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Slant Curves and Contact Magnetic Curves in Sasakian Lorentzian 3-Manifolds

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Abstract: In this article, we define Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold. Using a Lorentzian cross product, we prove that the ratio of κ and $\tau - 1$ is constant along a Frenet slant curve in a Sasakian Lorentzian three-manifold. Moreover, we prove that γ is a slant curve if and only if M is Sasakian for a contact magnetic curve γ in contact Lorentzian three-manifold M. As an example, we find contact magnetic curves in Lorentzian Heisenberg three-space.

Keywords: slant curves; Legendre curves; magnetic curves; Sasakian Lorentzian manifold

1. Introduction

As a generalization of Legendre curve, we defined the notion of slant curves in [1,2]. A curve in a contact three-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field. For a contact Riemannian manifold, we proved that a slant curve in a Sasakian three-manifold is that its ratio of κ and $\tau - 1$ is constant. Baikoussis and Blair proved that, on a three-dimensional Sasakian manifold, the torsion of the Legendre curve is $+1$ ([3]).

A *magnetic curve* represents a trajectory of a charged particle moving on the manifold under the action of a magnetic field in [4]. A *magnetic field* on a semi-Riemannian manifold (M, g) is a closed two-form F. The *Lorentz force* of the magnetic field F is a $(1, 1)$-type tensor field Φ given by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{1}$$

The magnetic trajectories of F are curves γ on M that satisfy the *Lorentz equation*

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'), \tag{2}$$

where ∇ is the Levi–Civita connection of g. The Lorentz equation generalizes the equation satisfied by the geodesics of M, namely $\nabla_{\gamma'}\gamma' = 0$. Since the Lorentz force Φ is skew-symmetric, we have

$$\frac{d}{dt}g(\gamma', \gamma') = 2g(\Phi(\gamma'), \gamma') = 0,$$

that is, magnetic curve have constant speed $\mid \gamma' \mid = v_0$. When the magnetic curve $\gamma(t)$ is arc-length parameterized, it is called a *normal magnetic curve*. Cabreizo et al. studied a contact magnetic field in three-dimensional Sasakian manifold ([5]).

In this article, we define the magnetic curve γ with contact magnetic field $F_{\xi,q}$ of the length q in three-dimensional Sasakian Lorentzian manifold M^3. We call it the *contact magnetic curve* or *trajectories* of $F_{\xi,q}$.

In Section 3, we define a Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold. Using the Lorentzian cross product, we prove that the ratio of κ and $\tau - 1$ is constant along a Frenet slant curve in a Sasakian Lorentzian three-manifold.

In Section 4, we prove that γ is a slant curve if and only if M is Sasakian for a contact magnetic curve γ in contact Lorentzian three-manifolds M. For example, we find contact magnetic curves in Lorentzian Heisenberg three-space.

2. Preliminaries

Contact Lorentzian Manifold

Let M be a $(2n+1)$-dimensional differentiable manifold. M has an almost contact structure (φ, ξ, η) if it admits a tensor field φ of $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1. \tag{3}$$

Suppose M has an almost contact structure (φ, ξ, η). Then, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, the endomorphism φ has rank $2n$.

If a $(2n+1)$-dimensional smooth manifold M with almost contact structure (φ, ξ, η) admits a compatible Lorentzian metric such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{4}$$

then we say M has an almost contact Lorentzian structure (η, ξ, φ, g). Setting $Y = \xi$, we have

$$\eta(X) = -g(X, \xi). \tag{5}$$

Next, if the compatible Lorentzian metric g satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \tag{6}$$

then η is a contact form on M, ξ is the associated Reeb vector field, g is an associated metric and $(M, \varphi, \xi, \eta, g)$ is called a *contact Lorentzian manifold*.

For a contact Lorentzian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}),$$

where X is a vector field tangent to M, t is the coordinate of $\mathbb{R}$ and f is a function on $M \times \mathbb{R}$. When the almost complex structure J is integrable, the contact Lorentzian manifold M is said to be *normal* or *Sasakian*. A contact Lorentzian manifold M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

Proposition 1 ([6,7])**.** *An almost contact Lorentzian manifold $(M^{2n+1}, \eta, \xi, \varphi, g)$ is Sasakian if and only if*

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X. \tag{7}$$

Using the similar arguments and computations in [8], we obtain

Proposition 2 ([6,7]). *Let $(M^{2n+1}, \eta, \xi, \varphi, g)$ be a contact Lorentzian manifold. Then,*

$$\nabla_X \xi = \varphi X - \varphi h X. \tag{8}$$

If ξ is a killing vector field with respect to the Lorentzian metric g. Then, we have

$$\nabla_X \xi = \varphi X. \tag{9}$$

3. Slant Curves in Contact Lorentzian Three-Manifolds

Let $\gamma : I \to M^3$ be a unit speed curve in Lorentzian three-manifolds M^3 such that γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant ε_1 is called the *causal character* of γ. A unit speed curve γ is said to be a spacelike or timelike if its causal character is 1 or -1, respectively.

A unit speed curve γ is said to be a *Frenet curve* if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve γ admits an orthonormal frame field $\{E_1 = \dot\gamma, E_2, E_3\}$ along γ. The constants ε_2 and ε_3 are defined by

$$g(E_i, E_i) = \varepsilon_i, \quad i = 2, 3$$

and called *second causal character* and *third causal character* of γ, respectively. Thus, $\varepsilon_1\varepsilon_2 = -\varepsilon_3$ is satisfied. Then, the *Frenet–Serret* equations are the following ([9,10]):

$$\begin{cases} \nabla_{\dot\gamma} E_1 = \quad \varepsilon_2 \kappa E_2, \\ \nabla_{\dot\gamma} E_2 = -\varepsilon_1 \kappa E_1 \quad - \varepsilon_3 \tau E_3, \\ \nabla_{\dot\gamma} E_3 = \quad \varepsilon_2 \tau E_2, \end{cases} \tag{10}$$

where $\kappa = |\nabla_{\dot\gamma}\dot\gamma|$ is the *geodesic curvature* of γ and τ its *geodesic torsion*. The vector fields E_1, E_2 and E_3 are called tangent vector field, principal normal vector field, and binormal vector field of γ, respectively.

A Frenet curve γ is a *geodesic* if and only if $\kappa = 0$. A Frenet curve γ with constant geodesic curvature and zero geodesic torsion is called a *pseudo-circle*. A *pseudo-helix* is a Frenet curve γ whose geodesic curvature and torsion are constant.

3.1. Lorentzian Cross Product

C. Camci ([11]) defined a cross product in three-dimensional almost contact Riemannian manifolds $(\widetilde{M}, \eta, \xi, \varphi, \widetilde{g})$ as following:

$$X \wedge Y = -\widetilde{g}(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y. \tag{11}$$

If we define the cross product $\wedge$ as Equation (11) in three-dimensional almost contact Lorentzian manifold $(M, \eta, \xi, \varphi, g)$, then

$$g(X \wedge Y, X) = 2\eta(X) g(X, \varphi Y) \neq 0.$$

In fact, we see already the cross product for a Lorentzian three-manifold as following:

Proposition 3. *Let $\{E_1, E_2, E_3\}$ be an orthonomal frame field in a Lorentzian three-manifold. Then,*

$$E_1 \wedge_L E_2 = \varepsilon_3 E_3, \quad E_2 \wedge_L E_3 = \varepsilon_1 E_1, \quad E_3 \wedge_L E_1 = \varepsilon_2 E_2. \tag{12}$$

Now, in three-dimensional almost contact Lorentzian manifold M^3, we define Lorentzian cross product as the following:

Definition 1. *Let $(M^3, \varphi, \xi, \eta, g)$ be a three-dimensional almost contact Lorentzian manifold. We define a Lorentzian cross product $\wedge_L$ by*

$$X \wedge_L Y = g(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y, \tag{13}$$

where $X, Y \in TM$.

The Lorentzian cross product $\wedge_L$ has the following properties:

Proposition 4. *Let $(M^3, \varphi, \xi, \eta, g)$ be a three-dimensional almost contact Lorentzian manifold. Then, for all $X, Y, Z \in TM$ the Lorentzian cross product has the following properties:*

(1) The Lorentzian cross product is bilinear and anti-symmetric.
(2) $X \wedge_L Y$ is perpendicular both of X and Y.
(3) $X \wedge_L \varphi Y = -g(X, Y)\xi - \eta(X)Y$.
(4) $\varphi X = \xi \wedge_L X$.
(5) Define a mixed product by $det(X, Y, Z) = g(X \wedge_L Y, Z)$ Then,

$$det(X, Y, Z) = -g(X, \varphi Y)\eta(Z) - g(Y, \varphi Z)\eta(X) - g(Z, \varphi X)\eta(Y)$$

and $det(X, Y, Z) = det(Y, Z, X) = det(Z, X, Y)$.
(6) $g(X, \varphi Y)Z + g(Y, \varphi Z)X + g(Z, \varphi X)Y = -(X, Y, Z)\xi$.

Proof. (We can prove by a similar way as in [11])
(1) and (2) are trivial.
(3) using Equations (3), (5) and (13),

$$\begin{aligned} X \wedge_L \varphi Y &= g(X, -Y + \eta(Y)\xi)\xi + \eta(X)(-Y + \eta(Y)\xi) \\ &= -g(X, Y)\xi - \eta(X)Y. \end{aligned}$$

(4) by Equation (13),

$$\xi \wedge_L X = g(\xi, \varphi X)\xi - \eta(X)\varphi\xi + \eta(\xi)\varphi X = \varphi X.$$

(5) from Equations (5) and (13),

$$\begin{aligned} g(X \wedge_L Y, Z) &= g(g(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y, Z) \\ &= -g(X, \varphi Y)\eta(Z) - g(Y, \varphi Z)\eta(X) - g(Z, \varphi X)\eta(Y). \end{aligned}$$

(6) is easily obtained by (5). □

From Equations (7) and (9), we have:

Proposition 5. *Let $(M^3, \varphi, \xi, \eta, g)$ be a three-dimensional Sasakian Lorentzian manifold. Then, we have*

$$\nabla_Z(X \wedge_L Y) = (\nabla_Z X) \wedge_L Y + X \wedge_L (\nabla_Z Y), \tag{14}$$

for all $X, Y, Z \in TM$.

Proof. From Equation (13), we get

$$\begin{aligned}\nabla_Z(X \wedge_L Y) &= \nabla_Z(-g(X,\varphi Y)\xi + \eta(Y)\varphi X - \eta(X)\varphi Y) \\ &= g(\nabla_Z X, \varphi Y)\xi + g(X, (\nabla_Z \varphi)Y)\xi + g(X, \varphi\nabla_Z Y)\xi + g(X,\varphi Y)\nabla_Z \xi \\ &\quad -\eta(\nabla_Z Y)\varphi X + g(Y, \nabla_Z \xi)\varphi X + \eta(Y)(\nabla_Z \varphi)X + \eta(Y)\varphi\nabla_Z X \\ &\quad +\eta(\nabla_Z X)\varphi Y - g(X, \nabla_Z \xi)\varphi Y - \eta(X)(\nabla_Z\varphi)Y - \eta(X)\varphi\nabla_Z Y \\ &= (\nabla_Z X) \wedge_L Y + X \wedge_L (\nabla_Z Y) + P(X,Y,Z),\end{aligned}$$

where

$$\begin{aligned}P(X,Y,Z) &= g(X,(\nabla_Z\varphi)Y)\xi + g(X,\varphi Y)\nabla_Z\xi + g(Y,\nabla_Z\xi)\varphi X - \eta(Y)(\nabla_Z\varphi)X \\ &\quad -g(X,\nabla_Z\xi)\varphi Y + \eta(X)(\nabla_Z\varphi)Y.\end{aligned}$$

Since M is a three-dimensional Sasakian Lorentzian manifold, it satisfies Equations (7) and (9). Hence, we have

$$P(X,Y,Z) = g(X,\varphi Y)\varphi Z + g(Y,\varphi Z)\varphi X + g(Z,\varphi X)\varphi Y.$$

Using Equation (6) of Proposition 4, we obtain $P(X,Y,Z) = 0$ and Equation (14). □

3.2. Frenet Slant Curves

In this subsection, we study a Frenet slant curve in contact Lorentzian three-manifolds.

A curve in a contact Lorentzian three-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field (i.e., $\eta(\gamma') = -g(\gamma',\xi)$ is a constant).

Since the Reeb vector field ξ is denoted by

$$\xi = \sum_{i=1}^{3} \varepsilon_i g(\xi, E_i)E_i = -\sum_{i=1}^{3} \varepsilon_i \eta(E_i)E_i,$$

using Equation (4) of Proposition 4 and Proposition 3, we have:

Proposition 6. *Let $(M^3, \varphi, \xi, \eta, g)$ be a three-dimensional almost contact Lorentzian manifold. Then, for a Frenet curve γ in M^3, we have*

$$\begin{aligned}\varphi E_1 &= \varepsilon_2\varepsilon_3(\eta(E_2)E_3 - \eta(E_3)E_2), \\ \varphi E_2 &= \varepsilon_3\varepsilon_1(\eta(E_3)E_1 - \eta(E_1)E_3), \\ \varphi E_3 &= \varepsilon_1\varepsilon_2(\eta(E_1)E_2 - \eta(E_2)E_1).\end{aligned}$$

By using Proposition 6, we find that differentiating $\eta(E_i)$ (for $i = 1,2,3$) along a Frenet curve γ

$$\begin{aligned}&\eta(E_1)' = \varepsilon_2\kappa\eta(E_2) + g(E_1, \varphi h E_1), \\ &\eta(E_2)' = -\varepsilon_1\kappa\eta(E_1) - \varepsilon_3(\tau - 1)\eta(E_3) + g(E_2, \varphi h E_1), \\ &\eta(E_3)' = \varepsilon_2(\tau - 1)\eta(E_2) + g(E_3, \varphi h E_1).\end{aligned}$$

Now, we assume that M^3 is a Sasakian Lorentzian manifold; then,

$$\eta(E_1)' = \varepsilon_2 \kappa \eta(E_2), \tag{15}$$
$$\eta(E_2)' = -\varepsilon_1 \kappa \eta(E_1) - \varepsilon_3(\tau - 1)\eta(E_3), \tag{16}$$
$$\eta(E_3)' = \varepsilon_2(\tau - 1)\eta(E_2). \tag{17}$$

From Equation (15), if γ is a geodesic curve, that is $\kappa = 0$, in a Sasakian Lorentzian three-manifold M^3, then γ is naturally a slant curve. Now, let us consider a non-geodesic curve γ; then, we have:

Proposition 7. *A non-geodesic Frenet curve γ in a Sasakian Lorentzian three-manifold M^3 is slant curve if and only if $\eta(E_2) = 0$.*

From Equations (15) and (17) and Proposition 7, we get that $\eta(E_1)$ and $\eta(E_3)$ are constants. Hence, using Equation (16), we obtain:

Theorem 1. *The ratio of κ and $\tau - 1$ is a constant along a non-geodesic Frenet slant curve in a Sasakian Lorentzian three-manifold M^3.*

Next, let us consider a Legendre curve γ as a spacelike curve with spacelike normal vector. For a Legendre curve γ, $\eta(\gamma') = \eta(E_1) = 0$, $\eta(E_2) = 0$ and $\eta(E_3)$ is a constant. Hence, using Equation (16), we have:

Corollary 1. *Let M be a three-dimensional Sasakian Lorentzian manifold $(M^3, \eta, \xi, \varphi, g)$. Then, the torsion of a Legendre curve is 1.*

From this, we see that the ratio of κ and $\tau - 1$ is a constant along non-geodesic Frenet slant curve containing Legendre curve.

3.3. Null Slant Curves

In this section, let us consider a null curve γ that has a null tangent vector field $g(\gamma', \gamma') = 0$ and γ is not a geodesic (i.e., $g(\nabla_{\gamma'}\gamma', \nabla_{\gamma'}\gamma') \neq 0$). We take a parameterization of γ such that $g(\nabla_{\gamma'}\gamma', \nabla_{\gamma'}\gamma') = 1$. Then, Duggal, K.L. and Jin, D.H ([12]) proved that there exists only one Cartan frame $\{T, N, W\}$ and the function τ along γ whose Cartan equations are

$$\nabla_T T = N, \quad \nabla_T W = \tau N, \quad \nabla_T N = -\tau T - W,$$

where

$$T = \gamma', \quad N = \nabla_T T, \quad \tau = \frac{1}{2} g(\nabla_T N, \nabla_T N), \quad W = -\nabla_T N - \tau T. \tag{18}$$

Hence,

$$g(T, W) = g(N, N) = 1, \quad g(T, T) = g(T, N) = g(W, W) = g(W, N) = 0.$$

For a null Legendre curve γ, we easily prove that γ is geodesic. Hence, we suppose that γ is non-geodesic; then, we have:

Theorem 2. *Let γ be a non-geodesic null slant curve in a Sasakian Lorentzian three-manifold. We assume that $\kappa = 1$, then we have*

$$N = \pm\frac{1}{a}\varphi\gamma', \quad \tau = \frac{1}{2a^2} \mp 1, \quad W = \frac{1}{2a^2}\gamma' - \frac{1}{a}\xi, \tag{19}$$

where $a = \eta(\gamma')$ is non-zero constant.

Proof. Let $\varphi T = lT + mN + nW$ for some l, m, n. We find $l = g(\varphi T, T) = 0$, then $\varphi T = mN + nW$. From this, we get

$$g(\varphi T, \varphi T) = m^2 = a^2 \quad and \quad 0 = g(\varphi T, \xi) = n(a\tau + m).$$

Hence, $m = \pm a$ and $n = 0$ or $m = -a\tau$.

If $n = 0$, then $N = \frac{1}{m}\varphi T = \pm\frac{1}{a}\varphi T$. Using the Cartan equation, we find that $\tau = \frac{1}{2a^2} \mp 1$ and $W = \frac{1}{2a^2}\gamma' - \frac{1}{a}\xi$.

Next, if $n \neq 0$ and $m = -a\tau$ then since γ is a slant curve, differentiating $g(\varphi T, N) = m = \pm a$, we have $n = g(\varphi T, W) = 0$, which gives a contradiction. □

From the second equation of Equation (19), we have:

Remark 1. *Let γ be a non-geodesic null slant curve in a Sasakian Lorentzian three-manifold. We assume that $\kappa = 1$ then τ is constant such that $\tau = \frac{1}{2a^2} \mp 1$.*

4. Contact Magnetic Curves

In a three-dimensional Sasakian Lorentzian manifold M^3, the Reeb vector field ξ is Killing. By Equation (6), the 2-form Φ is $d\eta$, that is $d\eta(X, Y) = g(X, \varphi Y)$, for all $X, Y \in \Gamma(TM)$.

Let $\gamma : I \to M$ be a smooth curve on a contact Lorentzian manifold $(M, \varphi, \xi, \eta, g)$. Then, we define a magnetic field on M by

$$F_{\xi,q}(X, Y) = -qd\eta(X, Y),$$

where $X, Y \in \mathbb{X}(M)$ and q is a non-zero constant. We call $F_{\xi,q}$ the *contact magnetic field* with strength q.

Using Equations (1), (4) and (6) we get $\Phi(X) = q\varphi X$. Hence, from Equation (2) the Lorentz equation is

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'. \tag{20}$$

This is the generalized equation of geodesics under arc length parameterization, that is $\nabla_{\gamma'}\gamma' = 0$. For $q = 0$, we find that the contact magnetic field vanishes identically and the magnetic curves are geodesics of M. The solutions of Equation (20) are called *contact magnetic curve* or *trajectories* of $F_{\xi,q}$.

By using Equations (8) and (20), differentiating $g(\xi, \gamma')$ along a contact magnetic curve γ in contact Lorentzian three-manifold

$$\begin{aligned}\frac{d}{dt}g(\xi, \gamma') &= g(\nabla_{\gamma'}\xi, \gamma') + g(\xi, \nabla_{\gamma'}\gamma') \\ &= g(\varphi\gamma' - \varphi h\gamma', \gamma') + g(\xi, q\varphi\gamma') \\ &= -g(\varphi h\gamma', \gamma').\end{aligned}$$

Hence, we have:

Theorem 3. *Let γ be a contact magnetic curve in a contact Lorentzian three-manifold M. γ is a slant curve if and only if M is Sasakian.*

Next, we find the curvature κ and torsion τ along non-geodesic Frenet contact magnetic curves γ. We suppose that $\eta(E_1) = a$, for a constant a. Then, using Equations (4), (10) and (20), we get

$$\varepsilon_2\kappa^2 = q^2 g(\varphi\gamma', \varphi\gamma') = q^2(\varepsilon_1 + a^2).$$

Hence, we find that γ has a constant curvature

$$\kappa = \mid q \mid \sqrt{\varepsilon_2(\varepsilon_1 + a^2)}, \tag{21}$$

and, from Equations (10), (20) and (21), the binormal vector field

$$E_2 = \frac{q}{\varepsilon_2\kappa}\varphi\gamma' = \frac{\delta\varepsilon_2}{\sqrt{\varepsilon_2(\varepsilon_1 + a^2)}}\varphi\gamma', \tag{22}$$

where $\delta = q / \mid q \mid$.

Using Proposition 3 and Equation (22), the binormal E_3 is computed as

$$\begin{aligned} \varepsilon_3 E_3 &= E_1 \wedge_L E_2 \\ &= \gamma' \wedge_L (\frac{\delta\varepsilon_2}{\sqrt{\varepsilon_2(\varepsilon_1 + a^2)}}\varphi\gamma') \\ &= -\frac{\delta\varepsilon_2}{\sqrt{\varepsilon_2(\varepsilon_1 + a^2)}}(\varepsilon_1\xi + a\gamma'). \end{aligned}$$

Differentiating binormal vector field E_3, we have

$$\begin{aligned} \nabla_{\gamma'} E_3 &= -\frac{\delta\varepsilon_2\varepsilon_3}{\sqrt{\varepsilon_2(\varepsilon_1 + a^2)}}\nabla_{\gamma'}(\varepsilon_1\xi + a\gamma') \\ &= -\frac{\delta\varepsilon_2\varepsilon_3}{\sqrt{\varepsilon_2(\varepsilon_1 + a^2)}}(\varepsilon_1 + qa)\varphi\gamma'. \end{aligned} \tag{23}$$

On the other hand, by Equation (10), we have

$$\nabla_{\gamma'} E_3 = \varepsilon_2\tau E_2 = \tau\frac{\delta\varphi\gamma'}{\sqrt{\varepsilon_2(\varepsilon_1 + a^2)}}. \tag{24}$$

From Equations (23) and (24), since $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, we obtain

$$\tau = 1 + \varepsilon_1 qa. \tag{25}$$

Moreover, if γ is a non-geodesic curve, then

$$\frac{\tau - 1}{\kappa} = \frac{\delta\varepsilon_1 a}{\sqrt{\varepsilon_2(\varepsilon_1 + a^2)}}.$$

Therefore, we obtain:

Theorem 4. *Let γ be a non-geodesic Frenet curve in a Sasakian Lorentzian three-manifold M. If γ is a contact magnetic curve, then it is slant pseudo-helix with curvature $\kappa = \mid q \mid \sqrt{\varepsilon_2(\varepsilon_1 + a^2)}$ and torsion $\tau = 1 + \varepsilon_1 qa$. Moreover, the ratio of κ and $\tau - 1$ is a constant.*

Since a Legendre curve is a spacelike curve with spacelike normal vector field and $\eta(\gamma') = a = 0$, we assume that γ is a Legendre curve and we have:

Corollary 2. *Let γ be a non-geodesic Legendre curve in a Sasakian Lorentzian three-manifold M. If γ is a contact magnetic curve, then it is Legendre pseudo-helix with curvature $\kappa = |q|$ and torsion $\tau = 1$.*

Now, from the geodesic curvature in Equation (21), if $\varepsilon_1 = 1$, then $\eta(\gamma') = a$ and $1 \leq 1 + a^2$, and we have $\varepsilon_2 = 1$. Moreover, using $\varepsilon_3 = -\varepsilon_1 \cdot \varepsilon_2$, we obtain $\varepsilon_3 = -1$. Next, if $\varepsilon_1 = -1$, then $\eta(\gamma') = a = \cosh \alpha_0$. Since γ is a geodesic for $a = \cosh \alpha_0 = 1$, we assume that γ is non-geodesic, and we get $a^2 > 1$. Hence, $-1 + a^2 > 0$ and we get $\varepsilon_2 = \varepsilon_3 = 1$. Therefore, we obtain:

Theorem 5. *Let γ be a non-geodesic Frenet curve in a Sasakian Lorentzian three-manifold M. If γ is a contact magnetic curve. then γ is one of the following:*

(i) *a spacelike curve with spacelike normal vector field; or*
(ii) *a timelike curve.*

Moreover, we have:

Corollary 3. *Let γ be a non-geodesic Frenet curve in a Sasakian Lorentzian three-manifold M. If γ is a contact magnetic curve, then there does not exist a spacelike curve with timelike normal vector field.*

In a similar with a Frenet curve, we study null contact magnetic curves in a Sasakian Lorentzian three-manifold M. Hence, we find that there exist a null contact magnetic curve with $q = \pm a$ and same the result with Theorem 2.

Example

The Heisenberg group $\mathbb{H}_3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined by

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} + \frac{x\overline{y}}{2} - \frac{\overline{x}y}{2}).$$

The mapping

$$\mathbb{H}_3 \to \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} : (x, y, z) \mapsto \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism between $\mathbb{H}_3$ and a subgroup of $GL(3, \mathbb{R})$.

Now, we take the contact form

$$\eta = dz + (ydx - xdy).$$

Then, the characteristic vector field of η is $\xi = \frac{\partial}{\partial z}$.

Now, we equip the Lorentzian metric as following:

$$g = dx^2 + dy^2 - (dz + (ydx - xdy))^2 .$$

We take a left-invariant Lorentzian orthonormal frame field (e_1, e_2, e_3) on $(\mathbb{H}_3, g)$:

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z},$$

and the commutative relations are derived as follows:

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = [e_3, e_1] = 0.$$

Then, the endomorphism field φ is defined by

$$\varphi e_1 = e_2,\ \varphi e_2 = -e_1,\ \varphi e_3 = 0.$$

The Levi–Civita connection ∇ of $(\mathbb{H}_3, g)$ is described as

$$\begin{aligned} \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0,\ \ \nabla_{e_1} e_2 = e_3 = -\nabla_{e_2} e_1, \\ \nabla_{e_2} e_3 = -e_1 = \nabla_{e_3} e_2,\ \ \nabla_{e_3} e_1 = e_2 = \nabla_{e_1} e_3. \end{aligned} \tag{26}$$

The contact form η satisfies $d\eta(X, Y) = g(X, \varphi Y)$. Moreover, the structure (η, ξ, φ, g) is Sasakian. The Riemannian curvature tensor R of $(\mathbb{H}_3, g)$ is given by

$$\begin{aligned} R(e_1, e_2)e_1 &= 3e_2, & R(e_1, e_2)e_2 &= -3e_1, \\ R(e_2, e_3)e_2 &= -e_3, & R(e_2, e_3)e_3 &= -e_2, \\ R(e_3, e_1)e_3 &= e_1, & R(e_3, e_1)e_1 &= e_3, \end{aligned}$$

and the other components are zero.

The sectional curvature is given by [6]

$$K(\xi, e_i) = -R(\xi, e_i, \xi, e_i) = -1,\ \ for\ i = 1, 2,$$

and

$$K(e_1, e_2) = R(e_1, e_2, e_1, e_2) = 3.$$

Thus, we see that the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is the Lorentzian Sasakian space forms with constant holomorphic sectional curvature $\mu = 3$.

Let γ be a Frenet slant curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ parameterized by arc-length. Then, the tangent vector field has the form

$$T = \gamma' = \sqrt{\varepsilon_1 + a^2}\cos\beta e_1 + \sqrt{\varepsilon_1 + a^2}\sin\beta e_2 + a e_3, \tag{27}$$

where $a = constant,\ \ \beta = \beta(s)$. Using Equation (26), we get

$$\nabla_{\gamma'}\gamma' = \sqrt{\varepsilon_1 + a^2}(\beta' + 2a)(-\sin\beta e_1 + \cos\beta e_2). \tag{28}$$

Since γ is a non-geodesic, we may assume that $\kappa = \sqrt{\varepsilon_1 + a^2}(\beta' + 2a) > 0$ without loss of generality. Then, the normal vector field

$$N = -\sin\beta e_1 + \cos\beta e_2.$$

The binormal vector field $\varepsilon_3 B = T \wedge_L N = -a\cos\beta e_1 - a\sin\beta e_2 - \sqrt{\varepsilon_1 + a^2}e_3$. From Theorem 5, we see that $\varepsilon_2 = 1$, thus we have $\varepsilon_3 = -\varepsilon_1$. Hence,

$$B = \varepsilon_1(a\cos\beta e_1 + a\sin\beta e_2 + \sqrt{\varepsilon_1 + a^2}e_3).$$

Using the Frenet–Serret Equation (10), we have

Lemma 1. *Let γ be a Frenet slant curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ parameterized by arc-length. Then, γ admits an orthonormal frame field $\{T, N, B\}$ along γ and*

$$\begin{aligned} \kappa &= \sqrt{\varepsilon_1 + a^2}(\beta' + 2a), \\ \tau &= 1 + \varepsilon_1 a(\beta' + 2a). \end{aligned} \tag{29}$$

Next, if γ is a null slant curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$, then the tangent vector field has the form

$$T = \gamma' = a\cos\beta e_1 + a\sin\beta e_2 + ae_3, \tag{30}$$

where $a = constant,\ \beta = \beta(s)$. Using Equation (26), we get

$$\nabla_{\gamma'}\gamma' = a(\beta' + 2a)(-\sin\beta e_1 + \cos\beta e_2). \tag{31}$$

Since γ is non-geodesic, using Equation (18) we have $\mid a(\beta' + 2a) \mid = 1$ and

$$N = -\sin\beta e_1 + \cos\beta e_2.$$

Differentiating N, we get

$$\nabla_{\gamma'}N = -(\beta' + a)\cos\beta e_1 - (\beta' + a)\sin\beta e_2 + ae_3.$$

From Equation (18), $\tau = \frac{1}{2}g(\nabla_{\gamma'}N, \nabla_{\gamma'}N) = \frac{1}{2}(\beta')^2 + a\beta'$. Since $W = -\nabla_{\gamma'}N - \tau T$, we have

$$W = \{-\frac{1}{2}(\beta')^2 + (\frac{1}{a} - a)\beta' + 1\}T - (\beta' + 2a)\xi = \frac{1}{2a}(\cos\beta e_1 + \sin\beta e_2 - e_3).$$

Therefore, we have

Lemma 2. *Let γ be a non-geodesic null slant curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. We assume that $\kappa = \mid a(\beta' + 2a) \mid = 1$. Then, its torsion is constant such that $\tau = \frac{1}{2a^2} \mp 1$.*

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then, the tangent vector field γ' of γ is

$$\gamma' = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

Using the relations:

$$\frac{\partial}{\partial x} = e_1 + ye_3,\ \frac{\partial}{\partial y} = e_2 - xe_3,\ \frac{\partial}{\partial z} = e_3,$$

if γ is a slant curve in $(\mathbb{H}_3, g)$, then from Equation (27) the system of differential equations for γ is given by

$$\frac{dx}{ds}(s) = \sqrt{\varepsilon_1 + a^2}\cos\beta(s), \tag{32}$$

$$\frac{dy}{ds}(s) = \sqrt{\varepsilon_1 + a^2}\sin\beta(s), \tag{33}$$

$$\frac{dz}{ds}(s) = a + \sqrt{\varepsilon_1 + a^2}(x(s)\sin\beta(s) - y(s)\cos\beta(s)).$$

Now, we construct a magnetic curve γ (containing Frenet and null curve) in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. From Equations (20) and (28), we have:

Proposition 8. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a magnetic curve parameterized by arc-length in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then,*

$$\beta' = q - 2a, \quad for\ a = \eta(\gamma').$$

Namely, β' is a constant, e.g., A, hence $\beta(s) = As + b$, $b \in \mathbb{R}$. If γ is a null curve, then $q = \pm\frac{1}{a}$. Finally, from Equations (32) and (33), we have the following result:

Theorem 6. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a non-geodesic curve parameterized by arc-length s in the Lorentzian Heisenberg group $(\mathbb{H}_3, g)$. If γ is a contact magnetic curve, then the parametric equations of γ are given by*

$$\begin{cases} x(s) = \frac{1}{A}\sqrt{\varepsilon_1 + a^2}\sin(As + b) + x_0, \\ y(s) = -\frac{1}{A}\sqrt{\varepsilon_1 + a^2}\cos(As + b) + y_0, \\ z(s) = \{a + \frac{\varepsilon_1 + a^2}{A}\}s - \frac{\sqrt{\varepsilon_1 + a^2}}{A}\{x_0\cos(As + b) + y_0\sin(As + b)\} + z_0, \end{cases}$$

where b, x_0, y_0, z_0 are constants. If $\varepsilon_1 = 0$ then γ is a null curve.

In particular, for a Frenet Legendre curve γ, we get $\beta' = q = A$.

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