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Sufficiency Criterion for A Subfamily of Meromorphic Multivalent Functions of Reciprocal Order with Respect to Symmetric Points

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Abstract: In the present research paper, our aim is to introduce a new subfamily of meromorphic p-valent (multivalent) functions. Moreover, we investigate sufficiency criterion for such defined family.

Keywords: meromorphic multivalent starlike functions; subordination

1. Introduction

Let the notation Ω_p be the family of meromorphic p-valent functions f that are holomorphic (analytic) in the region of punctured disk $\mathbb{E} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and obeying the following normalization

$$f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} a_{j+p}\, z^{j+p} \quad (z \in \mathbb{E}). \tag{1}$$

In particular $\Omega_1 = \Omega$, the familiar set of meromorphic functions. Further, the symbol $\mathcal{MS}^*$ represents the set of meromorphic starlike functions which is a subfamily of Ω and is given by

$$\mathcal{MS}^* = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) < 0 \ \ (z \in \mathbb{E}) \right\}.$$

Two points p and p' are said to be symmetrical with respect to o if o' is the midpoint of the line segment pp'. This idea was further nourished in [1,2] by introducing the family $\mathcal{MS}_s^*$ which is defined in set builder form as;

$$\mathcal{MS}_s^* = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(\frac{-2zf'(z)}{f(-z) - f(z)}\right) < 0 \ \ (z \in \mathbb{E}) \right\}.$$

Now, for $-1 \leq t < s \leq 1$ with $s \neq 0 \neq t$, $0 < \xi < 1$, λ is real with $|\lambda| < \frac{\pi}{2}$ and $p \in \mathbb{N}$, we introduce a subfamily of Ω_p consisting of all meromorphic p-valent functions of reciprocal order ξ, denoted by $\mathcal{NS}_p^{\lambda}(s,t,\xi)$, and is defined by

$$\mathcal{NS}_p^{\lambda}(s,t,\xi) = \left\{ f : f(z) \in \Omega_p \text{ and } \Re\left(e^{-i\lambda} \frac{ps^p t^p}{s^p - t^p} \frac{f(sz) - f(tz)}{zf'(z)} \right) > \xi \cos\lambda \ \ (z \in \mathbb{E}) \right\}.$$

We note that for $p = s = 1$ and $t = -1$, the class $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ reduces to the class $\mathcal{NS}_1^{\lambda}(1,-1,\xi) = \mathcal{NS}_*^{\lambda}(\xi)$ and is represented by

$$\mathcal{NS}_*^{\lambda}(\xi) = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(e^{-i\lambda} \frac{f(-z) - f(z)}{2zf'(z)} \right) > \xi \cos\lambda \ \ (z \in \mathbb{E}) \right\}.$$

For detail of the related topics, see the work of Al-Amiri and Mocanu [3], Rosihan and Ravichandran [4], Aouf and Hossen [5], Arif [6], Goyal and Prajapat [7], Joshi and Srivastava [8], Liu and Srivastava [9], Raina and Srivastava [10], Sun et al. [11], Shi et al. [12] and Owa et al. [13], see also [14–16].

For simplicity and ignoring the repetition, we state here the constraints on each parameter as $0 < \xi < 1$, $-1 \leq t < s \leq 1$ with $s \neq 0 \neq t$, λ is real with $|\lambda| < \frac{\pi}{2}$ and $p \in \mathbb{N}$.

We need to mention the following lemmas which will use in the main results.

Lemma 1. *"Let $H \subset \mathbb{C}$ and let $\Phi : \mathbb{C}^2 \times \mathbb{E}^* \to \mathbb{C}$ be a mapping satisfying $\Phi(ia, b : z) \notin H$ for $a, b \in \mathbb{R}$ such that $b \leq -n\frac{1+a^2}{2}$. If $p(z) = 1 + c_n z^n + \cdots$ is regular in $\mathbb{E}^*$ and $\Phi(p(z), zp'(z) : z) \in H \ \forall z \in \mathbb{E}^*$, then $\Re(p(z)) > 0$."*

Lemma 2. *"Let $p(z) = 1 + c_1 z + \cdots$ be regular in $\mathbb{E}^*$ and η be regular and starlike univalent in $\mathbb{E}^*$ with $\eta(0) = 0$. If $zp'(z) \prec \eta(z)$, then*

$$p(z) \prec 1 + \int_0^z \frac{\eta(t)}{t} dt.$$

This result is the best possible."

2. Sufficiency Criterion for the Family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$

In this section, we investigate the sufficiency criterion for any meromorphic p-valent functions belonging to the introduced family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$:

Now, we obtain the necessary and sufficient condition for the p-valent function f to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ as follows:

Theorem 1. *Let the function $f(z)$ be the member of the family Ω_p. Then*

$$f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi) \Leftrightarrow \left| \frac{e^{i\lambda}}{\mathcal{G}(z)} - \frac{1}{2\xi\cos\lambda} \right| < \frac{1}{2\xi\cos\lambda}, \tag{2}$$

where

$$\mathcal{G}(z) = \frac{p\, s^p t^p}{(s^p - t^p)} \frac{f(sz) - f(tz)}{zf'(z)}. \tag{3}$$

Proof. Suppose that inequality (2) holds. Then, we have

$$\begin{aligned}
\left|\frac{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)}{2\xi\cos\lambda e^{-i\lambda}\mathcal{G}(z)}\right| &< \frac{1}{2\xi\cos\lambda} \\
&\Leftrightarrow \left|\frac{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)}{2\xi\cos\lambda e^{-i\lambda}\mathcal{G}(z)}\right|^2 < \frac{1}{4\xi^2\cos^2\lambda} \\
&\Leftrightarrow \left(2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)\right)\left(\overline{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)}\right) < \left(e^{i\lambda}\overline{\mathcal{G}(z)}\right)e^{-i\lambda}\mathcal{G}(z) \\
&\Leftrightarrow 4\xi^2\cos^2\lambda - 2\xi\cos\lambda\left(e^{i\lambda}\overline{\mathcal{G}(z)} + e^{-i\lambda}\mathcal{G}(z)\right) < 0 \\
&\Leftrightarrow 2\xi\cos\lambda - 2\Re\left(e^{-i\lambda}\mathcal{G}(z)\right) < 0 \\
&\Leftrightarrow \Re\left(e^{-i\lambda}\mathcal{G}(z)\right) > \xi\cos\lambda,
\end{aligned}$$

and hence the result follows. □

Next, we investigate the sufficient condition for the p-valent function f to be in the family $\mathcal{NS}_p^\lambda(s,t,\xi)$ in the following theorem:

Theorem 2. *If $f(z)$ belongs to the family Ω_p of meromorphic p-valent functions and obeying*

$$\sum_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - \frac{n\beta\cos\lambda}{p}e^{i\lambda}\right)\right||a_n| < \frac{1}{2}\left(1 - \left|1 - 2\beta\cos\lambda e^{i\lambda}\right|\right), \tag{4}$$

then $f(z) \in \mathcal{NS}_p^\lambda(s,t,\xi)$.

Proof. To prove the required result we only need to show that

$$\left|\frac{2e^{i\lambda}\xi\cos\lambda z f'(z)/p - \frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}{\frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}\right| < 1. \tag{5}$$

Now consider the left hand side of (5), we get

$$\begin{aligned}
LHS &= \left|\frac{2e^{i\lambda}\xi\cos\lambda z f'(z)/p - \frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}{\frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}\right| \\
&= \left|\frac{\left(2e^{i\lambda}\xi\cos\lambda - 1\right) + \sum_{n=p+1}^{\infty}\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - \frac{2n\xi\cos\lambda}{p}e^{i\lambda}\right)a_n z^{n+p}}{1 + \sum_{n=p+1}^{\infty}\left(\frac{s^n - t^n}{s^p - t^p}\right)s^p t^p a_n z^{n+p}}\right| \\
&\le \frac{\left|2e^{i\lambda}\xi\cos\lambda - 1\right| + \sum_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - 2\beta\cos\lambda e^{i\lambda}\frac{n}{p}\right)\right||a_n||z^{n+p}|}{1 - \sum_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}\right)s^p t^p\right||a_n||z^{n+p}|} \\
&\le \frac{\left|2e^{i\lambda}\xi\cos\lambda - 1\right| + \sum_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - 2\beta\cos\lambda e^{i\lambda}\frac{n}{p}\right)\right||a_n|}{1 - \sum_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}\right)s^p t^p\right||a_n|}.
\end{aligned}$$

By virtue of inequality (4), we at once get the desired result. □

Also, we obtain another sufficient condition for the p-valent function f to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ by using Lemma 1, in the following theorem:

Theorem 3. *If $f(z) \in \Omega_p$ satisfies*

$$\Re\left\{e^{-i\lambda}\left(\alpha z\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}+1\right)\mathcal{G}(z)\right\} > \beta\cos\lambda - \frac{n}{2}\left((1-\beta)\,\alpha\cos\lambda\right),$$

then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, where $\mathcal{G}(z)$ is defined in Equation (3).

Proof. Let we choose the function $q(z)$ by

$$q(z) = \frac{e^{-i\lambda}\mathcal{G}(z) - \beta\cos\lambda + i\sin\lambda}{(1-\beta)\cos\lambda}, \tag{6}$$

then Equation (6) shows that $q(z)$ is holomorphic in $\mathbb{E}$ and also normalized by $q(0) = 1$.

From Equation (6), we can easily obtain that

$$e^{-i\lambda}\mathcal{G}(z)\left(1+\alpha z\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right) = \Phi\left(q(z), zq'(z), z\right),$$

where

$$\Phi\left(q(z), zq'(z), z\right) = \left[(1-\beta)\,\alpha zq'(z) + (1-\beta)\,q(z) + \beta\right]\cos\lambda - i\sin\lambda.$$

Now for all $a, b \in \mathbb{R}$ satisfying $2y \leq -n\left(1+a^2\right)$, we have

$$\begin{aligned}\Re\left\{\Phi(ia,b,z)\right\} &\leq \beta\cos\lambda - \frac{n}{2}\left(1+a^2\right)(1-\beta)\,\alpha\cos\lambda \\ &\leq \beta\cos\lambda - \frac{n}{2}(1-\beta)\,\alpha\cos\lambda.\end{aligned}$$

Now, let us define a set as

$$H = \left\{\zeta : \Re(\zeta) > \beta\cos\lambda - \frac{n}{2}\left((1-\beta)\,\alpha\cos\lambda\right)\right\},$$

then, we see that $\Phi(ia,b,z) \notin H$ and $\Phi\left(q(z), zq'(z), z\right) \in H$. Therefore, by using Lemma 1, we obtain that $\Re(q(z)) > 0$.
□

Further, in the next theorem, we obtain the sufficient condition for the p-valent function f to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ by using Lemma 2.

Theorem 4. *If $f(z)$ is a member of the family Ω_p of meromorphic p-valent functions and satisfies*

$$\left|\frac{e^{i\lambda}}{\mathcal{G}(z)}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\right| < \frac{1}{\beta\cos\lambda} - 1, \tag{7}$$

then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, where $\mathcal{G}(z)$ is given by Equation (3).

Proof. In order to prove the required result, we need to define the following function

$$q(z)\cos\lambda = e^{-i\lambda}\mathcal{G}(z) + i\sin\lambda,$$

then, Equation (6) shows that th function $q(z)$ is holomorphic in $\mathbb{E}$ and also normalized by $q(0) = 1$.

Now, by routine computations, we get

$$\frac{zq'(z)}{q(z) - i\tan\lambda} = \frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}.$$

Now, let us consider $z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)'$ and then by using inequality (7), we have

$$\left|z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)'\right| = \left|\frac{e^{i\lambda}}{\mathcal{G}(z)}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\right| < \frac{1}{\beta\cos\lambda} - 1,$$

therefore

$$z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)' \prec \frac{(1-\beta\cos\lambda)z}{\beta\cos\lambda}.$$

Using Lemma 2, we have

$$\frac{1}{(q(z) - i\tan\lambda)\cos\lambda} \prec 1 + \frac{(1-\beta\cos\lambda)}{\beta\cos\lambda}z,$$

equivalently

$$(q(z) - i\tan\lambda)\cos\lambda \prec \frac{\beta\cos\lambda}{\beta\cos\lambda + (1-\beta\cos\lambda)z} = H(z)\,(say). \tag{8}$$

After simplifications, we get

$$1 + \Re\left(\frac{zH''(z)}{H'(z)}\right) = 2\beta\cos\lambda - 1 > 0, \ \ for\ \frac{1}{2} < \beta < 1.$$

The region $H(\mathbb{E})$ shows that it is symmetric about the real axis and also $H(z)$ is convex. Hence

$$\Re\ (\mathcal{G}(z)) \geq H(1) > 0,$$

or

$$\Re\ (q(z)\cos\lambda - i\sin\lambda) > \beta\cos\lambda,$$

or

$$\Re\ \left(e^{-i\lambda}\mathcal{G}(z)\right) > \beta\cos\lambda,\ for\ \frac{1}{2} < \beta < 1.$$

□

Finally, we investigate the sufficient condition for the p-valent function f to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ in the following theorem:

Theorem 5. *If $f(z) \in \Omega_p$ satisfies*

$$\left|\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right)'\right| \leq \eta\,|z|^{\gamma},\ for\ \ 0 < \eta \leq \gamma + 1, \tag{9}$$

then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, where $\mathcal{G}(z)$ is defined in Equation (3).

Proof. Let us put

$$G(z) = z\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right).$$

Then $G(0) = 0$ and $G(z)$ is analytic in $\mathbb{E}$. Using inequality (9), we can write

$$\left|\left(\frac{G(z)}{z}\right)'\right| = \left|\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right)'\right| \leq \eta\,|z|^{\gamma}.$$

Now,

$$\left|\left(\frac{G(z)}{z}\right)\right| = \left|\int_0^z \left(\frac{G(t)}{t}\right)' dt\right| \leq \int_0^{|z|} \left|\left(\frac{G(t)}{t}\right)'\right| dt \leq \int_0^{|z|} \eta\,|t|^{\gamma}\,dt = \frac{\eta\,|z|^{\gamma+1}}{\gamma+1} < 1,$$

and this implies that

$$\left|\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right| < 1.$$

Now by using Theorem 1, we get the result which we needed. □

3. Conclusions

In our results, a new subfamily of meromorphic p-valent (multivalent) functions were introduced. Further, various sufficient conditions for meromorphic p-valent functions belonging to these subfamilies were obtained and investigated.

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