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Multi-Attribute Decision Making Based on Intuitionistic Fuzzy Power Maclaurin Symmetric Mean Operators in the Framework of Dempster-Shafer Theory

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Abstract: It is well known that there are some unfavorable shortcomings in the ordinary operational rules (OORs) of intuitionistic fuzzy number (IFN), and there exists a close and forceful connection between the intuitionistic fuzzy set (IFS) and Dempster-Shafer Theory (DST). We can utilize this relationship to present a transparent and fruitful semantic framework for IFS in terms of DST. In the framework of DST, an IFN can be converted into a basic probability assignment (BPA) and operations on IFNs can be represented as operations on a belief interval (BI), which can break away from the revealed shortcomings of the OORs of the IFN. Although there are many operators to aggregate the IFN, the operator to aggregate the BPA is rare. The Maclaurin symmetric mean (MSM) operator has the advantage of considering interrelationships among any number of attributes. The power average (PA) operator can reduce the influences of extreme evaluation values. In addition, for measuring the difference between IFNs, we replace the Hamming distance and Euclidean distance with the Jousselme distance (JD). In this paper, we develop an intuitionistic fuzzy power MSM ($\mathrm{IFPMSM}_{\mathrm{DST}}$) operator and an intuitionistic fuzzy weighted power MSM ($\mathrm{IFPWMSM}_{\mathrm{DST}}$) operator in the framework of the DST and provide their favorable properties. Then, we propose a novel method based on the proposed operators to solve multi-attribute decision-making (MADM) problems without intermediate defuzzification when their attributes and weights are both IFNs. Finally, some examples are utilized to demonstrate that the proposed methods outperform the previous ones.

Keywords: MADM; intuitionistic fuzzy number; Dempster-Shafer theory; power average operator; Maclaurin symmetric mean operator; Jousselme distance

1. Introduction

As the complexity of decision-making (DM) problems increases, it is generally complicated to express attribute values of alternatives by real numbers. Zadeh [1] initially introduced the fuzzy set (FS) theory, which is a successful tool in processing inaccurate and ambiguous information. However, it is obviously inadequate for processing the information with non-membership. As a generalized form of FS, the intuitionistic fuzzy set (IFS) developed by Atanassov [2] has a membership degree (MD), a non-membership degree (NMD), and a hesitancy degree (HD), which is free of the limitations of FS. Recently, studies on the DM methods based on the IFS have attracted substantial attention in various areas, such as supplier selection [3], "one belt, one road" investment selection [4], brand management [5], mine emergency DM [6], hospital performance evaluation [7], etc. Lately, a large number of contributions have focused on DM techniques based on IFSs, which are from three domains:

(1) The theory of foundations, for instance, operational rules (ORs) [8–10], distance and similarity measures [11], likelihood [12], consensus degree [13], accurate function [14], etc.
(2) The extended multi-attribute decision-making (MADM) techniques for IFS, such as TOPSIS [15], ELECTRE [16], VIKOR [17], TODIM [18], Entropy [19], and other techniques, such as choquet integral [20], multi-objective linear programming [21], etc.
(3) The MADM techniques based on the aggregation operators (OAs) of IFSs, such as arithmetic AOs [22], geometric AOs [23], the power average (PA) operator [24,25], Bonferroni mean (BM) operator [26–28], Heronian mean (HM) operator [29], Maclaurin symmetric mean (MSM) operator [30,31], etc.

Generally, the MADM techniques based on OAs are superior to than the traditional MADM techniques because they can obtain the comprehensive values of alternatives by aggregating all attribute values and then rank the alternatives. Thus, it is significant and valuable to research the OAs and then deal with the MADM problems.

The PA operator, initially investigated by Yager [24], can reduce the influences of the extreme evaluation values to a great extent by given different weights. When extreme evaluation values appear, the PA operator can supply them a smaller weight by allowing attribute values to support and complement each other, so that the influences of extreme evaluation values are reduced greatly. This favorable characteristic is extremely useful in real MADM problems, so the PA operator has attracted considerable attention from researchers. The PA operator has also been successfully extended to IFS. Xu [32] developed an intuitionistic fuzzy weighted PA (IFWPA) operator and intuitionistic fuzzy weighted geometric PA (IFWGPA) operator. He et al. [33] investigated a generalized interaction IFPA (GIIFPA) operator and weighted GIIFPA (WGIIFPA) operator. Jiang and Wei [11] presented an intuitionistic fuzzy evidential power average (IFEPA) operator in the framework of the Dempster-Shafer theory (DST).

In some MADM problems, for obtaining convincing aggregate results, we should be concerned about the interrelationships between attributes. In this case, the BM operator [26] and HM operator [29] were developed to solve these MADM problems. Then, Xu and Yager [27] further extended the BM operator to IFS and presented an intuitionistic fuzzy BM (IFBM) operator and weighted IFBM (WIFBM) operator. He and He [28] proposed the extended intuitionistic fuzzy interaction BM (EIFIBM) operator and weighted EIFIBM (WEIFIBM) operator. Liu and Chen [29] presented the intuitionistic fuzzy Archimedean HM (IFAHM) operator and weighted IFAHM (WIFAHM) operator. However, the BM operator and HM operator just take into account the interrelationships between attributes. In most MADM problems, we should fully incorporate the interrelationships among attributes. Obviously, the MSM operator can help us to solve these problems. The great advantage of the MSM operator is that it can neatly capture interrelationships among attributes by assigning a different value to the parameter κ. Recently, many researchers have payed considerable attention to extending the MSM operator to IFS. Qin and Liu [31] first investigated the intuitionistic fuzzy MSM (IFMSM) operator and the weighted IFMSM (WIFMSM) operator. Liu and Liu [8] developed an intuitionistic fuzzy interaction MSM (IFIMSM) operator and the weighted IFIMSM (WIFIMSM) operator.

Undoubtedly, we usually encounter some extraordinary MADM problems where the attributes are interrelated with each other, and some extreme evaluation values are provided. However, the OAs for the interrelationships among attributes, and reducing the influence of extreme evaluation values, are rare, so it is essential to combine the MSM operator with the PA operator in processing IFNs. However, with respect to the OAs for IFNs, we should not only consider the function, but also pay close attention to the ORs of the IFNs.

From [9,10], we know that the ordinary ORs (OORs) of IFNs possibly produce the unreasonable and counterintuitive aggregate results in the process of solving MADM problems because of some unfavorable properties (see Section 2.1; we have discerned and presented some credible critical numerical examples). For instance, let $\beta = \langle u, v \rangle$, $\beta_1 = \langle u_1, v_1 \rangle$ and $\beta_2 = \langle u_2, v_2 \rangle$ be three IFNs, when $\beta_1 < \beta_2$, we cannot always generate $(\beta_1 \oplus \beta_3) < (\beta_2 \oplus \beta_3)$, $(\beta_1 \otimes \beta_3) < (\beta_2 \otimes \beta_3)$ and $\delta\beta_1 < \delta\beta_2 \ \ (\delta > 0)$.

In addition, it is noteworthy that the existing OAs of IFNs with attribute weights represented by IFNs have not be defined, and this is also a greater disadvantage of IFS. Because in many real DM problems, it is troublesome for decision-makers to provide the exact significance of attributes by using real number, we can apply the weights denoted by IFNs to decrease the loss of information.

DST, which was initially introduced by Dempster [34] and then developed and expanded by Shafer [35], is a favorable tool for processing inaccurate or ambiguous information. DST gives BPA, which can express the occurrence rate of attributes in basic events. While DST is introduced, the belief function (BF) and plausibility function (PF) are likewise defined, and this constitutes the BI of the focal element (FE). The BI expresses the belief and uncertainty of the FE. In [9,10], we have seen that there is a strong and close connection between IFS and DST. This strong and close connection makes it certainly possible to immediately apply the aggregation rules (ARs) of DST to aggregate the attribute information of the alternatives represented by IFNs in a real MADM process. In the framework of DST, an IFN can be converted into a BPA, and operations on IFNs can be denoted as operations on the belief interval (BI), which can overcome the revealed drawbacks of the OORs of IFNs and get more convincing results. Although there are many operators to aggregate IFNs, the operators to aggregate the BPA is rare.

Based on the above discussions, the main purposes of this paper are as follows:

(1) For overcoming the revealed drawbacks of OORs of IFN and getting more convincing aggregate results, we convert an IFN into a BI and replace operations on IFNs with operations on BI;
(2) For utilizing ORs of BI to develop some PMSM operator for IFNs in the framework of DST, we convert an IFN into a BPA and replace Hamming distance and Euclidean distance with Jousselme distance (JD);
(3) For further reducing the loss of information, we use the presented operators to solve MADM problems without intermediate defuzzification when attributes and their weights are all IFNs.

Therefore, we will firstly propose the intuitionistic fuzzy power MSM ($\text{IFPMSM}_{\text{DST}}$) operator and intuitionistic fuzzy weighted power MSM ($\text{IFPWMSM}_{\text{DST}}$) operator in the framework of DST; then, based on the $\text{IFPMSM}_{\text{DST}}$ operator and $\text{IFPWMSBM}_{\text{DST}}$ operator, we develop a new MADM method. By comparing with the previous methods based on a intuitionistic fuzzy evidential power aggregation (IFEPA) operator [11], a weighted intuitionistic fuzzy MSM (WIFMSM) operator [28], and an extended weighted intuitionistic fuzzy interaction Bonferroni mean (EWIFIBM) operator [31], the advantages of the proposed methods are discussed.

The rest of this paper is organized as follows: In Section 2, we review concepts of the IFS and provide a critical analysis of OORs of the IFS, then give an interpretation of the IFS in the framework of DST. In Section 3, we propose $\text{IFPMSM}_{\text{DST}}$ operator and $\text{IFPWMSM}_{\text{DST}}$ operator based the PA operator and MSM operator. In Section 4, we develop a novel method with IFNs based on $\text{IFPMSM}_{\text{DST}}$ operator and $\text{IFPWMSM}_{\text{DST}}$ operator in the framework of DST. In Section 5, we utilize some numerical examples to demonstrate the reasonability and flexibility of presented operators. In Section 6, we discuss the conclusions.

2. Preliminaries

2.1. IFSs

Definition 1 [2]**.** *Let $A = \{\alpha_i | i = 1, 2, \cdots, t\}$ be a fixed set, and then the IFS B on A can be defined as follows: $B = \left\{\left\langle \alpha, u_\beta(\alpha), v_\beta(\alpha)\right\rangle\right\}$, where $u_\beta(\alpha) : A \to [0, 1]$ and $v_\beta(\alpha) : A \to [0, 1]$ are the MD and NMD of $\alpha \in A$ to B, respectively, and $0 \le u_\beta(\alpha) + v_\beta(\alpha) \le 1$. Moreover, $\pi_\beta(\alpha) = 1 - u_\beta(\alpha) - v_\beta(\alpha)$ denotes the HD of α to B.*

Usually, we use $\beta = \left\langle u_\beta, v_\beta \right\rangle$ to represent an IFN.

Definition 2 [22,23]**.** *Let* $\beta_1 = \langle u_1, v_1\rangle$ *and* $\beta_2 = \langle u_2, v_2\rangle$ *be two IFNs, then*

$$(1)\ \beta_1 \oplus \beta_2 = \langle u_1 + u_2 - u_1u_2, v_1v_2\rangle \tag{1}$$

$$(2)\ \beta_1 \otimes \beta_2 = \langle u_1u_2, v_1 + v_2 - v_1v_2\rangle \tag{2}$$

$$(3)\ \delta\beta_1 = \left\langle 1 - (1 - u_1)^{\delta}, {v_1}^{\delta}\right\rangle,\ \delta > 0 \tag{3}$$

$$(4)\ {\beta_1}^{\delta} = \left\langle {u_1}^{\delta}, 1 - (1 - v_1)^{\delta}\right\rangle,\ \delta > 0 \tag{4}$$

From the above operational Rules (1)–(4), we can get intuitionistic fuzzy weighted arithmetic mean (IFWAM) and the intuitionistic fuzzy weighted geometric mean (IFWGM). Let $\beta_i = \langle u_i, v_i\rangle$ be a group of IFNs and ω_i be the weight of β_i, $\sum_{i}^{t} \omega_i = 1$. Then

$$IFWAM(\beta_1, \beta_2, \cdots, \beta_t) = \oplus_{i=1}^{t}\omega_i\beta_i = \left\langle 1 - \prod_{i=1}^{t}(1 - u_i)^{\omega_i}, \prod_{i=1}^{t} {v_i}^{\omega_i}\right\rangle \tag{5}$$

$$IFWGM(\beta_1, \beta_2, \cdots, \beta_t) = \oplus_{i=1}^{t}{\beta_i}^{\omega_i} = \left\langle \prod_{i=1}^{t} {u_i}^{\omega_i}, 1 - \prod_{i=1}^{t}(1 - v_i)^{\omega_i}\right\rangle. \tag{6}$$

For IFN $\beta = \langle u, v\rangle$, the score function (SF) and accuracy function (AF) are defined by following form:

$$SF(\beta) = u - v \tag{7}$$

$$AF(\beta) = u + v. \tag{8}$$

Further, for IFNs $\beta_1 = \langle u_1, v_1\rangle$ and $\beta_2 = \langle u_2, v_2\rangle$, we can give the order relation between β_1 and β_2 as follows:

(1) If $SF(\beta_1) < SF(\beta_2)$, then $\beta_1 < \beta_2$;
(2) If $SF(\beta_1) = SF(\beta_2)$, then

   (i) $AF(h_1) < AF(h_2)$, then $\beta_1 < \beta_2$;
   (ii) $AF(\beta_1) = AF(\beta_2)$, then $\beta_1 = \beta_2$.

Let $\beta = \langle u, v\rangle$, $\beta_1 = \langle u_1, v_1\rangle$ and $\beta_2 = \langle u_2, v_2\rangle$ be three IFNs; then, it is straightforward to prove that Equations (1)–(4) have following properties:

$$(1)\ \beta_1 \oplus \beta_2 = \beta_2 \oplus \beta_1; \tag{9}$$

$$(2)\ \beta_1 \otimes \beta_2 = \beta_2 \otimes \beta_1; \tag{10}$$

$$(3)\ \delta(\beta_1 \oplus \beta_2) = \delta\beta_2 \oplus \delta\beta_1, \quad \delta > 0; \tag{11}$$

$$(4)(\delta + \xi)\beta = \delta\beta + \xi\beta, \quad \xi, \delta > 0; \tag{12}$$

$$(5)\ (\beta_1 \otimes \beta_2)^{\delta} = {\beta_1}^{\delta} \otimes {\beta_2}^{\delta}, \quad \delta > 0; \tag{13}$$

$$(6)\ \beta^{\xi} \otimes \beta^{\delta} = \beta^{\delta+\xi} \quad \delta, \xi > 0 \tag{14}$$

In [9,10], we have shown that Equations (1)–(8) possibly generate unreasonable and counterintuitive computation results due to some unfavorable properties in the practice DM environment.

(1) The Equation (1) is not a constant operation—that is to say, $\beta_1 < \beta_2$ cannot always generate $(\beta_1 \oplus \beta_3) < (\beta_2 \oplus \beta_3)$.

Example 1. *Let $\beta_1 = \langle 0.65, 0.35\rangle$, $\beta_2 = \langle 0.55, 0.15\rangle$, $\beta_3 = \langle 0.25, 0.25\rangle$, then $SF(\beta_1) = 0.3$, $SF(\beta_2) = 0.4$, we can get $\beta_1 < \beta_2$. On the other hand, $\beta_1 \oplus \beta_3 = \langle 0.8375, 0.0875\rangle$, $\beta_2 \oplus \beta_3 = \langle 0.6625, 0.0375\rangle$, $SF(\beta_1 \oplus \beta_3) = 0.75$, $SF(\beta_2 \oplus \beta_3) = 0.6624$, so we get $(h_1 \oplus h_3) > (h_2 \oplus h_3)$.*

(2) Equation (2) is not a constant operation, i.e., $\beta_1 < \beta_2$ cannot always generate $(\beta_1 \otimes \beta_3) < (\beta_2 \otimes \beta_3)$.

Example 2. *Let $\beta_1 = \langle 0.15, 0.45\rangle, h_2 = \langle 0.35, 0.55\rangle, \beta_3 = \langle 0.3, 0.3\rangle$, then $SF(\beta_1) = -0.3$, $SF(\beta_2) = -0.2$, we can get $\beta_1 < \beta_2$. On the other hand, $\beta_1 \otimes \beta_3 = \langle 0.045, 0.615\rangle$, $\beta_2 \otimes \beta_3 = \langle 0.105, 0.685\rangle$, $SF(\beta_1 \otimes \beta_3) = -0.57$, $SF(\beta_2 \otimes \beta_3) = -0.58$, so we get $(\beta_1 \otimes \beta_3) > (\beta_2 \otimes \beta_3)$.*

(3) *Equation (3) is not persistent under multiplication. In other words, $\beta_1 < \beta_2$ cannot always generate $\delta\beta_1 < \delta\beta_2$ $(\delta > 0)$.*

Example 3. *Let $\beta_1 = \langle 0.45, 0.35\rangle, \beta_2 = \langle 0.35, 0.25\rangle, \delta = 0.2$, then $SF(\beta_1) = 0.1$, $SF(\beta_2) = 0.1$, $AF(\beta_1) = 0.7$, $AF(\beta_2) = 0.5$ we can get $\beta_1 > \beta_2$. On the other hand, $\delta\beta_1 = \langle 0.1127, 0.8106\rangle$, $\delta\beta_2 = \langle 0.0825, 0.7579\rangle$, $SF(\eta h_1) = -0.6979$, $SF(\eta h_2) = -0.6754$, so we get $\delta\beta_1 < \delta\beta_2$.*

(4) IFWAM is not always monotone with respect to the SF and AF. In other words, $\beta_1 < \beta_2$ cannot invariably generate $IFWAM(\beta_1, \beta_3) < IFWAM(\beta_2, \beta_3)$.

Example 4. *Let $\beta_1 = \langle 0.5, 0.4\rangle$, $\beta_2 = \langle 0.3, 0.2\rangle$, $\beta_3 = \langle 0, 1\rangle$, $\omega_1 = \omega_2 = 0.5$, then $SF(\beta_1) = 0.1$, $SF(\beta_2) = 0.1$, $AF(\beta_1) = 0.9$, $AF(\beta_2) = 0.5$ we can get $\beta_1 > \beta_2$. On the other hand, $IFWAM(\beta_1, \beta_3) = \langle 0.29, 0.64\rangle$, $IFWAM(\beta_2, \beta_3) = \langle 0.16, 0.45\rangle$, $SF(IFWAM(\beta_1, \beta_3)) = -0.35$, $SF(IFWAM(\beta_2, \beta_3)) = -0.29$, so we get $IFWAM(\beta_1, \beta_3) < IFWAM(\beta_2, \beta_3)$.*

(5) IFWGM is not always monotone with respect to the SF and AF. In other words, $\beta_1 > \beta_2$ cannot invariably generate $IFWGM(h_1, h_3) > IFWGM(h_2, h_3)$.

Example 5. *Let $\beta_1 = \langle 0.39, 0.49\rangle$, $\beta_2 = \langle 0.35, 0.45\rangle$, $\beta_3 = \langle 0.2, 0.7\rangle$, then $SF(\beta_1) = -0.1$, $SF(\beta_2) = -0.1$, $AF(\beta_1) = 0.88$, $AF(\beta_2) = 0.80$, so we can get $\beta_1 > \beta_2$. On the other hand, when $\omega_1 = 0.4, \omega_3 = 0.6$, $IFWGM(\beta_1, \beta_3) = \langle 0.2613, 0.6291\rangle$, when $\omega_2 = 0.4$, $\omega_3 = 0.6$ $IFWAM(\beta_2, \beta_3) = \langle 0.2502, 0.6177\rangle$, and then $SF(IFWAM(\beta_1, \beta_3)) = -0.3678$, $SF(IFWAM(\beta_2, \beta_3)) = -0.3675$, so we get $IFWAM(\beta_1, \beta_3) < IFWAM(\beta_2, \beta_3)$.*

In some studies [9,10], we have found out that there is a very close connection between DST and IFS. With the help of this connection, we can provide transparent and fruitful semantics for IFN in terms of DST. Therefore, to reinforce the performance of operations on IFNs, we rewrite the definition and operations of IFS in the framework of DST, which break away from the above-listed limitations.

2.2. IFS in the Framework of Dempster-Shafer Theory

Let us assume Φ to be the set of κ mutually exclusive and exhaustive objects, which corresponds to κ hypotheses or propositions. Φ is called the frame of discernment and defined as follows: $\Phi = \{1, 2, 3, \cdots, \gamma\}$. $\Gamma(\Phi)$ is known as the power set of Φ containing all the possible subsets of Φ and defined as follows: $\Gamma(\Phi) = \left\{\phi, 1, 2, 3, \cdots, \gamma, (1, 2), (1, 3), \cdots, (\gamma - 1, \gamma), (1, 2, 3), \cdots, \Phi\right\}$. By definition, $\Gamma(\Phi)$ consists of 2^κ elements representing the event "the object is in X".

A BPA is a mapping from $\Gamma(\Phi)$ to [0,1] defined as follows: $\vartheta : \Gamma(\Phi) \rightarrow [0,1]$ and satisfies $\sum_{X \subseteq \Gamma(\Phi)} \vartheta(X) = 1$ and $\vartheta(\phi) = 0$.

Note that $\Gamma(\Phi)$ includes ϕ and the condition $\vartheta(\phi) = 0$ is required, but the subsets of Φ for which the mapping does not assume a 0 value are defined focal elements in classical DST. $\vartheta(X)$ denotes the degree of evidence support for the proposition of the object belongs to X. In a word, $\vartheta(X)$ is a measure of the belief attributed exactly to X, and to none of the subsets of X.

Definition 3 [35,36]**.** *Given a BPA ϑ on Φ, the belief function Bel can be defined as:*

$$Bel(X) = \sum_{Y \subseteq X} \vartheta(Y) \tag{15}$$

where $\vartheta(X) > 0$.

Definition 4 [35,36]**.** *Given a BPA ϑ on Φ, the plausibility function Pl can be defined as:*

$$Pl(X) = \sum_{Y \cap X \neq \phi} \vartheta(Y) = 1 - Bel\left(\overline{Y}\right) \tag{16}$$

where $\overline{Y}$ is the complementary set of X.

Therefore, the belief interval (BI) can be represented by interval $[Bel(X), Pl(X)]$. This may be explained as the interval enclosing the "true probability" of X.

In order to measure the similarity between two sets, we present a detailed description of JD between two bodies of evidence as follows:

Definition 5 [35,36]**.** *Let Φ be a frame of discernment including κ mutually exclusive and exhaustive hypothesis, and let $\Lambda_{\Gamma(\Phi)}$ be the space produced by all the subsets of Φ. A BPA is a vector $\vec{\vartheta}$ of $\Lambda_{\Gamma(\Phi)}$ with coordinates $\vartheta(X_i)$ such that:*

$$\sum_{i=1}^{2^{\kappa}} \vartheta(X_i) = 1$$

where $\vartheta(X_i) \geq 0, X_i \in \Gamma(\Phi)$.

In the above definition, $\vartheta(X_i) = 0$ is not necessarily required.

Definition 6 [36]**.** *Let ϑ_1 and ϑ_2 be two BPAs on the same frame of discernment Φ, including κ mutually exclusive and exhaustive hypotheses. The JD between ϑ_1 and ϑ_2 can be defined as follows:*

$$d_{BPA}(\vartheta_1, \vartheta_2) = \sqrt{\frac{1}{2}\left(\vec{\vartheta}_1 - \vec{\vartheta}_2\right)^T \underset{=}{D}\left(\vec{\vartheta}_1 - \vec{\vartheta}_2\right)} \tag{17}$$

where $\vec{\vartheta}_1$ and $\vec{\vartheta}_2$ are the BPAs according to Definition 5 and $\underset{=}{D}$ is a $2^{\kappa} * 2^{\kappa}$ matrix whose elements are

$$D(X, Y) = \frac{|X \cap Y|}{|X \cup Y|}, \ X, Y \in \Gamma(\Phi) \tag{18}$$

From Definition 6, another description of d_{BPA} is as follows:

$$d_{BPA}(\vartheta_1, \vartheta_2) = \sqrt{\frac{1}{2}\left(\|\vec{\vartheta}_1\|^2 + \|\vec{\vartheta}_2\|^2 - 2\left\langle \vec{\vartheta}_1, \vec{\vartheta}_2 \right\rangle\right)} \tag{19}$$

where $\left\langle \vec{s}_1, \vec{s}_2 \right\rangle$ is the scalar product defined by

$$\left\langle \vec{\vartheta}_1, \vec{\vartheta}_2 \right\rangle = \sum_{i=1}^{2^{\kappa}} \sum_{j=1}^{2^{\kappa}} \vartheta_1(\mathrm{X}_i)\vartheta_2(\mathrm{X}_j)\frac{|\mathrm{X}_i \cap \mathrm{X}_j|}{|\mathrm{X}_i \cup \mathrm{X}_j|} \tag{20}$$

with $\mathrm{X}_i, \mathrm{X}_j \in \Gamma(\Phi)$ for $i, j = 1, 2, \cdots, 2^{\kappa}$. $\|\vec{\vartheta}\|^2$ is the square norm of $\vec{\vartheta}$:

$$\|\vec{\vartheta}\|^2 = \left\langle \vec{\vartheta}, \vec{\vartheta} \right\rangle. \tag{21}$$

We know that the $u_{\beta}(\gamma)$, $v_{\beta}(\gamma)$ and $\pi_{\beta}(\gamma)$ of IFN β can represent a BPA of DST, respectively [9,10], so we can completely denote the IFS in the framework of DST based on above information of DST. In practice, when solving the DM problem with the information of IFNs, we implicitly confront three hypotheses as follows: $\gamma \in \beta$, $\gamma \notin \beta$ and $\gamma \in \beta$ or $\gamma \notin \beta$ (the case of hesitation). Therefore, these three hypotheses can be expressed as True ($\gamma \in \beta$), False ($\gamma \notin \beta$), and (True or False) (the case of hesitation). In such a case, $u_{\beta}(\gamma)$ signifies the probability or evidence of $\gamma \in \beta$, i.e., $\vartheta(True) = u_{\beta}(\gamma)$. By analogy, $\vartheta(False) = v_{\beta}(\gamma)$, $\vartheta(True\ \ or\ \ False) = \pi_{\beta}(\gamma)$. Because of $u_{\beta}(\gamma) + v_{\beta}(\gamma) + \pi_{\beta}(\gamma) = 1$ we can draw a conclusion that $u_{\beta}(\gamma)$,$v_{\beta}(\gamma)$ and $\pi_{\beta}(\gamma)$ denote a correct BPA, i.e.,

$$Bel_h(y) = \vartheta(True) = u_{\beta}(\gamma) \tag{22}$$

$$Pl_h(y) = \vartheta(True) + \vartheta(True\ \ or\ \ False) = u_{\beta}(\gamma) + \pi_{\beta}(\gamma) = 1 - v_{\beta}(\gamma). \tag{23}$$

Therefore, we rewrite the definition of IFS in the framework of DST.

Definition 7. *Let* $\mathrm{Y} = \left\{\gamma_i \middle| i = 1, 2, \cdots, t\right\}$ *be a fixed set; then, A IFS B on* Y *in the framework of DST can be defined as follows:* $\widetilde{B} = \left\{\left\langle \gamma_i, BI_{\beta}(\gamma_i)\right\rangle \middle| \gamma_i \in \mathrm{Y}\right\}$*, where* $BI_{\beta}(\gamma_i) = \left[Bel_{\beta}(\gamma_i), Pl_{\beta}(\gamma_i)\right] = \left[u_{\beta}(\gamma_i), 1 - v_{\beta}(\gamma_i)\right]$ *is a BI,* $u_{\beta}(\gamma_i) : \mathrm{Y} \to [0,1]$ *and* $v_{\beta}(\gamma_i) : \mathrm{Y} \to [0,1]$ *are the MD and NMD of* $\gamma \in \mathrm{Y}$ *to B, respectively, and* $0 \le u_{\beta}(\gamma_i) + v_{\beta}(\gamma_i) \le 1$.

For convenience, we represent an IFN in the framework of DST by $\widetilde{\beta} = BI_{\beta} = \left[Bel_{\beta}, Pl_{\beta}\right] = \left[u_{\beta}, 1 - v_{\beta}\right]$.

In order to enhance the performance of operations on IFNs, Dymova and Sevastjanov [9,10] redefined the operational rules on IFN in the framework of DST.

Definition 8. *Let* $\widetilde{\beta_1} = [Bel_1, Pl_1] = [u_1, 1 - v_1]$ *and* $\widetilde{\beta_2} = [Bel_2, Pl_2] = [u_2, 1 - v_2]$ *be two IFNs in the framework of DST; then*

$$(1)\ \widetilde{\beta_1} \oplus \widetilde{\beta_2} = \left[\frac{Bel_1 + Bel_2}{2}, \frac{Pl_1 + Pl_2}{2}\right] = \left[\frac{u_1 + u_2}{2}, 1 - \frac{v_1 + v_2}{2}\right] \tag{24}$$

$$(2)\ \widetilde{\beta_1} \otimes \widetilde{\beta_2} = [Bel_1 Bel_2, Pl_1 Pl_2] = [u_1 u_2, (1 - v_1)(1 - v_2)] \tag{25}$$

$$(3)\ \delta\widetilde{\beta_1} = [\delta Bel_1, \delta Pl_1] = [\delta u_1, \delta(1 - v_1)],\ \eta > 0 \tag{26}$$

$$(4)\ \widetilde{\beta_1}^{\delta} = \left[{Bel_1}^{\delta}, {Pl_1}^{\delta}\right] = \left[{u_1}^{\delta}, (1 - v_1)^{\delta}\right],\ \eta > 0 \tag{27}$$

$$(5)\ \widetilde{\beta_1}^{\widetilde{\beta_2}} = [Bel_1, Pl_1]^{[Bel_2, Pl_2]} = \left[{Bel_1}^{Pl_2}, {Pl_1}^{Bel_2}\right] = \left[{u_1}^{1 - v_2}, (1 - v_1)^{u_2}\right] \tag{28}$$

$$(6)\ \frac{\widetilde{\beta_1}}{\widetilde{\beta_2}} = \frac{[Bel_1, Pl_1]}{[Bel_2, Pl_2]} = \left[\frac{Bel_1}{Pl_2}, \frac{Pl_1}{Bel_2}\right] = \left[\frac{u_1}{1 - v_2}, \frac{1 - v_1}{u_2}\right] \tag{29}$$

From above operational Rules (24)–(27), we can get the $\text{IFWAM}_{\text{DST}}$ and $\text{IFWGM}_{\text{DST}}$ operators in the framework of DST. Let $\widetilde{\beta_i} = [u_i, 1 - v_i]$ be BI and w_i be the weight of $\widetilde{\beta_i}$, $\sum_{i}^{t} \omega_i = 1$. Then

$$IFWAM_{DST}(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}) = \left[\left[\frac{1}{t}\sum_{i=1}^{t} \omega_i Bel_i, \frac{1}{t}\sum_{i=1}^{t} \omega_i Pl_i\right]\right] = \left[\frac{1}{t}\sum_{i=1}^{t} \omega_i u_i, \frac{1}{t}\sum_{i=1}^{t} \omega_i (1 - v_i)\right] \tag{30}$$

$$IFWGM_{DST}(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}) = \left[\prod_{i=1}^{t} Bel_i^{\omega_i}, \prod_{i=1}^{t} Pl_i^{\omega_i}\right] = \left[\prod_{i=1}^{t} u_i^{\omega_i}, \prod_{i=1}^{t} (1 - v_i)^{\omega_i}\right] \tag{31}$$

For BI $\widetilde{\beta} = [u, 1 - v]$, the SF and AF in the framework of DST are defined by following form:

$$SF_{DST}\left(\widetilde{\beta}\right) = (Bel + Pl)/2 = (1 + u - v)/2 \tag{32}$$

$$AF_{DST}\left(\widetilde{\beta}\right) = Pl - Bel = 1 - u - v \tag{33}$$

Further, the order relation between $\widetilde{h}_1$ and $\widetilde{h}_2$ is denoted as follows:

(1) If $SF_{DST}\left(\widetilde{\beta_1}\right) < SF_{DST}\left(\widetilde{\beta_2}\right)$, then $\widetilde{\beta_1} < \widetilde{\beta_2}$;

(2) If $SF_{DST}\left(\widetilde{\beta_1}\right) = SF_{DST}\left(\widetilde{\beta_2}\right)$, then

  (i) $AF_{DST}\left(\widetilde{\beta_1}\right) > AF_{DST}\left(\widetilde{\beta_2}\right)$, then $\widetilde{\beta_1} < \widetilde{\beta_2}$

  (ii) $AF_{DST}\left(\widetilde{\beta_1}\right) = AF_{DST}\left(\widetilde{\beta_2}\right)$, then $\widetilde{\beta_1} = \widetilde{\beta_2}$

It is easy to discover that there is a very close connection between Rules (7) and (8) and Rules (32) and (33). However, it is not suitable that we use Rules (7) and (8) to compare BIs and use Rules (32) and (33) to compare IFNs.

Let $\widetilde{\beta} = [u, 1 - v]$, $\widetilde{\beta_1} = [u_1, 1 - v_1]$ and $\widetilde{\beta_2} = [u_2, 1 - v_2]$ be three Bis. In this way, it is easy to prove that (24)–(29) have the following properties:

$$(1)\ \widetilde{\beta_1} \oplus \widetilde{\beta_2} = \widetilde{\beta_2} \oplus \widetilde{\beta_1}; \tag{34}$$

$$(2)\ \widetilde{\beta_1} \otimes \widetilde{\beta_2} = \widetilde{\beta_2} \otimes \widetilde{\beta_1}; \tag{35}$$

$$(3)\ \delta(\widetilde{\beta_1} \oplus \widetilde{\beta_2}) = \delta\widetilde{\beta_2} \oplus \delta\widetilde{\beta_1}, \quad \delta > 0; \tag{36}$$

$$(4)\ (\xi + \delta)\widetilde{\beta} = \xi\widetilde{\beta} + \delta\widetilde{\beta}, \quad \xi, \delta > 0; \tag{37}$$

$$(5)\ \left(\widetilde{\beta_1} \otimes \widetilde{\beta_2}\right)^{\delta} = \widetilde{\beta_1}^{\delta} \otimes \widetilde{\beta_2}^{\delta}, \quad \delta > 0; \tag{38}$$

$$(6)\ \widetilde{\beta}^{\xi} \otimes \widetilde{\beta}^{\delta} = \widetilde{\beta}^{\xi + \delta} \quad \xi, \delta > 0. \tag{39}$$

The above new operational rules of IFNs in the framework of DST can overcome the drawbacks and shortcomings of the OORs of IFNs.

Theorem 1. *Equation (24) is a constant operation.*

Proof. Let $\widetilde{\beta} = [u, 1 - v]$, $\widetilde{\beta_1} = [u_1, 1 - v_1]$ and $\widetilde{\beta_2} = [u_2, 1 - v_2]$ be the corresponding BIs of IFNs $\beta = \langle u, v \rangle$, $\beta_1 = \langle u_1, v_1 \rangle$ and $\beta_2 = \langle u_2, v_2 \rangle$. Suppose $\widetilde{\beta_1} > \widetilde{\beta_2}$ and $u_1 - v_1 > u_2 - v_2$. We know that $\widetilde{\beta_1} + \widetilde{\beta} = \left[\frac{u_1 + u}{2}, 1 - \frac{v_1 + v}{2}\right]$ and $\widetilde{\beta_2} + \widetilde{\beta} = \left[\frac{u_2 + u}{2}, 1 - \frac{v_2 + v}{2}\right]$, so $u_1 - v_1 + u - v > u_2 - v_2 + u - v$. Then we can get $\widetilde{\beta_1} \oplus \widetilde{\beta} > \widetilde{\beta_2} \oplus \widetilde{\beta}$.

Suppose $\widetilde{\beta_1} > \widetilde{\beta_2}$, $u_1 - v_1 = u_2 - v_2$ and $u_1 + v_1 > u_2 + v_2$; then $u_1 - v_1 + u - v = u_2 - v_2 + u - v$ and $u_1 + v_1 + u + v > u_2 + v_2 + u + v$. Therefore, we can get $SF_{DST}(\widetilde{\beta_1} + \widetilde{\beta}) > SF_{DST}(\widetilde{\beta_2} + \widetilde{\beta})$ and $\widetilde{\beta_1} \oplus \widetilde{\beta} > \widetilde{\beta_2} \oplus \widetilde{\beta}$. □

Theorem 2. *Equation (25) is a constant operation.*

Proof. Theorem 2 is similar to Theorem 1; the proof is omitted here. □

Theorem 3. *Equation (26) is persistent under multiplication.*

Proof. Theorem 3 is similar to Theorem 1; the proof is omitted here. □

Theorem 4. *The $IFWAM_{DST}$ has monotonicity.*

Proof. (1) Let $\widetilde{\beta} = [u, 1-v]$, $\widetilde{\beta_1} = [u_1, 1-v_1]$ and $\widetilde{\beta_2} = [u_2, 1-v_2]$ be the corresponding BIs of IFNs $\beta = \langle u, v\rangle$, $\beta_1 = \langle u_1, v_1\rangle$ and $\beta_2 = \langle u_2, v_2\rangle$. Suppose $\widetilde{\beta_1} > \widetilde{\beta_2}$ and $u_1 - v_1 > u_2 - v_2$. Suppose ω_1, ω_2 are the weights, $\omega_1 + \omega_2 = 1$. Then

$$IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right) = \frac{1}{2}[\omega_1 u_1 + \omega_2 u, \omega_1(1-v_1) + \omega_2(1-v)]$$

$$IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right) = \frac{1}{2}[\omega_2 u_1 + \omega_2 u, \omega_1(1-v_2) + \omega_2(1-v)].$$

Since $\omega_1(u_1 - v_1) + \omega_2(u - v) > \omega_1(u_2 - v_2) + \omega_2(u - v)$, then $SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right)\right) > SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)\right)$ and $IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right) > IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)$.

Suppose $\widetilde{\beta_1} > \widetilde{\beta_2}$, $u_1 - v_1 = u_2 - v_2$ and $u_1 + v_1 > u_2 + v_2$; then, $\omega_1(u_1 - v_1) + \omega_2(u - v) = \omega_1(u_2 - v_2) + \omega_2(u - v)$ and $\omega_1(u_1 + v_1) + \omega_2(u + v) > \omega_1(u_2 + v_2) + \omega_2(u + v)$. Therefore, $SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right)\right) > SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)\right)$ and $IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right) > IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)$.

(2) Let $\widetilde{\beta} = [u, 1-v]$, $\widetilde{\beta_i} = [u_i, 1-v_i](i = 1, 2, \cdots, t)$, and $\widetilde{\beta}'_i = [u_i', 1-v_i'](i = 1, 2, \cdots, t)$ be the corresponding BIs of IFNs $\beta = \langle u, v\rangle$, $\beta_i = \langle u_i, v_i\rangle(i = 1, 2, \cdots, t)$ and $\beta_i' = \langle u_i', v_i'\rangle(i = 1, 2, \cdots, t)$. Suppose $\widetilde{\beta_i} > \widetilde{\beta_i}'$ and $u_i - v_i > u_i' - v_i'$. Suppose $\omega_i(i = 1, 2, \cdots, t)$ are the weights of β_i and β_i', ω is the weight of β, and $\left(\sum\limits_{i=1}^{t} \omega_i\right) + \omega = 1$. Then,

$$IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}, \widetilde{\beta}\right) = \frac{1}{t+1}\left[\left(\sum_{i=1}^{t} \omega_i u_i\right) + \omega u, \left(\sum_{i=1}^{t} \omega_i(1-v_i)\right) + \omega(1-v)\right]$$

$$IFWAM_{DST}\left(\widetilde{\beta_1}', \widetilde{\beta_2}', \cdots, \widetilde{\beta_m}', \widetilde{\beta}\right) = \frac{1}{t+1}\left[\left(\sum_{i=1}^{t} \omega_i u_i'\right) + \omega u, \left(\sum_{i=1}^{t} \omega_i(1-v_i')\right) + \omega(1-v)\right].$$

Since $\left(\sum\limits_{i=1}^{t} \omega_i u_i\right) + \omega u + \left(\sum\limits_{i=1}^{t} \omega_i(1-v_i)\right) + \omega(1-v) > \left(\sum\limits_{i=1}^{t} \omega_i u_i'\right) + \omega u, \left(\sum\limits_{i=1}^{t} \omega_i(1-v_i')\right) + \omega(1-v)$, then $SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right)\right) > SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)\right)$ and $IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right) > IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)$.

Suppose $\widetilde{\beta_i} > \widetilde{\beta_i}'$, $u_i - v_i = u_i' - v_i'$ and $u_i + v_i > u_i' + v_i'$; then, $\left(\sum\limits_{i=1}^{t} \omega_i u_i\right) + \omega u + \left(\sum\limits_{i=1}^{t} \omega_i(1-v_i)\right) + \omega(1-v) = \left(\sum\limits_{i=1}^{t} \omega_i u_i'\right) + \omega u, \left(\sum\limits_{i=1}^{t} \omega_i(1-v_i')\right) + \omega(1-v)$ and

$$\begin{array}{c}\left(\sum\limits_{i=1}^{t} \omega_i u_i\right) + \omega u + \left(\sum\limits_{i=1}^{t} \omega_i(1-v_i)\right) + \omega(1-v) = \left(\sum\limits_{i=1}^{t} \omega_i u_i'\right) + \omega u, \left(\sum\limits_{i=1}^{t} \omega_i(1-v_i')\right) + \omega(1-v) \\ \left(\sum\limits_{i=1}^{t} \omega_i(u_i + v_i)\right) + \omega(u+v) > \left(\sum\limits_{i=1}^{t} \omega_i(u_i' + v_i')\right) + \omega(u+v)\end{array}.$$

Therefore, $SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right)\right) > SF_{DST}\left(IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)\right)$ and $IFWAM_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta}\right) > IFWAM_{DST}\left(\widetilde{\beta_2}, \widetilde{\beta}\right)$. □

Theorem 5. *The IFWGM$_{DST}$ has monotonicity.*

Proof. Theorem 5 is similar to Theorem 4; the proof is omitted here. □

2.3. PA Operator

Yager [24] initially developed the PA operator, which permits the attribute values to assist and balance each other in the aggregation process.

Definition 9 [28,29]. *Let $\chi_\eta(\eta = 1, 2, \cdots t)$ be a set of evaluated values. The PA operator can be defined as follows:*

$$PA(\chi_1, \chi_2, \cdots, \chi_t) = \frac{\sum\limits_{\eta=1}^{t}\left(1 + \mathrm{T}\left(\chi_\eta\right)\right)\chi_\eta}{\sum\limits_{\eta=1}^{t}\left(1 + \mathrm{T}\left(\chi_\eta\right)\right)} \tag{40}$$

where $T\left(\chi_\eta\right) = \sum\limits_{\tau=1,\eta\neq\tau}^{t} Sup\left(\chi_\eta, \chi_\tau\right)$, $Sup\left(\chi_\eta, \chi_\tau\right) = 1 - d\left(\chi_\eta, \chi_\tau\right)$ and $Sup\left(\chi_\eta, \chi_\tau\right)$ denotes the grade of support for χ_η from χ_τ, which has the following properties:

(1) $Sup\left(\chi_\eta, \chi_\tau\right) \in [0, 1]$; (2) $Sup\left(\chi_\eta, \chi_\tau\right) = Sup\left(\chi_\eta, \chi_\tau\right)$; (3) $Sup\left(\chi_\eta, \chi_\tau\right) > Sup(\chi_k, \chi_l)$, if $d\left(\chi_\eta, \chi_\tau\right) < d(\chi_k, \chi_l)$, where $\eta, \tau, k, l \in \{1, 2, \cdots, t\}$.

2.4. MSM Operator

Qin and Liu [31] first proposed the MSM operator, which can capture the interrelationships among any number of multi-input attribute arguments by changing parameter values.

Definition 10 [31]. *Let $\chi_\eta(\eta = 1, 2, \cdots, t)$ be a set of positive numbers; then, the MSM operator of $\chi_\eta(\eta = 1, 2, \cdots, t)$ can be defined as follows:*

$$MSM^{(\kappa)}(\chi_1, \chi_2, \cdots, \chi_t) = \left(\frac{\sum\limits_{1\leq\eta_1<\cdots<\eta_\kappa\leq t}\prod\limits_{j=1}^{\kappa} x_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa} \tag{41}$$

where $C_t^\kappa = \frac{t!}{\kappa!(t-\kappa)!}$ is the binomial coefficient, and $(\eta_1, \eta_2, \cdots, \eta_\kappa)$ traverses all the $\kappa -$ tuple combination of $(1, 2, \cdots, t)$, where $1 \leq \kappa \leq t$.

Evidently, the MSM operator has some desirable properties [31]:

(1) $MSM^{(\kappa)}(0, 0, \cdots, 0) = 0, MSM^{(\kappa)}(\chi, \chi, \cdots, \chi) = \chi$;

(2) $MSM^{(\kappa)}(\chi_1, \chi_2, \cdots, \chi_t) \leq MSM^{(\kappa)}(\chi_1{'}, \chi_2{'}, \cdots, \chi_t{'})$, if $\chi_\eta \leq \chi_\eta{'}$ for all η;

(3) $\min\limits_\eta\left\{\chi_\eta\right\} \leq MSM^{(\kappa)}(\chi_1, \chi_2, \cdots, \chi_t) \leq \max\limits_\eta\left\{\chi_\eta\right\}$.

3. The IFPMSM$_{\mathrm{DST}}$ Operators

In this part, we initially propose the intuitionistic fuzzy (IF) power average (IFPA$_{\mathrm{DST}}$) operator, the IF power weighted average (IFPWA$_{\mathrm{DST}}$) operator, and the IF MSM (IFMSM$_{\mathrm{DST}}$) operator in the framework of the DST. Subsequently, we combine the MSM operator with the PA operator and extend them to IFNs to present the IF power MSM (IFPMSM$_{\mathrm{DST}}$) operator and IF power weighted MSM (IFPWMSM$_{\mathrm{DST}}$) operator.

3.1. PA Operator for IFNs in the Framework of DST

Definition 11. *Let* $\widetilde{B} = \left\{\widetilde{\beta_\eta}\middle|\widetilde{\beta_\eta} = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta\middle|\beta_\eta = \left\langle u_\eta, v_\eta\right\rangle, \eta = 1, 2, \cdots, t\right\}$. *The* $IFPA_{DST}$ *operator of* $\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots$, *and* $\widetilde{\beta_t}$ *can be defined as follows:*

$$IFPA_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \frac{\sum_{\eta=1}^{t}\left(1 + \mathrm{T}\left(\widetilde{\beta_\eta}\right)\right)\widetilde{\beta_\eta}}{\sum_{\eta=1}^{t}\left(1 + \mathrm{T}\left(\widetilde{\beta_\eta}\right)\right)} \tag{42}$$

$$\textit{where } \mathrm{T}\left(\widetilde{\beta_\eta}\right) = \sum_{\tau=1, j\neq\eta}^{t} Sup\left(\widetilde{\beta_\eta}, \widetilde{\beta_\tau}\right), \tag{43}$$

and $Sup\left(\widetilde{\beta_\eta}, \widetilde{\beta_\tau}\right)$ *denotes the support degree for* $\widetilde{\beta_\eta}$ *from* $\widetilde{\beta_\tau}$, *which satisfied the properties of Definition 9, where* $1 < \eta < t$, $1 < \tau < t$ *and* $\eta \neq \tau$.

For simplifying Equation (42), we indicate

$$\theta_\eta = \frac{1 + \mathrm{T}\left(\widetilde{\beta_\eta}\right)}{\sum_{\eta=1}^{t}\left(1 + \mathrm{T}\left(\widetilde{\beta_\eta}\right)\right)} \tag{44}$$

where $(\theta_1, \theta_2, \cdots, \theta_t)$ is the power-weighted vector of the $\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots$, and $\widetilde{\beta_t}$. Evidentially, $\theta_\eta \geq 0$ and $\sum_{\eta=1}^{t}\theta_\eta = 1$; then, Equation (43) can be simplified as follows:

$$IFPA_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \sum_{\eta=1}^{t}\theta_\eta\widetilde{\beta_\eta}. \tag{45}$$

Theorem 6. *Let* $\widetilde{B} = \left\{\widetilde{\beta_\eta}\middle|\widetilde{\beta_\eta} = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta\middle|\beta_\eta = \left\langle u_\eta, v_\eta\right\rangle, \eta = 1, 2, \cdots, t\right\}$. *The aggregated result based on the* $IFPA_{DST}$ *operator is also a BI and*

$$IFPA_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \left[\frac{1}{t}\sum_{\eta=1}^{t}\theta_\eta u_\eta, \frac{1}{t}\sum_{\eta=1}^{t}\theta_\eta\left(1 - v_\eta\right)\right]. \tag{46}$$

Proof. By the operational rules of IFNs in the framework of DST, we get $\theta_\eta\widetilde{\beta_\eta} = \theta_\eta\left[u_\eta, 1 - v_\eta\right] = \left[\theta_\eta u_\eta, \theta_\eta\left(1 - v_\eta\right)\right]$, where $1 < \eta < t$. Then, we get

$$IFPA_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \left[\frac{1}{t}\sum_{\eta=1}^{t}\theta_\eta u_\eta, \frac{1}{t}\sum_{\eta=1}^{t}\theta_\eta\left(1 - v_\eta\right)\right].$$

□

Definition 12. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \eta = 1, 2, \cdots, t\right\}$. *The aggregated result based on the* $IFPWA_{DST}$ *operator is also a BI and*

$$IFPWA_{DST}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \frac{\sum\limits_{\eta=1}^{t} \omega_\eta \left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right) \widetilde{\beta}_\eta}{\sum\limits_{\eta=1}^{t} \omega_\eta \left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)} \quad (47)$$

$$\mathrm{T}\left(\widetilde{\beta}_\eta\right) = \sum_{\tau=1, j \neq \eta}^{t} Sup\left(\widetilde{\beta}_\eta, \widetilde{\beta}_\tau\right) \quad (48)$$

where $Sup\left(\widetilde{\beta}_\eta, \widetilde{\beta}_\tau\right)$ *denotes the support degree for* $\widetilde{\beta}_\eta$ *from* $\widetilde{\beta}_\tau$, *which satisfied the properties of Definition 9, where* $1 < \eta < t$, $1 < \tau < t$ *and* $\eta \neq \tau$.

For simplifying Equation (41), we indicate

$$\overline{\theta}_\eta = \frac{\omega_\eta \left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)}{\sum\limits_{\eta=1}^{t} \omega_\eta \left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)} \quad (49)$$

where $\omega_\eta (\eta = 1, 2, \cdots, t)$ is the weight of $\widetilde{\beta}_\eta (\eta = 1, 2, \cdots, t)$. Evidentially, $\omega_\eta > 0$ and $\sum\limits_{\eta=1}^{t} \omega_\eta = 1$, so Equation (46) can be simplified as follows:

$$IFPWA_{DST}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \sum_{\eta=1}^{t} \overline{\theta}_\eta \widetilde{\beta}_\eta \quad (50)$$

Theorem 7. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \eta = 1, 2, \cdots, t\right\}$. *The aggregated result based on the* $IFPA_{DST}$ *operator is also a BI and*

$$IFPWA_{DST}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left[\frac{1}{t} \sum_{\eta=1}^{t} \overline{\theta}_\eta u_\eta, \frac{1}{t} \sum_{\eta=1}^{t} \overline{\theta}_\eta \left(1 - v_\eta\right)\right]. \quad (51)$$

The proof of Theorem 7 is similar to that of Theorem 6.

3.2. MSM Operator for IFNs in the Framework of DST

Definition 13. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \ \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \ \eta = 1, 2, \cdots, t\right\}$. *The* $IFMSM_{DST}$ *operator of* $\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots$, *and* $\widetilde{\beta}_t$ *can be defined as follows:*

$$IFMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} \widetilde{\beta}_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa}. \quad (52)$$

Theorem 8. *Let* $\widetilde{B} = \left\{ \widetilde{\beta_\eta} \middle| \widetilde{\beta_\eta} = \left[u_\eta, 1 - v_\eta \right],\ \eta = 1, 2, \cdots, t \right\}$ *be the corresponding BI set of IFS* $B = \left\{ \beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle,\ \eta = 1, 2, \cdots, t \right\}$. *The aggregated result based on the IFMSM$_{DST}$ operator is also a BI and*

$$IFMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \left[\left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} u_{\eta_j}}{\left(C_t^\kappa\right)^2} \right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} \left(1 - v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2} \right)^{1/\kappa} \right]. \tag{53}$$

Proof. By the operational rules of IFNs in the framework of DST, we get $\prod\limits_{j=1}^{\kappa} \widetilde{\beta_{\eta_j}} = \prod\limits_{j=1}^{\kappa} \left[u_\eta, 1 - v_\eta \right] = \left[\prod\limits_{j=1}^{\kappa} u_{\eta_j}, \prod\limits_{j=1}^{\kappa} \left(1 - v_{\eta_j}\right) \right]$ and $\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \left[\prod\limits_{j=1}^{\kappa} u_{\eta_j}, \prod\limits_{j=1}^{\kappa} \left(1 - v_{\eta_j}\right) \right] = \left[\frac{1}{C_t^\kappa} \sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} u_{\eta_j}, \frac{1}{C_t^\kappa} \sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} \left(1 - v_{\eta_j}\right) \right]$, so $IFMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \left[\left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} u_{\eta_j}}{\left(C_t^\kappa\right)^2} \right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} \left(1 - v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2} \right)^{1/\kappa} \right]$. □

3.3. Power MSM Operator for IFNs in the Framework of DST

Definition 14. *Let* $\widetilde{B} = \left\{ \widetilde{\beta_\eta} \middle| \widetilde{\beta_\eta} = \left[u_\eta, 1 - v_\eta \right],\ \eta = 1, 2, \cdots, t \right\}$ *be the corresponding BI set of IFS* $B = \left\{ \beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle,\ \eta = 1, 2, \cdots, t \right\}$. *The IFPMSM$_{DST}$ operator of* $\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots$, *and* $\widetilde{\beta_t}$ *can be defined as follows:*

$$IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} \frac{t\left(1 + \mathrm{T}\left(\widetilde{\beta_{\eta_j}}\right)\right)}{\sum\limits_{\eta=1}^{t} \left(1 + \mathrm{T}\left(\widetilde{\beta_\eta}\right)\right)} \widetilde{\beta_{\eta_j}}}{C_t^\kappa} \right)^{1/\kappa} \tag{54}$$

$$\text{where } \mathrm{T}\left(\widetilde{\beta_\eta}\right) = \sum_{\tau=1, \tau \ne \eta}^{t} Sup\left(\widetilde{\beta_\eta}, \widetilde{\beta_\tau}\right), \tag{55}$$

and $Sup\left(\widetilde{\beta_\eta}, \widetilde{\beta_\tau}\right)$ *denotes the support degree for* $\widetilde{\beta_\eta}$ *from* $\widetilde{\beta_\tau}$, *which satisfied the properties of Definition 9, where* $1 < \eta < t$, $1 < \tau < t$ *and* $\eta \ne \tau$.

For simplifying Equation (53), we indicate

$$\theta_\eta = \frac{1 + \mathrm{T}\left(\widetilde{\beta_\eta}\right)}{\sum\limits_{\eta=1}^{t} \left(1 + \mathrm{T}\left(\widetilde{\beta_\eta}\right)\right)} \tag{56}$$

where $(\theta_1, \theta_2, \cdots, \theta_t)$ is the power-weighted vector of the $\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots$, and $\widetilde{\beta}_t$. Evidentially, $\theta_\eta \geq 0$ and $\sum_{\eta=1}^{t} \theta_\eta = 1$, and then Equation (42) can be simplified as follows:

$$IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\widetilde{\beta}_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa}. \quad (57)$$

Theorem 9. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \eta = 1, 2, \cdots, t\right\}$. *The aggregated result based on the* $IFPMSM_{DST}$ *operator is also a BI and*

$$IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left[\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right]. \quad (58)$$

Proof. By the operational rules of IFNs in the framework of DST, we get $t\theta_{\eta_j}\widetilde{\beta}_{\eta_j} = t\theta_{\eta_j}\left[u_{\eta_j}, 1 - v_{\eta_j}\right] = \left[t\theta_{\eta_j}u_{\eta_j}, t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)\right]$ and $\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\widetilde{\beta}_{\eta_j} = \prod\limits_{j=1}^{\kappa}\left[t\theta_{\eta_j}u_{\eta_j}, t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)\right] = \left[\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}u_{\eta_j}, \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)\right]$, and then we can get $\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t}\left[\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}u_{\eta_j}, \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)\right] = \left[\frac{t^\kappa}{C_t^\kappa}\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t}\prod\limits_{j=1}^{\kappa}\theta_{\eta_j}u_{\eta_j}, \frac{t^\kappa}{C_t^\kappa}\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t}\prod\limits_{j=1}^{\kappa}\theta_{\eta_j}\left(1 - v_{\eta_j}\right)\right]$ and $IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left[\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right]$. □

Next, some desirable properties of the $\text{IFPMSM}_{\text{DST}}$ operator are proposed.

Theorem 10 (commutativity). *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \eta = 1, 2, \cdots, t\right\}$. *If* $\widetilde{B}' = \left\{\widetilde{\beta}'_\eta \middle| \widetilde{\beta}'_\eta = \left[u'_\eta, 1 - v'_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *is any permutation of* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$, *then*

$$IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}'_1, \widetilde{\beta}'_2, \cdots, \widetilde{\beta}'_t\right). \quad (59)$$

Proof. Since $\widetilde{B}' = \left\{\widetilde{\beta}'_\eta \middle| \widetilde{\beta}'_\eta = \left[u'_\eta, 1-v'_\eta\right], \eta = 1,2,\cdots,t\right\}$ denotes an any permutation of $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1-v_\eta\right], \eta = 1,2,\cdots,t\right\}$, based on the Theorem 9, we can get

$$\begin{aligned} IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) &= \left[\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1-v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right] \\ &= \left[\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta'_{\eta_j} u'_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta'_{\eta_j}\left(1-v'_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right] \\ &= IFMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}'_1, \widetilde{\beta}'_2, \cdots, \widetilde{\beta}'_t\right) \end{aligned}$$

□

Theorem 11 (Boundedness). *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1-v_\eta\right],\ \eta = 1,2,\cdots,t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta\right\rangle,\ \eta = 1,2,\cdots,t\right\}$. *Suppose* $\widetilde{\beta}_\eta^+ = \left[\max\limits_{\eta=1}^{t} u_\eta, \max\limits_{\eta=1}^{t}\left(1-v_\eta\right)\right]$ *and* $\widetilde{\beta}_\eta^- = \left[\min\limits_{\eta=1}^{t} u_\eta, \min\limits_{\eta=1}^{t}\left(1-v_\eta\right)\right]$. *Then,*

$$\widetilde{\beta}_\eta^- \le IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) \le \widetilde{\beta}_\eta^+. \tag{60}$$

Proof. Since $\theta_\eta = \frac{1+\mathrm{T}\left(\widetilde{\beta}_\eta\right)}{\sum\limits_{\eta=1}^{t}\left(1+\mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)}$ and $\sum\limits_{\eta=1}^{t}\theta_\eta = 1$, based on Equation (57) and the boundedness of the MSM operator, we can get $\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa} \le \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} \max\limits_{\eta=1}^{t} u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}$ and $\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1-v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa} \le \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} \max\limits_{\eta=1}^{t}\left(1-v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}$. From Equation (33),

$$\begin{aligned} SF\left(IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right)\right) &= \frac{1}{2}\left(\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa} + \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1-v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right) \\ &\le \frac{1}{2}\left(\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} \max\limits_{\eta=1}^{t} u_\eta}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa} + \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} \max\limits_{\eta=1}^{t}\left(1-v_\eta\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right) = SF\left(\widetilde{\beta}_\eta^+\right) \end{aligned}.$$

That is

$$\begin{aligned} IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) &= \left[\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1-v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right] \\ &\le \left[\left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} \max\limits_{\eta=1}^{t} u_\eta}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1\le\eta_1<\cdots<\eta_\kappa\le t}\prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} \max\limits_{\eta=1}^{t}\left(1-v_\eta\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right] = \widetilde{\beta}_\eta^+ \end{aligned}$$

Similarly, we can get $\widetilde{\beta_\eta^-} \leq IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right)$. Therefore, we have

$$\widetilde{\beta_\eta^-} \leq IFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) \leq \widetilde{\beta_\eta^+}.$$

□

We should see that the IFPMSM$_{\mathrm{DST}}$ operators do not have an idempotent property. However, if we make a slight modification (multiply by $\left(C_t^\kappa\right)^{\frac{1}{\kappa}}$), the modified IFPMSM$_{\mathrm{DST}}$ (MIFPMSM$_{\mathrm{DST}}$) operator will be an idempotent operator, as follows:

$$MIFPMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) = \left[\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} u_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)}{C_t^\kappa}\right)^{1/\kappa}\right]. \tag{61}$$

A similar expression was first obtained by Xu and Yager in [37]. It is evidential that we can obtain the same ordering result of the alternatives by using the IFPMSM$_{\mathrm{DST}}$ operator and the MIFPMSM$_{\mathrm{DST}}$ operator in a real DM. In addition, the IFPMSM$_{\mathrm{DST}}$ operator does not satisfy monotonicity because the d_{BPA} will also make a difference if the attribute values are changed.

Next, some special cases of the IFPMSM$_{\mathrm{DST}}$ operator are investigated by considering some diverse values of κ.

(1) When $\kappa = 1$, the IFPMSM$_{\mathrm{DST}}$ operator become the IFPA$_{\mathrm{DST}}$ operator, that is

$$\begin{aligned} IFPMSM_{DST}{}^{(1)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) &= \left[\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j} u_{\eta_j}}{\left(C_t^1\right)^2}, \frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right] \\ &= \left[\frac{1}{t}\sum_{\eta=1}^{t} \theta_\eta u_\eta, \frac{1}{t}\sum_{\eta=1}^{t} \theta_\eta\left(1 - v_\eta\right)\right] = IFPA_{DST}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) \end{aligned}$$

(2) When $\kappa = 2$, that is

$$\begin{aligned} IFPMSM_{DST}{}^{(2)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) &= \left[\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{2} t\theta_{\eta_j} u_{\eta_j}}{\left(C_t^2\right)^2}\right)^{1/2}, \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{2} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)}{\left(C_t^2\right)^2}\right)^{1/2}\right] \\ &= \left[\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} t^2\theta_{\eta_1} u_{\eta_1}\theta_{\eta_2} u_{\eta_2}}{\left(C_t^2\right)^2}\right)^{1/2}, \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} t^2\theta_{\eta_1}\left(1 - v_{\eta_1}\right)\theta_{\eta_1}\left(1 - v_{\eta_1}\right)}{\left(C_t^2\right)^2}\right)^{1/2}\right] \\ &= \left[\frac{t}{C_t^2}\left(\sum_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \theta_{\eta_1}\theta_{\eta_2} u_{\eta_1} u_{\eta_2}\right)^{1/2}, \frac{t}{C_t^2}\left(\sum_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \theta_{\eta_1}\theta_{\eta_1}\left(1 - v_{\eta_1}\right)\left(1 - v_{\eta_1}\right)\right)^{1/2}\right] \end{aligned}$$

(3) When $\kappa = n$, that is

$$\begin{aligned} IFPMSM_{DST}{}^{(t)}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_t}\right) &= \left[t\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_t \leq t} \prod\limits_{j=1}^{t} \theta_{\eta_j} u_{\eta_j}}{\left(C_t^t\right)^2}\right)^{1/t}, t\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_t \leq t} \prod\limits_{j=1}^{t} \theta_{\eta_j}\left(1 - v_{\eta_j}\right)}{\left(C_t^t\right)^2}\right)^{1/t}\right] \\ &= \left[t\left(\prod_{j=1}^{t} \theta_{\eta_j} u_{\eta_j}\right)^{1/t}, t\left(\prod_{j=1}^{t} \theta_{\eta_j}\left(1 - v_{\eta_j}\right)\right)^{1/t}\right] \end{aligned}$$

According to Equation (53), we discover that the $\mathrm{IFPMSM_{DST}}$ operator has a definite fault, i.e., it does not take the importance of the attributes into account. However, in many practice DM environments, the weights of attributes play a crucial role in the aggregate process. Therefore, we next present the weighted $\mathrm{IFPMSM_{DST}}$ ($\mathrm{IFPWMSM_{DST}}$) operator.

Definition 15. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta\right\rangle, \eta = 1, 2, \cdots, t\right\}$*. The* $IFPWMSM_{DST}$ *operator of* $\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots$ *, and* $\widetilde{\beta}_t$ *can be defined as follows:*

$$IFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} \frac{t\omega_{\eta_j}\left(1 + \mathrm{T}\left(\widetilde{\beta}_{\eta_j}\right)\right)}{\sum\limits_{\eta=1}^{t} \omega_\eta\left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)} \widetilde{\beta}_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa} \tag{62}$$

$$\text{where } \mathrm{T}\left(\widetilde{\beta}_\eta\right) = \sum_{\tau=1, \tau \ne \eta}^{t} Sup\left(\widetilde{\beta}_\eta, \widetilde{\beta}_\tau\right), \tag{63}$$

and $Sup\left(\widetilde{\beta}_\eta, \widetilde{\beta}_\tau\right)$ *denotes the support degree for* $\widetilde{\beta}_\eta$ *from* $\widetilde{\beta}_\tau$*, which satisfies the properties of Definition 9, where* $1 < \eta < t$*,* $1 < \tau < t$ *and* $\eta \ne \tau$*.*

For simplifying Equation (61), we indicate

$$\overline{\theta}_\eta = \frac{\omega_\eta\left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)}{\sum\limits_{\eta=1}^{t} \omega_\eta\left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)} \tag{64}$$

where $(\theta_1, \theta_2, \cdots, \theta_t)$ is the power-weighted vector of the $\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots$, and $\widetilde{\beta}_t$. Evidentially, $\theta_\eta \ge 0$ and $\sum\limits_{\eta=1}^{t} \theta_\eta = 1$; then, Equation (62) can be simplified as follows:

$$IFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} t\overline{\theta}_{\eta_j} \widetilde{\beta}_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa}. \tag{65}$$

Theorem 12. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta\right\rangle, \eta = 1, 2, \cdots, t\right\}$*. The aggregated result based on the* $IFPWMSM_{DST}$ *operator is also a BI and*

$$IFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left[\left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} t\overline{\theta}_{\eta_j} u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \le \eta_1 < \cdots < \eta_\kappa \le t} \prod\limits_{j=1}^{\kappa} t\overline{\theta}_{\eta_j}\left(1 - v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right]. \tag{66}$$

Proof. The proof is similar to Theorem 9. Therefore, it is omitted here. □

Likewise, the $\mathrm{IFPWMSM_{DST}}$ operator has also some desirable properties, shown as follows:

Theorem 13 (commutativity). *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \eta = 1, 2, \cdots, t\right\}$. *If* $\widetilde{B}' = \left\{\widetilde{\beta}'_\eta \middle| \widetilde{\beta}'_\eta = \left[u'_\eta, 1 - v'_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *is any permutation of* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$, *then*

$$IFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = IFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}'_1, \widetilde{\beta}'_2, \cdots, \widetilde{\beta}'_t\right). \tag{67}$$

Theorem 14 (Boundedness). *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \eta = 1, 2, \cdots, t\right\}$. *Suppose* $\widetilde{\beta}_\eta^+ = \left[\max_{\eta=1}^{t} u_\eta, \max_{\eta=1}^{t}\left(1 - v_\eta\right)\right]$ *and* $\widetilde{\beta}_\eta^- = \left[\min_{\eta=1}^{t} u_\eta, \min_{\eta=1}^{t}\left(1 - v_\eta\right)\right]$. *Then,*

$$\widetilde{\beta}_\eta^- \leq IFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) \leq \widetilde{\beta}_\eta^+. \tag{68}$$

We can also see that the $\text{IFPWMSM}_{\text{DST}}$ operators do not have an idempotent property. However, if we make a slight modification (multiply by $\left(C_t^\kappa\right)^{\frac{1}{\kappa}}$), the modified $\text{IFPWMSM}_{\text{DST}}$ ($\text{MIFPMSM}_{\text{DST}}$) operator will be an idempotent operator, as follows:

$$MIFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left[\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\overline{\theta}_{\eta_j} u_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\overline{\theta}_{\eta_j}\left(1 - v_{\eta_j}\right)}{C_t^\kappa}\right)^{1/\kappa}\right]. \tag{69}$$

The advantages of operational rules of IFNs in the framework of DST is that we can solve MADM problems without intermediate defuzzification when input arguments and their weights are both IFNs, which are not defined in ordinary operators of IFNs. Therefore, we firstly define the $\text{IFWPMSM}_{\text{DST}}$ operator where the attribute weights are IFN as follows.

Definition 16. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta \right\rangle, \eta = 1, 2, \cdots, t\right\}$. $\hat{\omega}_i = \langle \hat{u}_i, \hat{v}_i \rangle$ *is the weight of* β_i *in the form of IFN,* $\hat{\omega}_i = [\hat{u}_i, 1 - \hat{v}_i]$ *is the corresponding BI of* $\hat{\omega}_i$. $\overline{\overline{\omega}} = \left(\overline{\overline{\omega}}_1, \overline{\overline{\omega}}_2, \cdots, \overline{\overline{\omega}}_m\right)$ *is a normalized interval weight vector if and only if satisfies two conditions [38]: (1) There exists at least a normalized weight vector* $a = (a_1, a_2, \cdots, a_m) \in A$, *and* $A = \left\{a = (a_1, a_2, \cdots, a_t) \middle| \hat{u}_i \leq a_i \leq (1 - \hat{v}_i), i = 1, 2, \cdots, t, \sum\limits_{i=1}^{t} a_i = 1\right\}$ *is a set of normalized weight vectors;* (2) $\hat{u}_i$ *and* $1 - \hat{v}_i$, $(i = 1, 2, \cdots, m)$ *are all attainable in A. The* $IFWPMSM_{DST}$ *operator of* $\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots$, *and* $\widetilde{\beta}_t$ *can be defined as follows:*

$$IFP\overline{W}MSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} \frac{t\omega_{\eta_j}\left(1 + \mathrm{T}\left(\widetilde{\beta}_{\eta_j}\right)\right)}{\sum\limits_{\eta=1}^{t} \omega_\eta\left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)} \widetilde{\beta}_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa} \tag{70}$$

$$\text{where } \mathrm{T}\left(\widetilde{\beta}_\eta\right) = \sum_{\tau=1, \tau \neq \eta}^{t} Sup\left(\widetilde{\beta}_\eta, \widetilde{\beta}_\tau\right), \tag{71}$$

and $Sup\left(\widetilde{\beta}_\eta, \widetilde{\beta}_\tau\right)$ *denotes the support degree for* $\widetilde{\beta}_\eta$ *from* $\widetilde{\beta}_\tau$*, which satisfied the properties of Definition 9, where* $1 < \eta < t$*,* $1 < \tau < t$ *and* $\eta \neq \tau$*.*

For simplifying Equation (63), we indicate

$$\overline{\overline{\theta}}_\eta = \frac{\omega_\eta\left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)}{\sum\limits_{\eta=1}^{t} \omega_\eta\left(1 + \mathrm{T}\left(\widetilde{\beta}_\eta\right)\right)} \tag{72}$$

where $(\theta_1, \theta_2, \cdots, \theta_t)$ is the power-weighted vector of the $\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots$, and $\widetilde{\beta}_t$. Evidentially, $\theta_\eta \geq 0$ and $\sum\limits_{\eta=1}^{t} \theta_\eta = 1$. Then, Equation (70) can be simplified as follows:

$$IFP\overline{W}MSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\widetilde{\beta}_{\eta_j}}{C_t^\kappa}\right)^{1/\kappa}. \tag{73}$$

Theorem 15. *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta\right\rangle, \eta = 1, 2, \cdots, t\right\}$*.* $\hat{\omega}_i = \langle \hat{u}_i, \hat{v}_i\rangle$ *is the weight of* β_i *in the form of IFN,* $\hat{\omega}_i = [\hat{u}_i, 1 - \hat{v}_i]$ *is the corresponding BI of* $\hat{\omega}_i$*.* $\overline{\overline{\omega}} = \left(\overline{\overline{\omega}}_1, \overline{\overline{\omega}}_2, \cdots, \overline{\overline{\omega}}_m\right)$ *is a normalized interval weight vector. The aggregated result based on the* $IFPWMSM_{DST}$ *operator is also a BI and*

$$IFP\overline{W}MSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = \left[\left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}u_{\eta_j}}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}, \left(\frac{\sum\limits_{1 \leq \eta_1 < \cdots < \eta_\kappa \leq t} \prod\limits_{j=1}^{\kappa} t\theta_{\eta_j}\left(1 - v_{\eta_j}\right)}{\left(C_t^\kappa\right)^2}\right)^{1/\kappa}\right]. \tag{74}$$

Theorem 16 (commutativity)**.** *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta\right\rangle, \eta = 1, 2, \cdots, t\right\}$*.* $\hat{\omega}_i = \langle \hat{u}_i, \hat{v}_i\rangle$ *is the weight of* β_i *in the form of IFN,* $\hat{\omega}_i = [\hat{u}_i, 1 - \hat{v}_i]$ *is the corresponding BI of* $\hat{\omega}_i$*.* $\overline{\overline{\omega}} = \left(\overline{\overline{\omega}}_1, \overline{\overline{\omega}}_2, \cdots, \overline{\overline{\omega}}_m\right)$ *is a normalized interval weight vector. If* $\widetilde{B}' = \left\{\widetilde{\beta}'_\eta \middle| \widetilde{\beta}'_\eta = \left[u'_\eta, 1 - v'_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *is any permutation of* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$*. Then,*

$$IFP\overline{W}MSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) = IFP\overline{W}MSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}'_1, \widetilde{\beta}'_2, \cdots, \widetilde{\beta}'_t\right). \tag{75}$$

Theorem 17 (Boundedness)**.** *Let* $\widetilde{B} = \left\{\widetilde{\beta}_\eta \middle| \widetilde{\beta}_\eta = \left[u_\eta, 1 - v_\eta\right], \eta = 1, 2, \cdots, t\right\}$ *be the corresponding BI set of IFS* $B = \left\{\beta_\eta \middle| \beta_\eta = \left\langle u_\eta, v_\eta\right\rangle, \eta = 1, 2, \cdots, t\right\}$*.* $\hat{\omega}_i = \langle \hat{u}_i, \hat{v}_i\rangle$ *is the weight of* β_i *in the form of IFN,* $\hat{\omega}_i = [\hat{u}_i, 1 - \hat{v}_i]$ *is the corresponding BI of* $\hat{\omega}_i$*.* $\overline{\overline{\omega}} = \left(\overline{\overline{\omega}}_1, \overline{\overline{\omega}}_2, \cdots, \overline{\overline{\omega}}_m\right)$ *is a normalized interval weight vector. If* $\widetilde{\beta}_\eta^+ = \left[\max\limits_{\eta=1}^{t} u_\eta, \max\limits_{\eta=1}^{t}\left(1 - v_\eta\right)\right]$ *and* $\widetilde{\beta}_\eta^- = \left[\min\limits_{\eta=1}^{t} u_\eta, \min\limits_{\eta=1}^{t}\left(1 - v_\eta\right)\right]$*, then*

$$\widetilde{\beta}_\eta^- \leq IFPWMSM_{DST}{}^{(\kappa)}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots, \widetilde{\beta}_t\right) \leq \widetilde{\beta}_\eta^+. \tag{76}$$

4. A Novel MADM Method with IFNs in the Framework of DST

In this section, a new MADM method based on the proposed IFPWMSM$_{DST}$ operator and IFP$\overline{\text{W}}$MSM$_{DST}$ operator is developed to solve the MADM problems with IFNs in the framework of DST.

Considering a MADM problem, there are m alternatives, denoted by $E = \{e_1, e_2, \cdots, e_m\}$ and n attributes, denoted by $C = \{c_1, c_2, \cdots, c_n\}$. If the attribute weights are crisp numbers, then the weight vector is denoted by $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, satisfying $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1 (i = 1, 2, \cdots, n)$, or if the attribute weights are IFNs, $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \cdots, \overline{\omega}_n)$ with $\overline{\omega}_i = [u_i, 1 - v_i]$ for $i = 1, 2, \cdots, n$ is an interval weight vector. Let $R = \left(r_{ij}\right)_{m \times n}$ be a decision matrix, where r_{ij} is an evaluation value, which takes the form of a crisp number or IFN, given by decision makers for the alternative $e_i \in E$ with respect to the attribute $c_i \in C$. The purpose of this problem is to select the optimal alternatives.

The method for this MADM problem is shown in detail as follows:

Step 1. Normalize the decision matrix $R = \left(r_{ij}\right)_{m \times n}$. Only the cost criterion c_j, r_{ij} is normalized by using the converted formula (Note: The value converted using $r_{ij} = \left\langle v_{ij}, u_{ij}\right\rangle$ is still denoted by r_{ij}).

Step 2. Convert IFN r_{ij} to BI $\widetilde{\beta}_{ij}$.

Step 3. Calculate $Sup\left(\widetilde{\beta}_{ij}, \widetilde{\beta}_{i\varepsilon}\right)(i = 1, 2, \cdots m; j, \varepsilon = 1, 2, \cdots, n; j \neq \varepsilon)$, that is,

$$Sup\left(\widetilde{\beta}_{ij}, \widetilde{\beta}_{i\varepsilon}\right) = 1 - d_{BPA}\left(\widetilde{\beta}_{ij}, \widetilde{\beta}_{i\varepsilon}\right)(i = 1, 2, \cdots m; j, \varepsilon = 1, 2, \cdots, n; j \neq \varepsilon),$$

where $d_{BPA}\left(\widetilde{\beta}_{ij}, \widetilde{\beta}_{i\varepsilon}\right)$ is JD between $\widetilde{\beta}_{ij}$ and $\widetilde{\beta}_{i\varepsilon}$ in the framework of the DST.

Step 4. Calculate $T\left(\widetilde{\beta}_{ij}\right)$ of $\widetilde{\beta}_{ij}$ by the other $\widetilde{\beta}_{i\varepsilon}$, that is,

$$T\left(\widetilde{\beta}_{i\varepsilon}\right) = \sum_{\varepsilon=1, j\neq\varepsilon}^{m} Sup\left(\widetilde{\beta}_{ij}, \widetilde{\beta}_{i\varepsilon}\right)(i = 1, 2, \cdots m; j, \varepsilon = 1, 2, \cdots, n; j \neq \varepsilon) \quad .$$

Step 5. Calculate $\overline{\theta}_i$ or $\overline{\overline{\theta}}_i$. If $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \cdots, \overline{\omega}_n)$ with $\overline{\omega}_i = [u_i, 1 - v_i]$ for $i = 1, 2, \cdots, n$, satisfies $\sum_{i=1}^{n} u_i + \max_j\left(1 - v_j - u_j\right) \leq 1$ and $\sum_{i=1}^{n} (1 - v_i) - \max_j\left(1 - v_j - u_j\right) \geq 1$, then $\overline{\omega}$ is normalized. If $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \cdots, \overline{\omega}_n)$ is not a normalized interval weight vector (NIWV), then convert into a normalized weight vector (see [38]), that is,

$$u_i{}' = \min_{\substack{\overline{\omega} \in [u_j, (1 - v_j)], j = 1, 2, \cdots, n \\ \sum_{j=1}^{n} \overline{\omega}_j = 1}} \overline{\omega}_i = \max\left\{u_i, 1 - \sum_{j=1, j\neq i}^{n} \left(1 - v_j\right)\right\}, \quad i = 1, 2, \cdots, n \tag{77}$$

$$(1 - v_i)' = \max_{\substack{\overline{\omega} \in [u_j, (1 - v_j)], j = 1, 2, \cdots, n \\ \sum_{j=1}^{n} \overline{\omega}_j = 1}} \overline{\omega}_i = \max\left\{1 - v_i, 1 - \sum_{j=1, j\neq i}^{n} u_j\right\}, \quad i = 1, 2, \cdots, n \tag{78}$$

and then,

$$\overline{\overline{\theta}}_{ij} = \frac{\overline{\omega}_i\left(1 + T\left(\widetilde{\beta}_{ij}\right)\right)}{\sum_{k=1}^{n} \overline{\omega}_k\left(1 + T\left(\widetilde{\beta}_{ik}\right)\right)} \text{ or } \overline{\theta}_{ij} = \frac{\omega_i\left(1 + T\left(\widetilde{\beta}_{ij}\right)\right)}{\sum_{k=1}^{n} \overline{\omega}_k\left(1 + T\left(\widetilde{\beta}_{ik}\right)\right)}.$$

Step 6. Apply the proposed IFPWMSM$_{DST}$ operator or IFP$\overline{\text{W}}$MSM$_{DST}$ operator to acquire the comprehensive value $\widetilde{\beta}_i$ $(i = 1, 2, \cdots m)$ of each alternative.

Step 7. Calculate the $SF_{DST}\left(\widetilde{\beta}_i\right)$, $AF_{DST}\left(\widetilde{\beta}_i\right)$ by Equation (32) and Equation (33), respectively.

Step 8. Rank the alternatives and obtain the best alternative.

5. Practical Application

In this section, we apply an illustrated example of the share-bike evaluation to demonstrate the process of the novel method.

Example 6. *By increasing the value of customer experience, a share-bike operation company plans to put new share-bikes into market. Now, there are four different share-bikes ($E = \{e_1, e_2, e_3, e_4\}$) from four different share-bike manufacturers, but the share-bike operation company does not know which one is best. Therefore, they invite a tester to evaluate the four different share-bikes. Suppose four attributes are considered, containing safety (c_1), comfortability (c_2), convenience (c_3) and aesthetic (c_4). The weight vector ω of the attribute is $\omega = (0.4, 0.3, 0.2, 0.1)^T$. The assessment value r_{ij} of criterion $c_j (j = 1, 2, 3, 4)$ with the alternative $e_i (i = 1, 2, 3, 4)$ takes the form of the IFN, and the collected and processed decision matrix is constructed, as shown in Table 1.*

Table 1. The decision matrix represented by intuitionistic fuzzy number (IFNs).

	c_1	c_2	c_3	c_4
e_1	$\langle 0.6, 0.1 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.7, 0.1 \rangle$	$\langle 0.4, 0.3 \rangle$
e_2	$\langle 0.7, 0.2 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.5, 0.4 \rangle$	$\langle 0.5, 0.3 \rangle$
e_3	$\langle 0.3, 0.3 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.6, 0.1 \rangle$
e_4	$\langle 0.6, 0.3 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.4, 0.4 \rangle$	$\langle 0.5, 0.3 \rangle$

5.1. Rank the Alternatives by the New Method Based on $IFWPMSM_{DST}$ Operator

In this section, we present the detailed calculation process of the novel method based on the $\mathrm{IFWPMSM}_{DST}$ operator.

Step 1: Normalize the IFN matrix $R = \left(r_{ij}\right)_{m \times n}$.

Because the four attributes are beneficial, it is not essential to perform normalization.

Step 2: Convert IFN r_{ij} to BI $\widetilde{\beta}_{ij}$ and BPAs.

The converted results of the IFNs are shown in Table 2.

Table 2. The belief intervals (Bis) from the IFNs.

	c_1	c_2	c_3	c_4
b_1	$[0.6, 0.9]$	$[0.7, 0.7]$	$[0.7, 0.9]$	$[0.4, 0.7]$
b_2	$[0.7, 0.8]$	$[0.6, 0.9]$	$[0.5, 0.6]$	$[0.5, 0.7]$
b_3	$[0.3, 0.7]$	$[0.6, 0.8]$	$[0.7, 0.8]$	$[0.6, 0.9]$
b_4	$[0.6, 0.7]$	$[0.5, 0.8]$	$[0.4, 0.6]$	$[0.5, 0.7]$

The BPAs for $r_{ij} (i, j = 1, 2, 3, 4)$ are shown in Table 3.

Table 3. The basic probability assignments (BPas) for $r_{ij}(i,j=1,2,3,4)$.

	$s(T)$	$s(F)$	$s(T\ or\ F)$		$s(T)$	$s(F)$	$s(T\ or\ F)$
f_{11}	0.6	0.1	0.3	f_{31}	0.3	0.3	0.4
f_{12}	0.7	0.3	0	f_{32}	0.6	0.2	0.2
f_{13}	0.7	0.1	0.2	f_{33}	0.7	0.2	0.1
f_{14}	0.4	0.3	0.3	f_{34}	0.6	0.1	0.3
	$s(T)$	$s(F)$	$s(T\ or\ F)$		$s(T)$	$s(F)$	$s(T\ or\ F)$
f_{21}	0.7	0.2	0.1	f_{41}	0.6	0.3	0.1
f_{22}	0.6	0.1	0.3	f_{42}	0.5	0.2	0.3
f_{23}	0.5	0.4	0.1	f_{43}	0.4	0.4	0.2
f_{24}	0.5	0.3	0.2	f_{44}	0.5	0.3	0.2

Step 3: Calculating $Sup\left(\widetilde{\beta}_{ij},\widetilde{\beta}_{i\varepsilon}\right)$ $(i=1,2,\cdots 4;\ j,\varepsilon=1,2,\cdots,4;\ j\neq\varepsilon)$, we have

$$Sup\left(\widetilde{\beta}_{1j},\widetilde{\beta}_{1r}\right)=Sup\left(\widetilde{\beta}_{1r},\widetilde{\beta}_{1j}\right)=\begin{bmatrix}1.0000 & 0.9186 & 0.9065 & 0.7439\\ 0.9186 & 1.0000 & 0.7976 & 0.7095\\ 0.9065 & 0.7976 & 1.0000 & 0.6485\\ 0.7439 & 0.7095 & 0.6485 & 1.0000\end{bmatrix},$$

$$Sup\left(\widetilde{\beta}_{2j},\widetilde{\beta}_{2r}\right)=Sup\left(\widetilde{\beta}_{2r},\widetilde{\beta}_{2j}\right)=\begin{bmatrix}1.0000 & 0.7959 & 0.7127 & 0.6905\\ 0.7959 & 1.0000 & 0.7087 & 0.8463\\ 0.7127 & 0.7087 & 1.0000 & 0.8069\\ 0.6905 & 0.8463 & 0.8069 & 1.0000\end{bmatrix},$$

$$Sup\left(\widetilde{\beta}_{3j},\widetilde{\beta}_{3r}\right)=Sup\left(\widetilde{\beta}_{3r},\widetilde{\beta}_{3j}\right)=\begin{bmatrix}1.0000 & 0.8063 & 0.7962 & 0.8659\\ 0.8063 & 1.0000 & 0.6980 & 0.7743\\ 0.7962 & 0.6980 & 1.0000 & 0.8964\\ 0.8659 & 0.7743 & 0.8964 & 1.0000\end{bmatrix},$$

$$Sup\left(\widetilde{\beta}_{4j},\widetilde{\beta}_{4r}\right)=Sup\left(\widetilde{\beta}_{4r},\widetilde{\beta}_{4j}\right)=\begin{bmatrix}1.0000 & 0.8863 & 0.7085 & 0.9065\\ 0.8863 & 1.0000 & 0.6996 & 0.8549\\ 0.7085 & 0.6996 & 1.0000 & 0.6900\\ 0.9065 & 0.8549 & 0.6900 & 1.0000\end{bmatrix}.$$

Step 4: Calculate the $T\left(\widetilde{\beta}_{ij}\right)$ of $\widetilde{\beta}_{ij}$ by the other $\widetilde{\beta}_{ik}$, that is,

$$T\left(\widetilde{h}_{ij}\right)=\begin{bmatrix}2.4977 & 2.5630 & 2.3329 & 2.0064\\ 2.2296 & 2.3027 & 2.1284 & 2.3344\\ 2.3964 & 2.3753 & 2.6608 & 2.6570\\ 2.3806 & 2.4469 & 2.2081 & 2.3597\end{bmatrix}.$$

Step 5. Calculate $\overline{\theta}_{ij}$, and we have

$$\overline{\theta}_{ij}=\begin{bmatrix}0.4103 & 0.3157 & 0.1916 & 0.0824\\ 0.3978 & 0.3082 & 0.1899 & 0.1041\\ 0.3882 & 0.2886 & 0.2155 & 0.1076\\ 0.4028 & 0.3105 & 0.1868 & 0.0998\end{bmatrix}.$$

Step 6: Apply the proposed IFPWMSM$_{\text{DST}}$ operator shown in Equation (50) to get the comprehensive value $\widetilde{\beta_i}$ $(i = 1, 2, \cdots 4)$ of each alternative $(\kappa = 2)$.

$$IFPWMSM_{DST}\left(\widetilde{\beta_1}\right) = [0.1958, 0.2387],\quad IFPWMSM_{DST}\left(\widetilde{\beta_2}\right) = [0.1609, 0.2037],$$
$$IFPWMSM_{DST}\left(\widetilde{\beta_3}\right) = [0.1496, 0.2102],\quad IFPWMSM_{DST}\left(\widetilde{\beta_4}\right) = [0.1105, 0.1864].$$

Step 7: Calculate the $SF_{DST}\left(\widetilde{\beta_i}\right)$ by Equation (31), and we have

$$SF_{DST}\left(\widetilde{\beta_1}\right) = 0.2173, SF_{DST}\left(\widetilde{\beta_2}\right) = 0.1823, SF_{DST}\left(\widetilde{\beta_3}\right) = 0.1800, SF_{DST}\left(\widetilde{\beta_4}\right) = 0.1485.$$

Step 8: Rank the alternatives and obtain the best alternative.

Because $SF_{DST}\left(\widetilde{\beta_1}\right) > SF_{DST}\left(\widetilde{\beta_2}\right) > SF_{DST}\left(\widetilde{\beta_3}\right) > SF_{DST}\left(\widetilde{\beta_4}\right)$, the ranking order is $e_1 > e_2 > e_3 > e_4$, and the best share-bike is e_1.

5.2. The Influence of the Parameter κ on Ranking Results

Further, to analyze the influence of parameter κ on the ranking results, we assign a distinct parameter κ in the presented novel method to solve the above example, and the ranking orders are shown in Table 4.

Table 4. Ranking orders of the alternatives for different parameter κ.

κ	$SF_{DST}\left(\widetilde{\beta_i}\right)$	**Ranking Orders**
$\kappa = 1$	$\widetilde{S_1} = 0.1859, \widetilde{S_2} = 0.1605, \widetilde{S_3} = 0.1553, \widetilde{S_4} = 0.1468$	$e_1 > e_2 > e_4 > e_3$
$\kappa = 2$	$\widetilde{S_1} = 0.2173, \widetilde{S_2} = 0.1823, \widetilde{S_3} = 0.1800, \widetilde{S_4} = 0.1485$	$e_1 > e_2 > e_3 > e_4$
$\kappa = 3$	$\widetilde{S_1} = 0.2012, \widetilde{S_2} = 0.1760, \widetilde{S_3} = 0.1627, \widetilde{S_4} = 0.1377$	$e_1 > e_2 > e_3 > e_4$
$\kappa = 4$	$\widetilde{S_1} = 0.1986, \widetilde{S_2} = 0.1605, \widetilde{S_3} = 0.1613, \widetilde{S_4} = 0.1285$	$e_1 > e_3 > e_2 > e_4$

Note: $\widetilde{S_i}$ is abbreviation of score value $SF_{DST}(b_i)$.

From Table 4, we can see that the ranking orders of the four alternatives with different κ parameters are different. However, the best share-bike does not change by a different parameter κ, which is still e_1. This is, in all probability, because of the fact that the proposed novel method allows for more interacted attributes with an increase of the value of parameter κ. When $\kappa = 1$, the proposed novel method does not take interrelationship among attributes into account, and the ranking order is separate from the ones when $\kappa = 2$, $\kappa = 3$, and $\kappa = 4$. Clearly, this can illuminate the significance of taking interrelationship among attributes into account because there is a universal interrelationship among more than two attributes in the practice DM environment.

In addition, we also can find that the $SF_{DST}\left(\widetilde{\beta_i}\right)$ $(i = 1, 2, \cdots 4)$ of $e_i (i = 1, 2, \cdots 4)$ decreases with an increase of value of parameter κ. Based on this case, parameter κ can be used as the risk preference of the tester. For example, if the tester is risk-averse, then he/she can select a smaller value for parameter κ. Under normal circumstances, $\kappa = [n/2]$ is a suitable value, where n is the number of attributes and the symbol $[\cdot]$ is the round function.

5.3. The Verification of the Effectiveness

To demonstrate the plausibility and validity of the presented novel method based on the IFPWMSM$_{\text{DST}}$ operator, we deal with the same share-bike problem in Section 5.1 by applying the four existing methods, Jiang and Wei's method [11], based on the intuitionistic fuzzy evidential power aggregation (IFEPA) operator, He and He's method [28], based on the extended weighted intuitionistic fuzzy interaction Bonferroni mean (EWIFIBM) operator, Qin and Liu's method [31], based on the weighted intuitionistic fuzzy MSM (WIFMSM) operator, where we let $\kappa = 2$ for the presented novel method, and Qin and Liu's method [28]; let $\lambda = 1, p = 1$, and $q = 1$ for He and He's method [31]. The ranking orders of different methods are shown in Table 5.

Table 5. The ranking orders of different methods for Example 6.

Methods	Score Values	Ranking Orders
Jiang and Wei's method [11] based on IFEPA operator	$S_1 = 0.4629,\ S_2 = 0.4086,\ S_3 = 0.3807,\ S_4 = 0.3518$	$e_1 \succ e_2 \succ e_4 \succ e_3$
He and He's method [28] based on EWIFIBM operator	$S_1 = 0.2950,\ S_2 = 0.2749,\ S_3 = 0.2664,\ S_4 = 0.2215$	$e_1 \succ e_3 \succ e_2 \succ e_4$
Qin and Liu's method [31] based on WIFMSM	$S_1 = 0.1994,\ S_2 = 0.1806,\ S_3 = 0.1594,\ S_4 = 0.1447$	$e_1 \succ e_2 \succ e_3 \succ e_4$
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator	$\widetilde{S}_1 = 0.2173,\ \widetilde{S}_2 = 0.1823,\ \widetilde{S}_3 = 0.1800,\ \widetilde{S}_4 = 0.1485$	$e_1 \succ e_3 \succ e_2 \succ e_4$

Note: S_i is an abbreviation of score value $SF(b_i)$, $\widetilde{S}_i$ is an abbreviation of score value $SF_{DST}(b_i)$.

From Table 5, we get a desirable outcome, that is, the ranking order based on proposed novel method is same as existing three methods [11,28,31]. Therefore, the proposed novel method is effective.

5.4. The Advantages Compared with the Existing Methods

In Section 5.3, the effectiveness of the proposed novel method is verified. However, owing to the same ranking orders, it is not possible to highlight visually the advantages of the proposed novel method and limitations of the existing some method in [11,28,31]. Accordingly, we apply the proposed novel method and the existing three methods in [11,28,31] to deal with new numerical practice examples.

5.4.1. Considering the Interrelationship among Attributes

Example 7. *In order to maintain the long-term stability of their high-quality talent, a company wants to lease a dorm to them. Now, there are four alternatives ($E = \{e_1, e_2, e_3, e_4\}$) from four different residential districts. Suppose four attributes are considered, consisting of cost (c_1), comfortability (c_2), convenience (c_3), and living spaces (c_4). The weight vector ω of the attribute is $\omega = (0.4, 0.1, 0.2, 0.3)^T$. The assessment value r_{ij} of criterion $c_j (j = 1, 2, 3, 4)$ with alternative $e_i (i = 1, 2, 3, 4)$ takes the form of IFN, and the collected and processed decision matrix is constructed, as shown in Table 6. The ranking orders are shown in Table 7.*

Table 6. The decision matrix represented by IFNs.

	c_1	c_2	c_3	c_4
e_1	$\langle 0.4, 0.3 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.6, 0.1 \rangle$
e_2	$\langle 0.6, 0.3 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.6, 0.3 \rangle$	$\langle 0.6, 0.2 \rangle$
e_3	$\langle 0.4, 0.3 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.5, 0.2 \rangle$
e_4	$\langle 0.6, 0.3 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.4, 0.2 \rangle$	$\langle 0.6, 0.2 \rangle$

From Table 6, it is easy to see that when $\kappa = 1$ the ranking order based on the presented method is the same as the one based on Jiang and Wei's method [11], i.e., $e_3 \succ e_2 \succ e_1 \succ e_4$. To put it other way, when $\kappa = 1$, the presented method does not take into consideration the interrelationship of attributes. In this instance, the presented method is similar to Jiang and Wei's method [11], which simply offers a weighted average function. It is obvious that these ranking orders may not be reasonable, because, in this example, there is a dependable interrelationship between cost, comfort, and living spaces. In a real DM environment, we should consider this interaction among attributes. At the same time, it is also easy to find that when $\kappa = 2$ and $\kappa = 3$, the ranking orders based on presented method is the same as the one based on Qin and Liu's method [31] and He and He's method [28], i.e., $e_3 \succ e_2 \succ e_4 \succ e_1$. Obviously, when $\kappa = 2$ and $\kappa = 3$, the presented method considers this interaction among attributes, which is similar to Qin and Liu's method [31] and He and He's method [28]. Undoubtedly, these ranking orders are more reasonable.

Table 7. The ranking orders of the different methods for Example 7.

Methods	Score Values	Ranking Orders
Jiang and Wei's method [11] based on IFEPA operator	$S_1 = 0.2180, S_2 = 0.2219, S_3 = 0.2309, S_4 = 0.1994$	$e_3 \succ e_2 \succ e_1 \succ e_4$
Qin and Liu's method [31] based on WIFMSM ($\kappa = 2$)	$S_1 = 0.1769, S_2 = 0.1905, S_3 = 0.1976, S_4 = 0.1870$	$e_3 \succ e_2 \succ e_4 \succ e_1$
He and He's method [28] based on EWIFIBM operator ($\lambda = 1, p = 1$ and $q = 1$)	$S_1 = 0.3017, S_2 = 0.3428, S_3 = 0.3592, S_4 = 0.3307$	$e_3 \succ e_2 \succ e_4 \succ e_1$
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator ($\kappa = 1$)	$\widetilde{S}_1 = 0.2180, \widetilde{S}_2 = 0.2286, \widetilde{S}_3 = 0.2375, \widetilde{S}_4 = 0.2005$	$e_3 \succ e_2 \succ e_1 \succ e_4$
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator ($\kappa = 2$)	$\widetilde{S}_1 = 0.1802, \widetilde{S}_2 = 0.1973, \widetilde{S}_3 = 0.2095, \widetilde{S}_4 = 0.1965$	$e_3 \succ e_2 \succ e_4 \succ e_1$
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator ($\kappa = 3$)	$\widetilde{S}_1 = 0.1758, \widetilde{S}_2 = 0.1906, \widetilde{S}_3 = 0.1984, \widetilde{S}_4 = 0.1863$	$e_3 \succ e_2 \succ e_4 \succ e_1$

Note: S_i is an abbreviation of score value $SF(b_i)$, $\widetilde{S}_i$ is an abbreviation of score value $SF_{DST}(b_i)$.

5.4.2. Reducing the Influence of Extreme Evaluation Values

Example 8. *In many cases, due to the preferences of decision makers, extreme evaluation values of attributes may be provided, i.e., values that are too high or too low. Thus, it is possible to get some unreasonable ranking orders. To illustrate this case, based on Example 7, we change the value of* r_{11} *to* $\langle 0.99, 0.01 \rangle$ *and change the value of* r_{44} *to* $\langle 0.01, 0.01 \rangle$*. Then, we obtain a new decision matrix, which is shown in Table 8. The ranking orders are shown in Table 9.*

Table 8. The changed decision matrix represented by IFNs based on Example 7.

	c_1	c_2	c_3	c_4
e_1	$\langle 0.99, 0.01 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.6, 0.1 \rangle$
e_2	$\langle 0.6, 0.3 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.6, 0.3 \rangle$	$\langle 0.6, 0.2 \rangle$
e_3	$\langle 0.4, 0.3 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.5, 0.2 \rangle$
e_4	$\langle 0.6, 0.3 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.4, 0.2 \rangle$	$\langle 0.01, 0.01 \rangle$

From Table 9, it is easy to see that too high an evaluation value $\langle 0.99, 0.01 \rangle$ and too low an evaluation value $\langle 0.01, 0.01 \rangle$ have pivotal effects on the ranking orders based on Qin and Liu's method [31] and He and He's method [28]. Their ranking orders are changed from $e_3 \succ e_2 \succ e_4 \succ e_1$ to $e_1 \succ e_3 \succ e_2 \succ e_4$, and the best alternative is also changed from e_3 to e_1. Evidentially, the ranking order may be unreasonable. In other words, in real DM problems, decision-makers may become a manipulator by giving some extreme and biased evaluation values. Nevertheless, the ranking orders based on Jiang and Wei's method [11] and the presented method are still reasonable. Although some ranking orders have changed, their best alternative is still the same as the one acquired in Example 7, i.e., e_3 and the influence of extreme evaluation values is not very strong.

Obviously, the reason for the favorable ranking orders is that the PA operator can significantly reduce the influence of extreme evaluation values by inputting different weights. When decision-makers give extreme evaluation values, the PA operator can give these attributes a relatively smaller weight by the support degrees (SDs) between attributes. Under such circumstances, the influence of extreme evaluation values on ranking orders fades.

Table 9. The ranking orders of the different methods for Example 8.

Methods	Score Values	Ranking Orders
Jiang and Wei's method [11] based on IFEPA operator	$S_1 = 0.2216,\ S_2 = 0.2219,\ S_3 = 0.2309,\ S_4 = 0.1549$	$e_3 \succ e_2 \succ e_1 \succ e_4$
Qin and Liu's method [31] based on WIFMSM ($\kappa = 2$)	$S_1 = 0.2185,\ S_2 = 0.1905,\ S_3 = 0.1976,\ S_4 = 0.1439$	$e_1 \succ e_3 \succ e_2 \succ e_4$
He and He's method [28] based on EWIFIBM operator ($\lambda = 1, p = 1$ and $q = 1$)	$S_1 = 0.3682,\ S_2 = 0.3428,\ S_3 = 0.3592,\ S_4 = 0.2805$	$e_1 \succ e_3 \succ e_2 \succ e_4$
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator ($\kappa = 1$)	$\widetilde{S}_1 = 0.2281,\ \widetilde{S}_2 = 0.2286,\ \widetilde{S}_3 = 0.2375,\ \widetilde{S}_4 = 0.1759$	$e_3 \succ e_2 \succ e_1 \succ e_4$
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator ($\kappa = 2$)	$\widetilde{S}_1 = 0.1965,\ \widetilde{S}_2 = 0.1973,\ \widetilde{S}_3 = 0.2095,\ \widetilde{S}_4 = 0.1630$	$e_3 \succ e_2 \succ e_4 \succ e_1$
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator ($\kappa = 3$)	$\widetilde{S}_1 = 0.1896,\ \widetilde{S}_2 = 0.1906,\ \widetilde{S}_3 = 0.1984,\ \widetilde{S}_4 = 0.1649$	$e_3 \succ e_2 \succ e_4 \succ e_1$

Note: S_i is an abbreviation of score value $SF(b_i)$, $\widetilde{S}_i$ is an abbreviation of score value $SF_{DST}(b_i)$.

5.4.3. The Attribute Weights can be Denoted by IFNs

Example 9. *From Section 2, we know that the ordinary operation laws of IFS may lead to unreasonable ranking orders because of the unfavorable properties and because the AOs presented by IFNs are not developed. Undoubtedly, these are critical defects in the ordinary operation laws of the IFS. However, the operation laws of the IFS in the framework of the DST proposed in this paper can overcome these critical problems by solving the MADM problems without intermediate defuzzification when the attributes and their weights are IFNs. To illustrate the advantage of this proposed method based on the* $\text{IFPWMSM}_{\text{DST}}$ *operator, and based on Example 7, we initially change* $\omega = (0.4, 0.1, 0.2, 0.3)^T$ *to* $\hat{\omega} = (\langle 0.3, 0.3\rangle, \langle 0.2, 0.6\rangle, \langle 0.1, 0.4\rangle, \langle 0.3, 0.5\rangle)^T$. *The corresponding interval weights of* $\hat{\omega}$ *are* $\hat{\omega} = ([0.3, 0.7], [0.2, 0.4], [0.1, 0.6], [0.3, 0.5])$. *From Step 5 in Section 4, we know that* $\hat{\omega}$ *is NIWV. The ranking orders are listed in Table 10.*

Table 10. The ranking orders of different methods for Example 9.

Methods	Score Values	Ranking Orders
Jiang and Wei's method [11] based on IFEPA operator	Cannot be counted	Cannot be ranked
He and He's method [28] based on EWIFIBM operator	Cannot be counted	Cannot be ranked
Qin and Liu's method [31] based on WIFMSM	Cannot be counted	Cannot be ranked
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator	$\widetilde{S}_1 = 0.1904,\ \widetilde{S}_2 = 0.2296,\ \widetilde{S}_3 = 0.2408,\ \widetilde{S}_4 = 0.2075$	$e_3 \succ e_2 \succ e_4 \succ e_1$

Note: $\widetilde{S}_i$ is an abbreviation of score value $SF_{DST}(b_i)$.

From Table 10, we can see that only the proposed method based on the $\text{IFPWMSM}_{\text{DST}}$ operator can give a ranking order, i.e., $e_3 \succ e_2 \succ e_4 \succ e_1$. Consequently, the operation laws of the IFS in the framework of DST proposed in this paper can augment the function of AOs.

On the strength of the above three examples, the limitations of the existing methods [11,28,31] are analyzed and summarized as follows:

(1) With regard to method [11], on the one hand, this method does not take into account the interrelationships of attributes. In Example 7, we point out that in some real circumstances, it is meaningful to consider the interrelationships of attributes, but this method can only deal with MADM problems in which attributes are independent of each other. On the other hand, because there are not variable parameters, this method cannot manifest the decision-makers' subject preference, so it does not apply to some experts with risk attitudes.

(2) With regard to method [28], for one thing, this method cannot reduce the influence of extreme evaluation values. For another, it only considers the interrelationships between attributes, but it cannot capture the interrelationships among attributes.
(3) With regard to method [31], although this method can take into account the interrelationships among attributes, it similarly cannot overcome the drawbacks from the influence of extreme evaluation values.

Most crucially, compared with the presented method based on the $\text{IFPWMSM}_{\text{DST}}$ operator, the most obvious limitations of methods [11,28,31] is that they cannot be calculated when attribute weights are IFNs and may obtain some unfavorable ranking orders because of operation laws.

In the following, we make a comparison of the characteristics of the presented method based on the $\text{IFPWMSM}_{\text{DST}}$ operator with existing methods [11,28,31]. This comparison is shown in Table 11. We can see that the presented method based on the $\text{IFPWMSM}_{\text{DST}}$ operator is free from the drawbacks of the three existing methods [11,28,31] and is more extensive and flexible in dealing with MADM problems.

Table 11. The comparison results of the characteristics for the different methods.

Method	Whether it Eliminates the Effects of Biased Values	Whether it Considers the Interrelationships among Attributes	Whether Attribute Weights Can Be Denoted by IFNs	Whether it Overcomes the Drawbacks of the Ordinary Operation Laws of the IFS
Jiang and Wei's method [11] based on IFEPA operator	Yes	No	No	No
He and He's method [28] based on EWIFIBM operator	No	No	No	No
Qin and Liu's method [31] based on WIFMSM	No	Yes	No	No
The proposed method based on $\text{IFPWMSM}_{\text{DST}}$ operator	Yes	Yes	Yes	Yes

6. Conclusions

There exists a close and forceful relationship between the IFS and the DST. In the framework of the DST, an IFN can be converted into a BPA, and mathematical operations on IFNs are represented as operations on Bis, which can overcome the drawbacks of the OORs of IFNs. In this case, we can utilize JD to measure the differences between IFNs. We all know that decision-makers may provide extreme evaluation values in practice MADM problems, and we should consider the interrelationships among attributes in many cases. Therefore, based on the characteristics of the MSM operator and PA operator, we developed two novel aggregate operators in the framework of the DST, i.e., an $\text{IFPMSM}_{\text{DST}}$ operator and an $\text{IFPWMSM}_{\text{DST}}$ operator. In addition, we discussed the properties of above two new aggregate operators. Then, we proposed a novel MADM method based on the $\text{IFPWMSM}_{\text{DST}}$ operator, which can overcome drawbacks of some existing methods [11,28,31], where they cannot be calculated when attribute weights are IFNs and may obtain some unfavorable ranking orders because of operation laws. Finally, some examples were utilized to demonstrate that the presented methods outperform the previous ones [11,28,31]. In the future, we will apply the $\text{IFPWMSM}_{\text{DST}}$ operator to solve multi-attribute group decision-making problems in the framework of the DST. We will also power the MSM operator to aggregate other fuzzy information, such as interval intuitionistic fuzzy sets, hesitant fuzzy sets, and so on. Further, we will use the presented method to deal with some practice MADM problems, such as green supplier selection, disease diagnosis, and so on.

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