^{\infty}\eta_{n,i}$, *for* $M \geq 0$, $\eta_{n-1,0} \leq \sum_{i=1}^{\infty}\eta_{n-1,i} \leq \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\eta_{n-1,i}M < \infty$, $\sum_{i=0}^{\infty}\eta_{n,i} = \sum_{i=0}^{\infty}\alpha_{n,i} = \sum_{i=0}^{\infty}\beta_{n,i} = 1$ *and* $k_n = \max_{i\in I}\{k_{n,i}\}$*. If* $\Gamma = \Omega\cap\omega\cap\Im \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to* $x^* \in \Gamma$*, where* $\sum_{i=0}^{\infty}\beta_{n,i}\Pi_{AK}^{p}B_{\delta_n}^{T_i}(x^*) = \sum_{i=0}^{\infty}\beta_{n,i}B_{\delta_n}^{T_i}(x^*)$*, for each* $i \in I$*.*

Proof. For $x, y \in K$ and $i \in I$, let $\Phi_i(x,y) = g_i(x,y) + \langle J_{E_1}^{p}C_i x, y - x\rangle + \theta_i(y) - \theta_i(x)$. Since g_i are bi-functions satisfying $(A1)-(A4)$ in (3) and C_i are monotone and Lipschitz continuous mappings, and θ_i are convex and lower semicontinuous functions, therefore $\Phi_i (i \in I)$ satisfy the conditions $(A1)-(A4)$ in (3), and hence the algorithm (29) can be written as follows:

$$\begin{cases} \Phi_i(u_{n,i},y) + \langle J_{E_1}^{p}G_{n,i}x_n, y-u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y-u_{n,i}, J_{E_1}^{p}u_{n,i} - J_{E_1}^{p}x_n\rangle \geq 0 \\ \forall y\in K,\ \forall i\in I, \\ y_n = J_{E_1^*}^{q}\left(\sum_{i=0}^{\infty}\alpha_{n,i}B_{\delta_n}^{U_i}\left(J_{E_1}^{p}u_{n,i} - \sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J_{E_2}^{p}(I-\Pi_{AK}^{p}B_{\delta_n}^{T_i})Au_{n,i}\right)\right), \\ x_{n+1} = J_{E_1^*}^{q}\left(\eta_{n,0}J_{E_1}^{p}(f(x_n)) + \sum_{i=1}^{\infty}\eta_{n,i}J_{E_1}^{p}(S_{n,i}(y_n))\right) n \geq 1. \end{cases} \tag{32}$$

We will divide the proof into four steps.

Step One: We show that $\{x_n\}$ is a bounded sequence.

Assume that $\|\sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^{p}(I-\Pi_{AK}^{p}B_{\delta_n}^{T_i})Au_{n,i}\| = 0$ and $\|J_{E_1}^{p}G_{n,i}x_n\| = 0$. Then, by (32), we have

$$\Phi_i(u_{n,i},y) + \frac{1}{r_{n,i}}\left\langle y-u_{n,i}, J_{E_1}^{p}u_{n,i} - J_{E_1}^{p}x_n\right\rangle \geq 0\ \forall y\in K,\ \forall i\in I. \tag{33}$$

By (33) and Lemma 13, for each $i\in I$, we have that $u_{n,i} = J_{E_1^*}^{q}(\Upsilon_{r_{n,i}}(J_{E_1}^{p}x_n))$. By Lemma 4 and for $v\in\Gamma$ and $v = \Upsilon_{r_{n,i}}v$, we have

$$\triangle_p(u_{n,i},v) = V_p(\Upsilon_{r_{n,i}}(J_{E_1}^{p}x_n),v) \leq V_p(J_{E_1}^{p}x_n,v) = \triangle_p(x_n,v). \tag{34}$$

In addition, for each $i\in I$, let $v = B_{\gamma}^{U_i}v$. By Lemma 4 and for $v\in\Gamma$, we have

$$\triangle_p(y_n,v) = V_p\left(\sum_{i=0}^{\infty}\alpha_{n,i}B_{\delta_n}^{U_i}J_{E_1}^{p}u_{n,i},v\right) \leq \triangle_p(u_{n,i},v). \tag{35}$$

Now assume that $\|\sum_{i=0}^{\infty}\beta_{n,i}J_{E_2}^{p}(I-\Pi_{AK}^{p}B_{\delta_n}^{T_i})Au_{n,i}\| \neq 0$ and $\|J_{E_1}^{p}G_{n,i}x_n\| \neq 0$. Then by (32), we have that

$$\Phi_i(u_{n,i},y) + \frac{1}{r_{n,i}}\left\langle y-u_{n,i}, J_{E_1}^{p}u_{n,i} - (J_{E_1}^{p}x_n - r_{n,i}J_{E_1}^{p}G_{n,i}x_n)\right\rangle \geq 0\ \forall y\in K,\ \forall i\in I. \tag{36}$$

By (36) and Lemma 13, for each $i\in I$, we have $u_{n,i} = J_{E_1^*}^{q}(\Upsilon_{r_{n,i}}(J_{E_1}^{p}x_n - r_{n,i}J_{E_1}^{p}G_{n,i}x_n))$. For $v\in\Gamma$, by (22) in Lemma 15, we get

$$\triangle_p(u_{n,i},v) \leq \triangle_p(x_n,v). \tag{37}$$

In addition, for each $i\in I$, $v\in\Gamma$, (21) in Lemma 15 gives

$$\triangle_p(y_n,v) \leq \triangle_p(u_{n,i},v). \tag{38}$$

Let $u_{n,i} = 0$. By Lemma 1, we have

$$\triangle_p(u_{n,i}, v) = \frac{1}{p}\|v\|^p \tag{39}$$

and by (27), (39), Lemmas 4 and 15, we have

$$\begin{aligned}\triangle_p(y_n, v) \leq & \frac{1}{q}\left\|\sum_{i=0}^{\infty}\beta_{n,i}\lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\right\|^p \\ & + \triangle_p(u_{n,i}, v) + \lambda_n \left\langle \sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}, Au_{n,i}\right\rangle \\ & - \lambda_n \left\langle \sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}, \sum_{i=0}^{\infty}\beta_{n,i}(I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\right\rangle.\end{aligned} \tag{40}$$

However, by (30) and (40), we have

$$\begin{aligned}&\triangle_p(y_n, v) \\ &\leq \frac{1}{q}\frac{1}{\|A\|^p}\frac{\left\langle \sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}, \sum_{i=0}^{\infty}\beta_{n,i}(I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\right\rangle^p}{\left\|\sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\right\|^p} \\ &+ \triangle_p(u_{n,i}, v) + \lambda_n \langle \sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}, Au_{n,i}\rangle \\ &- \lambda_n \langle \sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}, \sum_{i=0}^{\infty}\beta_{n,i}(I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\rangle \\ &\leq \triangle_p(u_{n,i}, v) \\ &- \frac{1}{\|A\|^p}\frac{\langle \sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}, \sum_{i=0}^{\infty}\beta_{n,i}(I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\rangle^p}{\|\sum_{i=0}^{\infty}\beta_{n,i} J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{T_i})Au_{n,i}\|^p}.\end{aligned} \tag{41}$$

This implies that

$$\triangle_p(y_n, v) \leq \triangle_p(u_{n,i}, v). \tag{42}$$

By (42) and (37), we get

$$\triangle_p(y_n, v) \leq \triangle_p(x_n, v). \tag{43}$$

In addition, it follows from the assumption $\eta_{n,0} \leq \sum_{i=1}^{\infty}\eta_{n,i}$, (43), Definition 3, Lemmas 9 and 4

$$
\begin{aligned}
&\triangle_p(x_{n+1}, v)\\
&= \triangle_p \left(J^q_{E_1^*} \left(\eta_{n,0} J^p_{E_1}(f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} J^p_{E_1}(S_{n,i}(y_n)) \right), v \right)\\
&= V_p \left(\eta_{n,0} J^p_{E_1}(f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} J^p_{E_1}(S_{n,i}(y_n)), v \right)\\
&\leq \eta_{n,0} V_p \left(J^p_{E_1}(f(x_n)), v \right) + \sum_{i=1}^{\infty} \eta_{n,i} V_p \left(J^p_{E_1}(S_{n,i}(y_n)), v \right)\\
&\leq \eta_{n,0} \zeta \triangle_p (x_n, v) + \eta_{n,0}(\triangle_p(f(v), v)\\
&+ \langle J^p_{E_1} x_n - J^p_{E_1} f(v), f(v) - v \rangle) + \sum_{i=1}^{\infty} \eta_{n,i} k_{n,i} \triangle_p (y_n, v)\\
&\leq \eta_{n,0} \left(\triangle_p(f(v), v) + \langle J^p_{E_1} x_n - J^p_{E_1} f(v), f(v) - v \rangle \right)\\
&+ \left(\eta_{n,0} \zeta + \sum_{i=1}^{\infty} \eta_{n,i} k_{n,i} \right) \triangle_p (x_n, v)\\
&\leq \eta_{n,0} \left(\triangle_p(f(v), v) + \langle J^p_{E_1} x_n - J^p_{E_1} f(v), f(v) - v \rangle \right)\\
&+ \left(\sum_{i=1}^{\infty} \eta_{n,i} (\zeta + k_{n,i}) \right) \triangle_p (x_n, v)\\
&\leq max \left\{ \frac{\left(\triangle_p(f(v), v) + \langle J^p_{E_1} x_1 - J^p_{E_1} f(v), f(v) - v \rangle \right)}{\zeta + k_{1,i}}, \triangle_p(x_1, v) \right\}.
\end{aligned} \tag{44}
$$

By (44), we conclude that $\{x_n\}$ is bounded, and hence, from (42), (34), (35), (44), (38), and (37), $\{y_n\}$ and $\{u_{n,i}\}$ are also bounded.

Step Two: We show that $\lim_{m \to \infty} \triangle_p (x_{n+1}, x_n) = 0$. By Lemmas 1, 4, 10, and 7, we have, by the convexity of $\triangle_p$ in the first argument and for $\eta_{n-1,0} \leq \sum_{i=1}^{\infty} \eta_{n-1,i}$,

$$
\begin{aligned}
&\triangle_p(x_{n+1}, x_n) = \triangle_p(J^q_{E_1^*} \left(\eta_{n,0} J^p_{E_1}(f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} J^p_{E_1}(S_{n,i}(y_n)) \right),\\
&J^q_{E_1^*} \left(\eta_{n-1,0} J^p_{E_1}(f(x_{n-1})) + \sum_{i=1}^{\infty} \eta_{n-1,i} J^p_{E_1}(S_{n-1,i}(y_{n-1})) \right))\\
&\leq \eta_{n,0} \triangle_q^* (J^p_{E_1}(f(x_n)), \eta_{n-1,0} J^p_{E_1}(f(x_{n-1})) + \sum_{i=1}^{\infty} \eta_{n-1,i} J^p_{E_1}(S_{n-1,i}(y_{n-1})))\\
&+ \sum_{i=1}^{\infty} \eta_{n,i} \triangle_q^* (J^p_{E_1}(S_{n,i}(y_n)), \eta_{n-1,0} J^p_{E_1}(f(x_{n-1})) + \sum_{i=1}^{\infty} \eta_{n-1,i} J^p_{E_1}(S_{n-1,i}(y_{n-1})))\\
&\leq \eta_{n,0} \left(\triangle_q^*(J^p_{E_1}(f(x_n), J^p_{E_1}(f(x_{n-1}))) \right)\\
&+ \sum_{i=1}^{\infty} \eta_{n-1,i} \left(\sum_{i=1}^{\infty} \eta_{n,i} \frac{1}{p} \|S_{n-1,i}(y_{n-1})\|^p + \eta_{n,0} \|f(x_n)\| \left\| J^p_{E_1}(S_{n-1,i}(y_{n-1})) \right\| \right)\\
&+ \eta_{n-1,0} \left(\eta_{n,0} \frac{1}{p} \|f(x_{n-1})\|^p + \sum_{i=1}^{\infty} \eta_{n,i} \|S_{n,i}(y_n)\| \left\| J^p_{E_1}(f(x_{n-1})) \right\| \right)\\
&+ \sum_{i=1}^{\infty} \eta_{n,i} \triangle_q^* \left((J^p_{E_1} S_{n,i}(y_n), J^p_{E_1} S_{n-1,i}(y_{n-1}) \right)\\
&\leq (1 - \eta_{n,0}(1 - \zeta)) \triangle_p (x_n, x_{n-1}) + \sum_{i=1}^{\infty} \eta_{n,i} \sup_{n, n-1 \geq 1} \{ \triangle_p(S_{n,i}(y_n), S_{n-1,i}(y_{n-1})) \}\\
&+ \sum_{i=1}^{\infty} \eta_{n-1,i} M,
\end{aligned} \tag{45}
$$

where
$$M = max\left\{max\{\|f(x_n))\|, \|S_{n-1,i}(y_{n-1})\|\}, max\{\|f(x_{n-1})\|, \|S_{n,i}(y_n)\|\}\right\}.$$
In view of the assumption $\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\eta_{n-1,i}M < \infty$ and (45), Lemmas 11 and 8 imply
$$\lim_{n\to\infty} \triangle_p(x_{n+1}, x_n) = 0. \tag{46}$$

Step Three: We show that $\lim_{n\to\infty} \triangle_p(S_{n,i}y_n, y_n) = 0$.
For each $i \in I$, we have
$$\triangle_p(S_i(y_n), v) \le \triangle_p(y_n, v).$$

Then,
$$\begin{aligned}
0 &\le \triangle_p(y_n, v) - \triangle_p(S_i(y_n), v)\\
&= \triangle_p(y_n, v) - \triangle_p(x_{n+1}, v) + \triangle_p(x_{n+1}, v) - \triangle_p(S_i(y_n), v)\\
&\le \triangle_p(x_n, v) - \triangle_p(x_{n+1}, v) + \triangle_p(x_{n+1}, v) - \triangle_p(S_i(y_n), v)\\
&= \triangle_p(x_n, v) - \triangle_p(x_{n+1}, v) + \triangle_p\left(J^q_{E_1^*}\left(\eta_{n,0}J^p_{E_1}(f(x_n)) + \sum_{i=1}^{\infty}\eta_{n,i}J^p_{E_1}(S_i(y_n))\right), v\right)\\
&- \triangle_p(S_i(y_n), v)\\
&\le \triangle_p(x_n, v) - \triangle_p(x_{n+1}, v) + \eta_{n,0}\triangle_p(f(x_n), v) - \eta_{n,0}\triangle_p(S_i(y_n), v)\\
&\longrightarrow 0 \; as \; n \to \infty.
\end{aligned}\tag{47}$$

By (47) and Definition 2, we get
$$\lim_{n\to\infty} \triangle_p(S_i y_n, y_n) = 0. \tag{48}$$

By uniform continuity of S, we have
$$\lim_{n\to\infty} \triangle_p(S_{n,i}y_n, y_n) = 0. \tag{49}$$

Step Four: We show that $x_n \to x^* \in \Gamma$.
Note that,
$$\begin{aligned}
\triangle_p(x_{n+1}, y_n) &= \triangle_p(J^q_{E_1^*}\left(\eta_{n,0}J^p_{E_1}(f(x_n)) + \sum_{i=1}^{\infty}\eta_{n,i}J^p_{E_1}(S_{n,i}(y_n))\right), y_n)\\
&\le \eta_{n,0}\triangle_p(f(x_n), y_n) + \sum_{i=1}^{\infty}\eta_{n,i}\triangle_p(S_{n,i}(y_n), y_n)\\
&\le \eta_{n,0}(\zeta\triangle_p(x_n, y_n) + \triangle_p(f(y_n), y_n) + \langle f(x_n) - f(y_n), J^p_{E_1}f(y_n) - J^p_{E_1}y_n\rangle)\\
&+ \sum_{i=1}^{\infty}\eta_{n,i}\triangle_p(S_{n,i}(y_n), y_n)\\
&\le (1 - \eta_{n,0}(1-\zeta))\triangle_p(x_n, y_n)\\
&+ \eta_{n,0}(\triangle_p(f(y_n), y_n) + \langle f(x_n) - f(y_n), J^p_{E_1}f(y_n) - J^p_{E_1}y_n\rangle)\\
&+ \sum_{i=1}^{\infty}\eta_{n,i}\triangle_p(S_{n,i}(y_n), y_n).
\end{aligned}\tag{50}$$

By (49), (50), and Lemma 8, we have

$$\lim_{n\to\infty} \triangle_p (x_n, y_n) = 0. \tag{51}$$

Therefore, by (51) and the boundedness of $\{y_n\}$, and since by (46), $\{x_n\}$ is Cauchy, we can assume without loss of generality that $y_n \rightharpoonup x^*$ for some $x^* \in E_1$. It follows from Lemmas 2, 3, and (48) that $x^* = S_i x^*$, for each $i \in I$. This means that $x^* \in \Im$.

In addition, by (31) and the fact that $u_{n,i} \to x^*$ *as* $n \to \infty$, we arrive at

$$\frac{(J^p_{E_1} u_{n,i} - J^p_{E_1} y_n) - \sum_{i=0}^{\infty} \beta_{n,i} \lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK} B^{T_i}_{\delta_n}) A u_{n,i}}{\delta_n} \in \sum_{i=0}^{\infty} \alpha_{n,i} U_i(y_n). \tag{52}$$

By (21), we have

$$\| \sum_{i=0}^{\infty} \beta_{n,i}(I - \Pi^p_{AK} B^{T_i}_{\delta_n}) A u_{n,i} \| \le \left[\frac{\triangle_p(u_{n,i}, v) - \triangle_p(y_n, v)}{\|A\|^{-1}[1-\iota]} \right] \longrightarrow 0 \ as\ n \to \infty, \tag{53}$$

and by (41), we have

$$\| \sum_{i=0}^{\infty} \beta_{n,i}(I - \Pi^p_{AK} B^{T_i}_{\delta_n}) A u_{n,i} \| \le \left[\frac{\triangle_p(u_{n,i}, v) - \triangle_p(y_n, v)}{(p\|A\|)^{-1}} \right]^{\frac{1}{p}} \longrightarrow 0 \ as\ n \to \infty. \tag{54}$$

From (53), (54), and (52), by passing n to infinity in (52), we have that $0 \in \sum_{i=0}^{\infty} \alpha_{n,i} U_i(x^*)$. This implies that $x^* \in SOLVIP(U_i)$. In addition, by (48), we have $Ay_n \rightharpoonup Ax^*$. Thus, by (53), (54) and an application of the demi-closeness of $\sum_{i=0}^{\infty} \beta_{n,i}(I - \Pi^p_{AK} B^{T_i}_{\delta_n})$ at zero, we have that $0 \in \sum_{i=0}^{\infty} \beta_{n,i} T_i(Ax^*)$. Therefore, $Ax \in SOLVIP(T_i)$ as $\sum_{i=0}^{\infty} \beta_{n,i} \Pi^p_{AK} B^{T_i}_{\delta}(Ax^*) = \sum_{i=0}^{\infty} \beta_{n,i} B^{T_i}_{\delta}(Ax^*)$. This means that $x^* \in \Omega$.

Now, we show that $x^* \in (\cap_{i=1}^{\infty} GMEP(\theta_i, C_i, G_i, g_i)$. By (32), we have

$$\Phi_i(u_{n,i}, y) + \langle J^p_{E_1} G_{n,i} x_n, y - u_{n,i} \rangle + \frac{1}{r_{n,i}} \langle y - u_{n,i}, J^p_{E_1} u_{n,i} - J^p_{E_1} x_n \rangle \ge 0 \qquad \forall y \in K,\ \forall i \in I,$$

Since Φ_i, for each $i \in I$, are monotone, that is, for all $y \in K$,

$$\begin{aligned} &\Phi_i(u_{n,i}, y) + \Phi_i(y, u_{n,i}) \le 0 \\ &\Rightarrow \frac{1}{r_{n,i}} \langle y - u_{n,i}, J^p_{E_1} u_{n,i} - J^p_{E_1} x_n \rangle \\ &\ge \Phi_i(y, u_{n,i}) + \langle J^p_{E_1} G_{n,i} x_n, y - u_{n,i} \rangle, \end{aligned}$$

therefore,

$$\frac{1}{r_{n,i}} \langle y - u_{n,i}, J^p_{E_1} u_{n,i} - J^p_{E_1} x_n \rangle \ge \Phi_i(y, u_{n,i}) + \langle J^p_{E_1} G_{n,i} x_n, y - u_{n,i} \rangle.$$

By the lower semicontinuity of Φ_i, for each $i \in I$, the weak upper semicontinuity of G, and the facts that, for each $i \in I$, $u_{n,i} \to x^*$ as $n \to \infty$ and J^p is $norm - to - weak^*$ uniformly continuous on a bounded subset of E_1, we have

$$0 \ge \Phi_i(y, x^*) + \langle J^p_{E_1} G_{n,i} x^*, y - x^* \rangle. \tag{55}$$

Now, we set $y_t = ty + (1-t)x^* \in K$. From (55), we get

$$0 \geq \Phi_i(y_t, x^*) + \langle J^p_{E_1} G_{n,i} x^*, y_t - x^* \rangle. \tag{56}$$

From (56), and by the convexity of Φ_i, for each $i \in I$, in the second variable, we arrive at

$$\begin{aligned} 0 = \Phi_i(y_t, y_t) &\leq t\Phi_i(y_t, y) + (1-t)\Phi_i(y_t, x^*) \\ &\leq t\Phi_i(y_t, y) + (1-t)\langle J^p_{E_1} G_{n,i} x^*, y_t - x^* \rangle \\ &\leq t\Phi_i(y_t, y) + (1-t)t\langle J^p_{E_1} G_{n,i} x^*, y - x^* \rangle, \end{aligned}$$

which implies that

$$\Phi_i(y_t, y) + (1-t)\langle J^p_{E_1} G_{n,i} x^*, y - x^* \rangle \geq 0. \tag{57}$$

From (57), by the lower semicontinuity of Φ_i, for each $i \in I$, we have for $y_t \to x^*$ as $t \to 0$

$$\Phi_i(x^*, y) + \langle J^p_{E_1} G_{n,i} x^*, y - x^* \rangle \geq 0. \tag{58}$$

Therefore, by (58) we can conclude that $x^* \in (\cap_{i=1}^{\infty} GMEP(\theta_i, C_i, G_i, g_i)$. This means that $x^* \in \omega$. Hence, $x^* \in \Gamma$.

Finally, we show that $x_n \to x^*$, as $n \to \infty$. By Definition 3, we have

$$\begin{aligned} &\triangle_p\,(x_{n+1}, x^*) \\ &= \triangle_p(J^q_{E_1^*}\left(\eta_{n,0} J^p_{E_1}(f(x_n)) + \sum_{i=1}^{\infty} \eta_{n,i} J^p_{E_1}(G_{n,i}(y_n))\right), x^*) \\ &\leq \eta_{n,0} \triangle^*_q\,(J^p_{E_1}(f(u_n)), J^p_{E_1} x^*) + \sum_{i=1}^{\infty} \eta_{n,i} \triangle^*_q\,(J^p_{E_1}(G_{n,i}(y_n)), J^p_{E_1} x^*) \\ &\leq \eta_{n,0}\zeta \triangle_p\,(x_n, x^*) + \eta_{n,0}(\triangle_p(f(x^*), x^*) \\ &+ \langle J^p_{E_1} x_n - J^p_{E_1} f(x^*), f(x^*) - x^* \rangle) + \sum_{i=1}^{\infty} \eta_{n,i} k_n \triangle_p\,(y_n, x^*) \\ &\leq \eta_{n,0}\left(\triangle_p(f(x^*), x^*) + \langle J^p_{E_1} x_n - J^p_{E_1} f(x^*), f(x^*) - x^* \rangle\right) \\ &+ \left(1 - \sum_{i=1}^{\infty} \eta_{n,i}\,(1-k_n)\right) \triangle_p\,(x_n, x^*). \end{aligned} \tag{59}$$

By (59) and Lemma 8, we have that

$$\lim_{n\to\infty} \triangle_p\,(x_n, x^*) = 0.$$

The proof is completed. □

In Theorem 1, $i = 0$ leads to the following new result.

Corollary 1. *Let $g : K \times K \to R$ be bifunctions satisfying $(A1)-(A4)$ in (3). Let $(I - \Pi^p_{AK} B^T_\delta)$ be demiclosed at zero. Suppose that $x_1 \in E_1$ is chosen arbitrarily and the sequence $\{x_n\}$ is defined as follows:*

$$\begin{cases} g(u_n, y) + \langle J^p_{E_1} C u_n + J^p_{E_1} G_n x_n, y - u_n \rangle + \theta(y) - \theta(u_n) \\ +\frac{1}{r_n}\langle y - u_n, J^p_{E_1} u_n - J^p_{E_1} x_n \rangle \geq 0 \; \forall y \in K, \\ y_n = J^q_{E_1^*}\left(B^U_{\delta_n}\left(J^p_{E_1} u_n - \lambda_n A^* J^p_{E_2}(I - \Pi^p_{AK} B^T_{\delta_n}) A u_n\right)\right), \\ x_{n+1} = J^q_{E_1^*}\left(\eta_n J^p_{E_1}(f(x_n)) + (1-\eta_n) J^p_{E_1}(S_n(y_n))\right) n \geq 1, \end{cases} \tag{60}$$

where $r_n = \frac{1}{\|J^p_{E_1} G_n x_n\|}$, $\mu_n = \frac{1}{\|x_n\|^{p-1}}$ *and* $\gamma \in (0,1)$ *such that* $\rho_{E_1^*}(\mu_n) = \frac{\gamma \langle J^p_{E_1} G_n x_n, x_n - v\rangle}{2^q G_q \|x_n\|^p \|J^p_{E_1} G_n x_n\|}$, *and*

$$\lambda_n = \begin{cases} \frac{1}{\|A\|} \frac{1}{\|J^p_{E_2}(I - \Pi^p_{AK} B^T_{\delta_n}) A u_n\|}, & u_n \neq 0 \\ \frac{1}{\|A\|^p} \frac{\|J^p_{E_2}(I - \Pi^p_{AK} B^T_{\delta_n}) A u_n\|^{p(p-1)}}{\|J^p_{E_2}(I - \Pi^p_{AK} B^T_{\delta_n}) A u_n\|^p}, & u_n = 0, \end{cases} \tag{61}$$

and $\iota \in (0,1)$ *and* $\tau_n = \frac{1}{\|u_n\|^{p-1}}$ *are chosen such that*

$$\rho_{E_1^*}(\tau_n) = \frac{\iota}{2^q G_q \|A\|} \times \frac{\|J^p_{E_2}(I - \Pi^p_{AK} B^T_{\delta_n}) A u_n\|^p}{\|u_n\|^p \|J^p_{E_2}(I - \Pi^p_{AK} B^T_{\delta_n}) A u_n\|^{p-1}}, \tag{62}$$

and $\lim_{n\to\infty} \eta_n = 0$, *for* $M \geq 0$, $\sum_{n=1}^{\infty} \eta_{n-1} M < \infty$, *and* $\eta_n \leq \frac{1}{2}$. *If* $\Gamma = \Omega \cap \omega \cap \Im \neq \emptyset$, *then* $\{x_n\}$ *converges strongly to* $x^* \in \Gamma$, *where* $\Pi^p_{AK} B^T_{\delta_n}(x^*) = B^T_{\delta_n}(x^*)$.

3. Application to Generalized Mixed Equilibrium Problem, Split Hammerstein Integral Equations and Fixed Point Problem

Definition 4. *Let* $C \subset \mathbb{R}^n$ *be bounded. Let* $k : C \times C \to \mathbb{R}$ *and* $f : C \times \mathbb{R} \to \mathbb{R}$ *be measurable real-valued functions. An integral equation of Hammerstien-type has the form*

$$u(x) + \int_C k(x,y) f(y, u(y)) dy = w(x),$$

where the unknown function u *and non-homogeneous function* w *lies in a Banach space* E *of measurable real-valued functions. By transforming the above equation, we have that*

$$u + KFu = w,$$

and therefore, without loss of generality, we have

$$u + KFu = 0. \tag{63}$$

The split Hammerstein integral equations problem is formulated as finding $x^* \in E_1$ and $y^* \in E_1^*$ such that

$$x^* + KFx^* = 0 \text{ with } Fx^* = y^* \text{ and } Ky^* + x^* = 0$$

and $Ax^* \in E_2$ and $Ay^* \in E_2^*$ such that

$$Ax^* + K'F'Ax^* = 0 \text{ with } F'Ax^* = Ay^* \text{ and } K'Ay^* + Ax^* = 0$$

where $F : E_1 \to E_1^*$, $K : E_1^* \to E_1$ and $F' : E_2 \to E_2^*$, $K' : E_2^* \to E_2$ are maximal monotone mappings.

Lemma 16 ([21]). *Let* E *be a Banach space. Let* $F : E \to E^*$, $K : E^* \to E$ *be bounded and maximal monotone operators. Let* $D : E \times E^* \to E^* \times E$ *be defined by* $D(x,y) = (Fx - y, Ky + x)$ *for all* $(x,y) \in E \times E^*$. *Then, the mapping* D *is maximal monotone.*

By Lemma 16, if K, K', and F, F' are multi-valued maximal monotone operators then, we have two resolvent mappings,

$$B_\delta^D = (J_{E_1}^p + \delta J_{E_1}^p D)^{-1} J_{E_1}^p \text{ and } B_\delta^{D'} = (J_{E_2}^p + \delta J_{E_2}^p D')^{-1} J_{E_2}^p,$$

where $F : E_1 \to E_1^*$, $K : E_1^* \to E_1$ are multi-valued and maximal monotone operators, $D : E_1 \times E_1^* \to E_1^* \times E_1$ is defined by $D(x,y) = (Fx - y, Ky + x)$ for all $(x,y) \in E_1 \times E_1^*$, and $F' : E_2 \to E_2^*$, $K' : E_2^* \to E_2$ are multi-valued and maximal monotone operators, $D' : E_2 \times E_2^* \to E_2^* \times E_2$ is defined by $D'(Ax, Ay) = (F'Ax - Ay, K'Ay + Ax)$ for all $(Ax, Ay) \in E_2 \times E_2^*$. Then D and D' are maximal monotone by Lemma 16.

When $U = D$ and $T = D'$ in Corollary 1, the algorithm (60) becomes

$$\begin{cases} g(u_n, y) + \langle J_{E_1}^p C_n u_n + J_{E_1}^p G_n x_n, y - u_n \rangle + \theta(y) - \theta(u_n) \\ + \frac{1}{r_n} \langle y - u_n, J_{E_1}^p u_n - J_{E_1}^p x_n \rangle \geq 0 \ \forall y \in K, \\ y_n = J_{E_1^*}^q \left(B_{\delta_n}^{D_n} \left(J_{E_1}^p u_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^{D'_n}) A u_n \right) \right) \\ x_{n+1} = J_{E_1^*}^q \left(\eta_n J_{E_1}^p (f(x_n)) + (1 - \eta_n) J_{E_1}^p (S_n(y_n)) \right) n \geq 1; \end{cases}$$

and its strong convergence is guaranteed, which solves the problem of a common solution of a system of generalized mixed equilibrium problems, split Hammerstein integral equations, and fixed-point problems for the mappings involved in this algorithm.

4. A Numerical Example

Let $i = 0$, $E_1 = E_2 = \mathbb{R}$, and $K = AK = [0, \infty)$, for $Ax = x \ \forall x \in E_1$. The generalized mixed equilibrium problem is formulated as finding a point $x \in K$ such that,

$$g_0(x,y) + \langle G_0 x, y - x \rangle + \theta_0(y) - \theta_0(x) \geq 0, \ \forall y \in K. \tag{64}$$

Let $r_0 \in (0,1]$ and define $\theta_0 = 0$, $g_0(x,y) = \frac{y^2}{r_0} + \frac{2x^2}{r_0}$ and $G_0(x) = S_0(x) = \frac{1}{r_0} x$.

Clearly, $g_0(x,y)$ satisfies the conditions $(A1) - (A4)$ and $G_0(x) = S_0(x)$ is a Bregman asymptotically non-expansive mapping, as well as a $1-$ inverse strongly monotone mapping. Since Υ_{r_0} is single-valued, therefore for $y \in K$, we have that

$$\begin{aligned} & g_0(u_0, y) + \langle G_0 x, y - u_0 \rangle + \frac{1}{r_0} \langle y - u_0, u_0 - x \rangle \geq 0 \\ \Leftrightarrow & \frac{y^2}{r_0} + \frac{2u_0^2}{r_0} + \frac{1}{r_0} \langle y - u_0, u_0 \rangle \geq 0 \\ \Leftrightarrow & \frac{y^2}{r_0} + \frac{2|yu_0|}{r_0^{\frac{3}{2}}} + \frac{x^2}{r_0} \geq 0. \end{aligned} \tag{65}$$

As (65) is a nonnegative quadratic function with respect to y variable, so it implies that the coefficient of y^2 is positive and the discriminant $\frac{4u_0^2}{r_0^3} - \frac{4x^2}{r_0^2} \leq 0$, and therefore $u_0 = x\sqrt{r_0}$. Hence,

$$\Upsilon_{r_0}(x) = x\sqrt{r_0}. \tag{66}$$

By Lemma 13 and (66), $F(\Upsilon_{r_0}) = GEP(g_0, G_0) = \{0\}$ and $F(S_0) = \{0\}$. Define

$$U_0, T_0 : \mathbb{R} \longrightarrow \mathbb{R} \; by \; U_0(x) = T_0(Ax) \begin{cases} (0,1), x \geq 0 \\ \{1\}, x < 0, \end{cases}$$

$$P_{[0,\infty)} : \mathbb{R} \longrightarrow [0,\infty) \; by \; P_{[0,\infty)}(Ax) = \begin{cases} 0, Ax \in (-\infty, 0) \\ Ax, Ax \in [0,\infty), \end{cases}$$

$$B_\delta^{U_0} = B_\delta^T : \mathbb{R} \longrightarrow \mathbb{R} \; by \; B_\delta^T(Ay) = B_\delta^{U_0}(y) = \begin{cases} \frac{y}{1+(0,\delta)}, y \geq 0 \\ \frac{y}{1+\delta}, y < 0, \end{cases}$$

$$P_{[0,\infty)} B_\delta^T : \mathbb{R} \longrightarrow [0,\infty) \; by \; P_{[0,\infty)} B_\delta^T(Ay) = \begin{cases} \frac{Ay}{1+(0,\delta)}, Ay \geq 0 \\ 0, Ay < 0. \end{cases}$$

It is clear that U_0 and T_0 are multi-valued maximal monotone mappings, such that $0 \in SOLVIP(U_0)$ and $0 \in SOLVIP(T_0)$. We define the $\zeta-$contraction mapping by $f(x) = \frac{x}{2}$, $\delta_n = \frac{1}{2^{n+1}}$, $\eta_{n,0} = \frac{1}{n+1}$, $r_{n,0} = \frac{1}{2^{2n}}$ and $\zeta = \frac{1}{2}$. Hence, for

$$\lambda_n = \begin{cases} \frac{1+\left(0,\frac{1}{2^{n+1}}\right)}{\left|u_{n,0}\left(1+\left(0,\frac{1}{2^{n+1}}\right)\right)-u_{n,0}\right|}, \; u_{n,0} > 0, \\ 1, u_{n,0} = 0, \\ \frac{1}{|u_{n,0}|}, \; u_{n,0} < 0, \end{cases}$$

$$\begin{cases} u_{n,0} = \frac{1}{2^n} x_n, \\ y_n^1 = \frac{u_{n,0}}{1+\left(0,\frac{1}{2^{n+1}}\right)}(u_{n,0} - 1), \; u_{n,0} > 0, \\ y_n^2 = \left[\frac{u_{n,0}}{1+\left(0,\frac{1}{2^{n+1}}\right)}\right]^2, \; u_{n,0} = 0, \\ y_n^3 = \frac{2^{n+1} u_{n,0}}{2^{n+1}+1}(u_{n,0} + 1), \; u_{n,0} < 0, \\ x_{n+1} = \frac{x_n}{2(n+1)} + \frac{2^{2n} n y_n}{(n+1)}, \; n \geq 1, \end{cases}$$

we get,

$$x_{n+1} = \begin{cases} \frac{x_n}{2(n+1)} + \frac{n x_n^2 - 2^n x_n}{(n+1)\left(1+\left(0,\frac{1}{2^{n+1}}\right)\right)}, \; x_n > 0, \\ \frac{x_n}{2(n+1)} + \frac{n x_n^2}{(n+1)\left(1+\left(0,\frac{1}{2^{n+1}}\right)\right)}, x_n = 0, \\ \frac{x_n}{2(n+1)} + \frac{n 2^{n+1}(x_n^2 + x_n)}{2^{n+1}+1}, \; x_n < 0. \end{cases}$$

In particular,

$$x_{n+1} = \begin{cases} \frac{x_n}{2(n+1)} + \frac{5(n x_n^2 - 2^n x_n)}{6(n+1)}, \; x_n > 0, \\ \frac{x_n}{2(n+1)} + \frac{5 n x_n^2}{6(n+1)}, x_n = 0, \\ \frac{x_n}{2(n+1)} + \frac{n 2^{n+1}(x_n^2 + x_n)}{2^{n+1}+1}, \; x_n < 0. \end{cases}$$

By Theorem 1, the sequence $\{x_n\}$ converges strongly to $0 \in \Gamma$. The Figures 1 and 2 below obtained by $(MATLAB)$ software indicate convergence of $\{x_n\}$ given by (32) with $x_1 = -10.0$ and $x_1 = 10.0$, respectively.

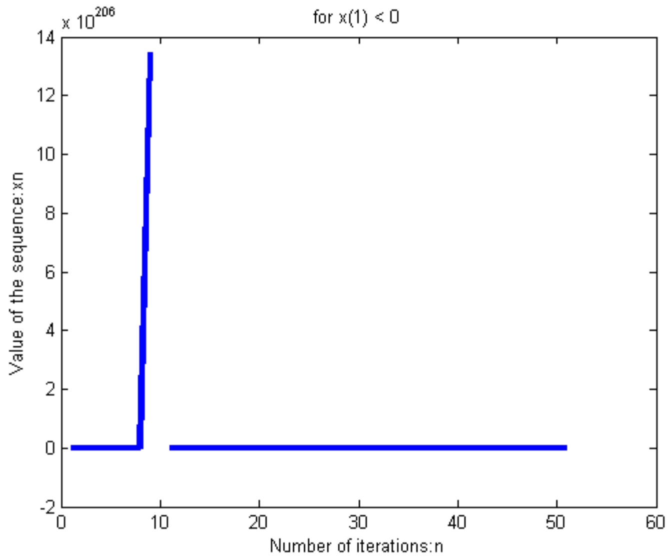

Figure 1. Sequence convergence with initial condition −10.0.

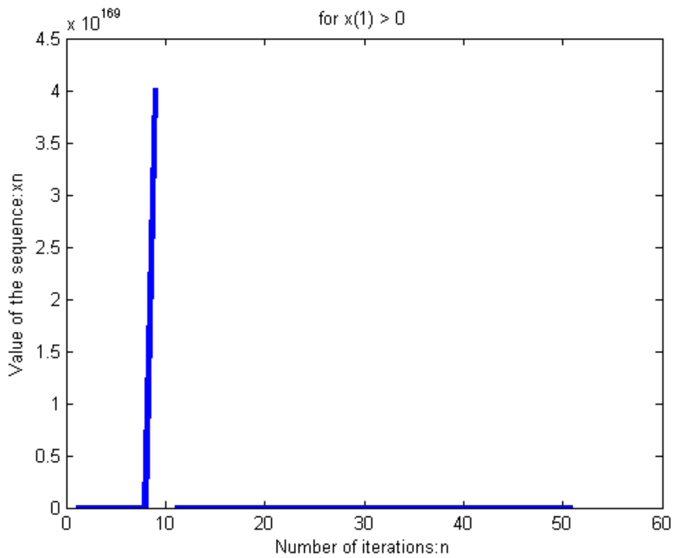

Figure 2. Sequence convergence with initial condition 10.0

Remark 1. *Our results generalize and complement the corresponding ones in [2,7,9,10,22,23].*

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