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Iterative Algorithms for a System of Variational Inclusions in Banach Spaces

Lu-Chuan Ceng [1], Mihai Postolache [2,3,4,*] and Yonghong Yao [5,6]

1 Department of Mathematics, Shanghai Normal University, Shanghai 200234, China; zenglc@hotmail.com
2 Center for General Education, China Medical University, Taichung 40402, Taiwan
3 Romanian Academy, Gh. Mihoc-C. Iacob Institute of Mathematical Statistics and Applied Mathematics, 050711 Bucharest, Romania
4 University "Politehnica" of Bucharest, Department of Mathematics and Informatics, 060042 Bucharest, Romania
5 School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China; yaoyonghong@aliyun.com
6 The Key Laboratory of Intelligent Information and Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China
* Correspondence: emscolar@yahoo.com

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Abstract: A system of variational inclusions (GSVI) is considered in Banach spaces. An implicit iterative procedure is proposed for solving the GSVI. Strong convergence of the proposed algorithm is given.

Keywords: system of variational inclusions; accretive mapping; strict pseudocontraction; implicit iterative procedure

MSC: 47H05; 47H10; 47J25

1. Introduction

Let X be a smooth Banach space and $\emptyset \neq C \subset X$ a closed convex set. Let $A_1, A_2 : C \to X$ and $M_1, M_2 : C \to 2^X$ be nonlinear mappings. In the present article, we consider the following system of variational inclusions (GSVI, for short) which aims to seek $(u^*, v^*) \in C \times C$ verifying

$$\begin{cases} 0 \in \varsigma_1(A_1 v^* + M_1 u^*) + u^* - v^*, \\ 0 \in \varsigma_2(A_2 u^* + M_2 v^*) + v^* - u^*, \end{cases} \tag{1}$$

where ς_1 and ς_2 are two positive constants.

Special cases: If $A_1 = A_2 = A$ and $M_1 = M_2 = M$, then the relation (1) reduces to seek $(u^*, v^*) \in C \times C$ verifying

$$\begin{cases} 0 \in \varsigma_1(Av^* + Mu^*) + u^* - v^*, \\ 0 \in \varsigma_2(Au^* + Mv^*) + v^* - u^*. \end{cases} \tag{2}$$

If $u^* = v^*$ in (2), then the relation (2) reduces to seek $(u^*, v^*) \in C \times C$ verifying

$$0 \in Au^* + Mu^*. \tag{3}$$

Especially, if $M = \partial\phi$, where $\phi : H \to R \cup +\infty$ is a proper convex lower semi-continuous function, then we have the following mixed quasi-variational inequality

$$\langle Au, y - u\rangle + \phi(y) - \phi(u)\rangle \geq 0, \forall y \in H.$$

Variational inequalities and variational inclusions have played vital roles in practical applications. Numerous iterative procedures for approaching variational inequalities and variational inclusions have been computed by the researchers [1–29].

In [4], the authors introduced an iterative procedure for approaching GSVI (1). Qin et al. [30] suggested an extragradient algorithm for solving GSVI (1), and demonstrated the strong convergence analysis of the presented algorithm. Lan et al. [28], Buong et al. [11], Zhang et al. [13] studied iterative procedures for approaching variational inclusion (3).

On the other hand, iterative computation of zeros or fixed points of nonlinear operators has been studied extensively in the literature [14,31–36]. Zhang et al. [37] introduced an iterative procedure for approaching a solution of the inclusion problem (3) and a fixed point of a nonexpansive mapping in Hilbert spaces. Peng et al. [38] presented a viscosity algorithm for finding a solution of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem and a fixed point of a nonexpansive mapping.

Motivated by the above work, in the present paper, we consider the GSVI (1) with the hierarchical variational inequality constraint for a strict pseudocontraction T in Banach spaces. We suggest an implicit iterative procedure for solving the GSVI (1) with the HVI constraint for strict pseudocontraction T. We show the strong convergence of the suggested procedure to a solution of the GSVI (1).

2. Preliminaries

Let X be a real Banach space and $\emptyset \neq C \subset X$ a closed convex set. A mapping $f : C \to C$ is said to be k-Lipschitz if $\|f(u) - f(v)\| \leq k\|u - v\|, \forall u, v \in C$ for some $k \geq 0$. If $k < 1$, then f is said to be a k-contraction. If $k = 1$, then f is said to be nonexpansive.

Recall that an operator $T : C \to X$ is called

(i) accretive if

$$\langle Tu - Tv, j(u - v)\rangle \geq 0, \ \forall u, v \in C,$$

where $j(u - v) \in J(u - v)$.

(ii) α-inverse-strongly accretive if

$$\langle Tu - Tv, j(u - v)\rangle \geq \alpha\|Tu - Tv\|^2, \ \forall u, v \in C,$$

where $j(u - v) \in J(u - v)$ and $\alpha > 0$.

(iii) strictly pseudocontractive if

$$\langle Tu - Tv, j(u - v)\rangle \leq \|u - v\|^2 - \beta\|u - v - (Tu - Tv)\|^2, \ \forall u, v \in C,$$

where $j(u - v) \in J(u - v)$ and $\beta > 0$.

If X is q-uniformly smooth with $1 < q \leq 2$, then

$$\|u + v\|^q + \|u - v\|^q \leq 2(\|u\|^q + \|cv\|^q), \ \forall u, v \in X,$$

where $c > 0$ is some constant.

Proposition 1 ([32])**.** *In a smooth and uniformly convex Banach space X, for all $u, v \in B_r = \{u \in X : \|u\| \leq r\}$, there holds*

$$g(\|u - v\|) \leq \|u\|^2 - 2\langle u, j(v)\rangle + \|v\|^2,$$

where $g : [0, 2r] \to \mathbf{R}$ *is a strictly increasing, continuous, and convex function satisfying* $g(0) = 0$.

Proposition 2 ([35])**.** *In a 2-uniformly smooth Banach space X, there holds*

$$\|u+v\|^2 \le \|u\|^2 + 2\langle v, j(u)\rangle + 2\|cv\|^2, \ \forall u, v \in X.$$

Let $D \subset C$ and $\Pi : C \to D$ be an operator. If $\Pi[(1-s)\Pi(u) + su] = \Pi(u)$, whenever $(1-s)\Pi(u) + su \in C$ for $u \in C$ and $s \ge 0$, we call Π is sunny.

Proposition 3 ([26])**.** *Let X be a smooth Banach space and* $\emptyset \ne C \subset X$ *a closed convex set. Let* $\emptyset \ne D \subset C$ *be a set and* $\Pi : C \to D$ *be a retraction. Then the following conclusions are equivalent:*

(i) $\langle \Pi(u) - u, j(v - \Pi(u))\rangle \ge 0, \forall u \in C$ *and* $\forall v \in D$;
(ii) $\|\Pi(u) - \Pi(v)\|^2 \le \langle u - v, j(\Pi(u) - \Pi(v))\rangle, \forall u, v \in C$;
(iii) Π *is sunny nonexpansive operator.*

If an accretive operator M satisfies $R(I + rM) = X$ for each $r > 0$, then M is said to be m-accretive. Assume that an accretive M satisfies the range condition $\overline{D(M)} \subset R(I + rM)$. Define the resolvent $J_r^M : R(I + rM) \to D(M)$ of M by $J_r^M = (I + rM)^{-1}$. Note that J_r^M is nonexpansive and $F(J_r^M) = M^{-1}0 = \{x \in D(M) : 0 \in Mx\}$ [31]. If $M^{-1}0 \ne \emptyset$, then the inclusion $0 \in Mx$ is solvable.

Lemma 1. *Let X be a smooth Banach space and* $\emptyset \ne C \subset X$ *a closed convex set. Let* $M : C \to 2^X$ *be an* m*-accretive operator. Then, for any given* $r > 0$,

$$\|J_r^M x - J_r^M y\|^2 \le \langle x - y, j(J_r^M x - J_r^M y)\rangle, \ \forall x, y \in X.$$

This means that $J_r^M : X \to C$ *is nonexpansive.*

Proof. Put $u = J_r^M x$ and $v = J_r^M y$. Then we have $x \in (I + rM)u$ and $y \in (I + rM)v$. Hence, there exist $\tilde{u} \in Mu$ and $\tilde{v} \in Mv$ such that $x = u + r\tilde{u}$ and $y = v + r\tilde{v}$. Utilizing the accretiveness of M, we obtain

$$\begin{aligned}\langle x - y, j(J_r^M x - J_r^M y)\rangle &= \langle u + r\tilde{u} - (v + r\tilde{v}), j(u - v)\rangle \\ &= \langle u - v, j(u - v)\rangle + r\langle \tilde{u} - \tilde{v}, j(u - v)\rangle \\ &= \|u - v\|^2 + r\langle \tilde{u} - \tilde{v}, j(u - v)\rangle \\ &\ge \|u - v\|^2 \\ &= \|J_r^M x - J_r^M y\|^2.\end{aligned}$$

□

Lemma 2. *Let* $M_1, M_2 : C \to 2^X$ *be two* m*-accretive operators and* $A_1, A_2 : C \to X$ *be two operators.* (x^*, y^*) *is a solution of the GSVI (1) iff* $Qx^* = J_{\varsigma_1}^{M_1}(I - \varsigma_1 A_1)J_{\varsigma_2}^{M_2}(I - \varsigma_2 A_2)x^*$, *where* $y^* = J_{\varsigma_2}^{M_2}(I - \varsigma_2 A_2)x^*$.

Proof. Observe that

$$\begin{cases} 0 \in x^* - y^* + \varsigma_1(A_1 y^* + M_1 x^*) \\ 0 \in y^* - x^* + \varsigma_2(A_2 x^* + M_2 y^*) \end{cases} \Leftrightarrow \begin{cases} x^* = J_{\varsigma_1}^{M_1}(I - \varsigma_1 A_1)y^*, \\ y^* = J_{\varsigma_2}^{M_2}(I - \varsigma_2 A_2)x^* \end{cases} \Leftrightarrow \ x^* = Qx^*.$$

□

Lemma 3 ([3])**.** *Let X be a strictly convex Banach space and* $\emptyset \ne C \subset X$ *a closed convex set. Let* $\mu \in (0, 1)$ *be a constant. Define an operator* $S : C \to X$ *by* $Sx = \mu T_1 x + (1 - \mu)T_2 x, \forall x \in C$, *where* $T_1, T_2 : C \to X$ *be two nonexpansive mappings with* $F(T_1) \cap F(T_2) \ne \emptyset$. *Then S is nonexpansive and* $F(S) = F(T_1) \cap F(T_2)$.

Lemma 4 ([3]). *Let X be a 2-uniformly smooth Banach space and $\emptyset \neq C \subset X$ a closed convex set. If the operator $A : C \to X$ is α-inverse-strongly accretive, then*

$$\|(I - \zeta A)u - (I - \zeta A)v\|^2 \leq \|u - v\|^2 + 2\zeta(c^2\zeta - \alpha)\|Au - Av\|^2, \ \forall u, v \in C.$$

Lemma 5 ([3]). *Let X be a 2-uniformly smooth Banach space and $\emptyset \neq C \subset X$ a closed convex set. Let $M_1, M_2 : C \to 2^X$ be two m-accretive operators and $A_i : C \to X (i = 1, 2)$ be ζ_i-inverse-strongly accretive operator. Define an operator $Q : C \to C$ by $Q := J^{M_1}_{\varsigma_1}(I - \varsigma_1 A_1) J^{M_2}_{\varsigma_2}(I - \varsigma_2 A_2)$. If $0 \leq \varsigma_i \leq \dfrac{\zeta_i}{c^2} (i = 1, 2)$, then $Q : C \to C$ is nonexpansive.*

Lemma 6 ([36]). *Let X be a uniformly smooth Banach space and $\emptyset \neq C \subset X$ a closed convex set. Let $A : C \to C$ be a nonexpansive mapping with $F(A) \neq \emptyset$, and $f : C \to X$ be a contraction. Let $t \in (0, 1)$. Define a net z_t by $z_t = tf(z_t) + (1 - t)Az_t$. Then $z_t \to x^* \in F(A)$ and*

$$\langle (I - f)x^*, j(x^* - x) \rangle \leq 0, \quad \forall x \in F(A).$$

Lemma 7 ([36]). *Assume the sequence $\{a_n\} \subset [0, \infty)$ satisfies $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \sigma_n (\forall n \geq 0)$, where the sequences $\{\lambda_n\} \subset (0, 1)$ and $\{\sigma_n\}$ satisfy*

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(ii) *either* $\sum_{n=0}^{\infty} |\lambda_n \sigma_n| < \infty$ *or* $\limsup_{n\to\infty} \sigma_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 8 ([33]). *Let X be a 2-uniformly smooth Banach space and $\emptyset \neq C \subset X$ a closed convex set. Let $T : C \to C$ be a λ-strict pseudocontraction. Define an operator T_α by $T_\alpha x = (1 - \alpha)x + \alpha T x, \alpha \in (0, 1)$. Then, $T_\alpha : C \to C$ is nonexpansive with $F(T_\alpha) = F(T)$ provided $\alpha \in (0, \dfrac{\lambda}{c^2}]$.*

3. Main Results

Theorem 1. *Let X be a uniformly convex and 2-uniformly smooth Banach space and $\emptyset \neq C \subset X$ a closed convex set. Let $M_1, M_2 : C \to 2^X$ be two m-accretive operators and $A_i : C \to X (i = 1, 2)$ be ζ_i-inverse-strongly accretive operator. Let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$. Let $V : C \to C$ be a nonexpansive operator and $T : C \to C$ be a λ-strict pseudocontraction with $\Omega := F(T) \cap F(Q) \neq \emptyset$, where the operator Q is defined as in Lemma 5. Assume that the sequences $\{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset (0, 1)$, $\{\delta_n\} \subset (0, 1)$ and $\{\gamma_n\} \subset (0, 1)$ satisfy*

(i) $\alpha_n + \delta_n + \beta_n + \gamma_n = 1 (\forall n \geq 1)$;
(ii) $\alpha_n \to 0$ *and* $\dfrac{\beta_n}{\alpha_n} \to 0$;
(iii) $\gamma_n \to 1$;
(iv) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Given $x_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} y_n = J^{M_2}_{\varsigma_2}(x_n - \varsigma_2 A_2 x_n), \\ x_n = \alpha_n f(x_{n-1}) + \delta_n x_{n-1} + \beta_n V x_{n-1} + \gamma_n [\mu S x_n + (1 - \mu) J^{M_1}_{\varsigma_1}(y_n - \varsigma_1 A_1 y_n)], \ \forall n \geq 1, \end{cases} \tag{4}$$

where $Sx = (1 - \alpha)x + \alpha T x$, $\forall x \in C$ with $0 < \alpha < \min\{1, \dfrac{\lambda}{c^2}\}$ and $\mu \in (0, 1)$. Then $x_n \to x^*$, $y_n \to y^*$ and

(a) (x^*, y^*) *solves the GSVI (1);*
(b) x^* *solves the variational inequality:* $\langle (I - f)x^*, j(u - x^*) \rangle \geq 0, \ \forall u \in \Omega$.

Proof. By Lemmas 5 and 8, Q and S are nonexpansive and $F(S) = F(T)$. Put $A := \mu S + (1-\mu)Q$ with $\mu \in (0,1)$. It is easy to see that the implicit iterative scheme (4) can be rewritten as

$$x_n = \alpha_n f(x_{n-1}) + \delta_n x_{n-1} + \beta_n V x_{n-1} + \gamma_n A x_n, \ \forall n \geq 1. \tag{5}$$

Consider the mapping $F_n u = \alpha_n f(x_{n-1}) + \delta_n x_{n-1} + \beta_n V x_{n-1} + \gamma_n A u, \forall u \in C$. According to Lemma 3, we have

$$\|F_n u - F_n v\| = \gamma_n \|Au - Av\| \leq \gamma_n \|u - v\|, \ \forall u, v \in C.$$

Hence F_n is a contraction. Thus, (5) and hence (4) are all well-posed.

Let $u^\dagger \in \Omega$. Thus, $Tu^\dagger = u^\dagger$ and $Qu^\dagger = u^\dagger$. It is clear that

$$\begin{aligned} x_n - u^\dagger &= \alpha_n f(x_{n-1}) + \delta_n x_{n-1} + \beta_n V x_{n-1} + \gamma_n A x_n - u^\dagger \\ &= \alpha_n (f(x_{n-1}) - u^\dagger) + \delta_n (x_{n-1} - u^\dagger) + \beta_n (V x_{n-1} - u^\dagger) + \gamma_n (A x_n - u^\dagger). \end{aligned}$$

By Lemma 3, we get

$$\begin{aligned} \|x_n - u^\dagger\| &\leq \alpha_n \|f(x_{n-1}) - u^\dagger\| + \delta_n \|x_{n-1} - u^\dagger\| + \beta_n \|V x_{n-1} - u^\dagger\| + \gamma_n \|A x_n - u^\dagger\| \\ &\leq \alpha_n (\|f(x_{n-1}) - f(u^\dagger)\| + \|f(u^\dagger) - u^\dagger\|) + \delta_n \|x_{n-1} - u^\dagger\| \\ &\quad + \beta_n (\|V x_{n-1} - V u^\dagger\| + \|V u^\dagger - u^\dagger\|) + \gamma_n \|x_n - u^\dagger\| \\ &\leq \alpha_n (k \|x_{n-1} - u^\dagger\| + \|f(u^\dagger) - u^\dagger\|) + \delta_n \|x_{n-1} - u^\dagger\| \\ &\quad + \beta_n (\|x_{n-1} - u^\dagger\| + \|V u^\dagger - u^\dagger\|) + \gamma_n \|x_n - u^\dagger\| \\ &= \alpha_n \|f(u^\dagger) - u^\dagger\| + (1 - (1-k)\alpha_n - \gamma_n) \|x_{n-1} - u^\dagger\| + \beta_n \|V u^\dagger - u^\dagger\| + \gamma_n \|x_n - u^\dagger\|. \end{aligned}$$

By condition (ii), without loss of generality, we assume that $\beta_n \leq \alpha_n$ for all $n \geq 1$. Hence,

$$\begin{aligned} \|x_n - u^\dagger\| &\leq [1 - (1-k)\frac{\alpha_n}{1-\gamma_n}] \|x_{n-1} - u^\dagger\| + \frac{\alpha_n}{1-\gamma_n} \|f(u^\dagger) - u^\dagger\| + \frac{\beta_n}{1-\gamma_n} \|V u^\dagger - u^\dagger\| \\ &\leq [1 - (1-k)\frac{\alpha_n}{1-\gamma_n}] \|x_{n-1} - u^\dagger\| + \frac{\alpha_n}{1-\gamma_n} \|f(u^\dagger) - u^\dagger\| + \frac{\alpha_n}{1-\gamma_n} \|V u^\dagger - u^\dagger\| \\ &= [1 - (1-k)\frac{\alpha_n}{1-\gamma_n}] \|x_{n-1} - u^\dagger\| + \frac{\alpha_n}{1-\gamma_n} (\|f(u^\dagger) - u^\dagger\| + \|V u^\dagger - u^\dagger\|) \\ &\leq \max\{\|x_{n-1} - u^\dagger\|, (\|f(u^\dagger) - u^\dagger\| + \|V u^\dagger - u^\dagger\|)/(1-k)\}. \end{aligned} \tag{6}$$

Thus, $\{x_n\}$, $\{Tx_n\}$, $\{Sx_n\}$, $\{y_n\}$, $\{Qx_n\}$ and $\{Ax_n\}$ are all bounded.

Set $q = J_{\varsigma_2}^{M_2}(u^\dagger - \varsigma_2 A_2 u^\dagger)$ and $z_n = J_{\varsigma_1}^{M_1}(y_n - \varsigma_1 A_1 y_n)$. Then $z_n = Qx_n, \forall n \geq 1$. By virtue of Lemma 4, we get

$$\begin{aligned} \|y_n - q\|^2 &= \|J_{\varsigma_2}^{M_2}(x_n - \varsigma_2 A_2 x_n) - J_{\varsigma_2}^{M_2}(u^\dagger - \varsigma_2 A_2 u^\dagger)\|^2 \\ &\leq \|x_n - u^\dagger - \varsigma_2 (A_2 x_n - A_2 u^\dagger)\|^2 \\ &\leq \|x_n - u^\dagger\|^2 - 2\varsigma_2 (\zeta_2 - c^2 \varsigma_2) \|A_2 x_n - A_2 u^\dagger\|^2, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \|z_n - u^\dagger\|^2 &= \|J_{\varsigma_1}^{M_1}(y_n - \varsigma_1 A_1 y_n) - J_{\varsigma_1}^{M_1}(q - \varsigma_1 A_1 q)\|^2 \\ &\leq \|y_n - q - \varsigma_1 (A_1 y_n - A_1 q)\|^2 \\ &\leq \|y_n - q\|^2 - 2\varsigma_1 (\zeta_1 - c^2 \varsigma_1) \|A_1 y_n - A_1 q\|^2. \end{aligned} \tag{8}$$

Substituting (7) for (8), we derive

$$\|z_n - u^\dagger\|^2 \leq \|x_n - u^\dagger\|^2 - 2\varsigma_2 (\zeta_2 - c^2 \varsigma_2) \|A_2 x_n - A_2 u^\dagger\|^2 - 2\varsigma_1 (\zeta_1 - c^2 \varsigma_1) \|A_1 y_n - A_1 q\|^2. \tag{9}$$

In view of (5) and (9), we obtain

$$
\begin{aligned}
\|x_n - u^\dagger\|^2 = {} & \alpha_n \langle f(x_{n-1}) - u^\dagger, j(x_n - u^\dagger)\rangle + \delta_n \langle x_{n-1} - u^\dagger, j(x_n - u^\dagger)\rangle \\
& + \beta_n \langle V x_{n-1} - u^\dagger, j(x_n - u^\dagger)\rangle + \gamma_n \langle (1-\mu) S x_n + \mu z_n - u^\dagger, j(x_n - u^\dagger)\rangle \\
\le {} & \alpha_n \langle f(x_{n-1}) - u^\dagger, j(x_n - u^\dagger)\rangle + \delta_n \|x_{n-1} - u^\dagger\| \|x_n - u^\dagger\| \\
& + \beta_n \|V x_{n-1} - u^\dagger\| \|x_n - u^\dagger\| + \gamma_n \|(1-\mu) S x_n + \mu z_n - u^\dagger\| \|x_n - u^\dagger\| \\
\le {} & \delta_n \|x_{n-1} - u^\dagger\| \|x_n - u^\dagger\| + \gamma_n [(1-\mu)\|x_n - u^\dagger\| + \mu \|z_n - u^\dagger\|] \|x_n - u^\dagger\| \\
& + \alpha_n [\langle f(x_{n-1}) - f(u^\dagger), j(x_n - u^\dagger)\rangle + \langle f(u^\dagger) - u^\dagger, j(x_n - u^\dagger)\rangle] \\
& + \beta_n (\|V x_{n-1} - V u^\dagger\| + \|V u^\dagger - u^\dagger\|) \|x_n - u^\dagger\| \\
\le {} & \delta_n \|x_{n-1} - u^\dagger\| \|x_n - u^\dagger\| + \gamma_n [(1-\mu)\|x_n - u^\dagger\| + \mu \|z_n - u^\dagger\|] \|x_n - u^\dagger\| \\
& + \alpha_n [k \|x_{n-1} - u^\dagger\| \|x_n - u^\dagger\| + \langle f(u^\dagger) - u^\dagger, j(x_n - u^\dagger)\rangle] \\
& + \beta_n (\|x_{n-1} - u^\dagger\| + \|V u^\dagger - u^\dagger\|) \|x_n - u^\dagger\| \\
\le {} & \alpha_n \langle f(u^\dagger) - u^\dagger, j(x_n - u^\dagger)\rangle + [1 - (1-k)\alpha_n - \gamma_n]/2 (\|x_{n-1} - u^\dagger\|^2 + \|x_n - u^\dagger\|^2) \\
& + \beta_n \|V u^\dagger - u^\dagger\| \|x_n - u^\dagger\| + \gamma_n \|x_n - u^\dagger\|^2 - \gamma_n \mu [\varsigma_2 (\zeta_2 - c^2 \varsigma_2) \|A_2 x_n - A_2 u^\dagger\|^2 \\
& + \varsigma_1 (\zeta_1 - c^2 \varsigma_1) \|A_1 y_n - A_1 q\|^2].
\end{aligned} \tag{10}
$$

It follows that

$$
\begin{aligned}
& \gamma_n \mu [\varsigma_2 (\zeta_2 - c^2 \varsigma_2) \|A_2 x_n - A_2 u^\dagger\|^2 + \varsigma_1 (\zeta_1 - c^2 \varsigma_1) \|A_1 y_n - A_1 q\|^2] \\
& \le [1 - (1-k)\alpha_n - \gamma_n]/2 (\|x_{n-1} - u^\dagger\|^2 + \|x_n - u^\dagger\|^2) + \alpha_n \langle f(u^\dagger) - u^\dagger, j(x_n - u^\dagger)\rangle \\
& \quad + \beta_n \|V u^\dagger - u^\dagger\| \|x_n - u^\dagger\| - (1 - \gamma_n) \|x_n - u^\dagger\|^2 \\
& \le \alpha_n \|f(u^\dagger) - u^\dagger\| \|x_n - u^\dagger\| + [1 - (1-k)\alpha_n - \gamma_n]/2 (\|x_{n-1} - u^\dagger\|^2 + \|x_n - u^\dagger\|^2) \\
& \quad + \beta_n \|V u^\dagger - u^\dagger\| \|x_n - u^\dagger\|.
\end{aligned}
$$

By the assumptions (ii) and (iii), we conclude

$$
\lim_{n\to\infty} \|A_2 x_n - A_2 u^\dagger\| = 0 \text{ and } \lim_{n\to\infty} \|A_1 y_n - A_1 q\| = 0. \tag{11}
$$

Utilizing Lemma 1 and Proposition 1, we have

$$
\begin{aligned}
\|y_n - q\|^2 &= \|J_{\varsigma_2}^{M_2}(x_n - \varsigma_2 A_2 x_n) - J_{\varsigma_2}^{M_2}(u^\dagger - \varsigma_2 A_2 u^\dagger)\|^2 \\
&\le \langle (x_n - \varsigma_2 A_2 x_n) - (u^\dagger - \varsigma_2 A_2 u^\dagger), j(y_n - q)\rangle \\
&= \langle x_n - u^\dagger, j(y_n - q)\rangle + \varsigma_2 \langle A_2 u^\dagger - A_2 x_n, j(y_n - q)\rangle \\
&\le [\|x_n - u^\dagger\|^2 + \|y_n - u^\dagger\|^2 - g_1(\|x_n - y_n - (u^\dagger - q)\|)]/2 + \varsigma_2 \|A_2 u^\dagger - A_2 x_n\| \|y_n - q\|.
\end{aligned}
$$

It follows that

$$
\|y_n - q\|^2 \le \|x_n - u^\dagger\|^2 - g_1(\|x_n - y_n - (u^\dagger - q)\|) + 2\varsigma_2 \|A_2 u^\dagger - A_2 x_n\| \|y_n - q\|. \tag{12}
$$

Similarly,

$$
\begin{aligned}
\|z_n - u^\dagger\|^2 &= \|J_{\varsigma_1}^{M_1}(y_n - \varsigma_1 A_1 y_n) - J_{\varsigma_1}^{M_1}(q - \varsigma_1 A_1 q)\|^2 \\
&\le \langle (y_n - \varsigma_1 A_1 y_n) - (q - \varsigma_1 A_1 q), j(z_n - u^\dagger)\rangle \\
&= \langle y_n - q, j(z_n - q)\rangle + \varsigma_1 \langle A_1 q - A_1 y_n, j(z_n - u^\dagger)\rangle \\
&\le \frac{1}{2}[\|y_n - q\|^2 + \|z_n - u^\dagger\|^2 - g_2(\|y_n - z_n + (u^\dagger - q)\|)] + \varsigma_1 \|A_1 q - A_1 y_n\| \|z_n - u^\dagger\|,
\end{aligned}
$$

which implies that

$$\|z_n - u^\dagger\|^2 \le \|y_n - q\|^2 - g_2(\|y_n - z_n + (u^\dagger - q)\|) + 2\varsigma_1\|A_1 q - A_1 y_n\|\|z_n - u^\dagger\|. \tag{13}$$

Substituting (12) for (13), we get

$$\begin{aligned}\|z_n - u^\dagger\|^2 \le{}& \|x_n - u^\dagger\|^2 - g_1(\|x_n - y_n - (u^\dagger - q)\|) - g_2(\|y_n - z_n + (u^\dagger - q)\|)\\ &+ 2\varsigma_2\|A_2 u^\dagger - A_2 x_n\|\|y_n - q\| + 2\varsigma_1\|A_1 q - A_1 y_n\|\|z_n - u^\dagger\|.\end{aligned} \tag{14}$$

From (10) and (14), we have

$$\begin{aligned}\|x_n - u^\dagger\|^2 \le{}& \alpha_n\langle f(u^\dagger) - u^\dagger, j(x_n - u^\dagger)\rangle + [1 - (1-k)\alpha_n - \gamma_n](\|x_{n-1} - u^\dagger\|^2 + \|x_n - u^\dagger\|^2)/2\\ &+ \beta_n\|Vu^\dagger - u^\dagger\|\|x_n - u^\dagger\| + \gamma_n(\|x_n - u^\dagger\|^2 + (1-\mu)\|x_n - u^\dagger\|^2 + \mu\|z_n - u^\dagger\|^2)/2\\ \le{}& \alpha_n\langle f(u^\dagger) - u^\dagger, j(x_n - u^\dagger)\rangle + [1 - (1-k)\alpha_n - \gamma_n](\|x_{n-1} - u^\dagger\|^2 + \|x_n - u^\dagger\|^2)/2\\ &+ \beta_n\|Vu^\dagger - u^\dagger\|\|x_n - u^\dagger\| + \gamma_n\|x_n - u^\dagger\|^2 - \frac{\gamma_n\mu}{2}[g_1(\|x_n - y_n - (u^\dagger - q)\|)\\ &+ g_2(\|y_n - z_n + (u^\dagger - q)\|)] + \gamma_n\mu(\varsigma_2\|A_2 u^\dagger - A_2 x_n\|\|y_n - q\|\\ &+ \varsigma_1\|A_1 q - A_1 y_n\|\|z_n - u^\dagger\|).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\gamma_n\mu}{2}[g_1(\|x_n - y_n - (u^\dagger - q)\|) + g_2(\|y_n - z_n + (u^\dagger - q)\|)]\\ &\le \alpha_n\langle f(u^\dagger) - u^\dagger, j(x_n - u^\dagger)\rangle + [1 - (1-k)\alpha_n - \gamma_n](\|x_{n-1} - u^\dagger\|^2 + \|x_n - u^\dagger\|^2)/2\\ &\quad + \beta_n\|Vu^\dagger - u^\dagger\|\|x_n - u^\dagger\| - (1-\gamma_n)\|x_n - u^\dagger\|^2 + \gamma_n\mu(\varsigma_2\|A_2 u^\dagger - A_2 x_n\|\|y_n - q\|\\ &\quad + \varsigma_1\|A_1 q - A_1 y_n\|\|z_n - u^\dagger\|)\\ &\le \alpha_n\|f(u^\dagger) - u^\dagger\|\|x_n - u^\dagger\| + [1 - (1-k)\alpha_n - \gamma_n](\|x_{n-1} - u^\dagger\|^2 + \|x_n - u^\dagger\|^2)/2\\ &\quad + \beta_n\|Vu^\dagger - u^\dagger\|\|x_n - u^\dagger\| + \varsigma_2\|A_2 u^\dagger - A_2 x_n\|\|y_n - q\| + \varsigma_1\|A_1 q - A_1 y_n\|\|z_n - u^\dagger\|.\end{aligned}$$

This together with conditions (ii) and (iii) implies that

$$\lim_{n\to\infty} g_1(\|x_n - y_n - (u^\dagger - q)\|) = 0 \text{ and } \lim_{n\to\infty} g_2(\|y_n - z_n + (u^\dagger - q)\|) = 0.$$

Hence,

$$\lim_{n\to\infty} \|x_n - y_n - (u^\dagger - q)\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - z_n + (u^\dagger - q)\| = 0. \tag{15}$$

In light of (15), we have

$$\|x_n - z_n\| \le \|x_n - y_n - (u^\dagger - q)\| + \|y_n - z_n + (u^\dagger - q)\| \to 0,$$

which means that

$$\lim_{n\to\infty} \|x_n - Qx_n\| = 0. \tag{16}$$

Note that

$$\begin{aligned}\gamma_n\|x_n - Ax_n\| &= \|\alpha_n(f(x_{n-1}) - x_n) + \delta_n(x_{n-1} - x_n) + \beta_n(Vx_{n-1} - x_n)\|\\ &\le \alpha_n\|f(x_{n-1}) - x_n\| + \delta_n\|x_{n-1} - x_n\| + \beta_n\|Vx_{n-1} - x_n\|.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|x_n - Ax_n\| = 0. \tag{17}$$

Also, observe that

$$\mu\|Sx_n - x_n\| = \|Ax_n - x_n - (1-\mu)(Qx_n - x_n)\| \leq \|Ax_n - x_n\| + \|Qx_n - x_n\|.$$

In terms of (16) and (17), we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{18}$$

Since $S = (1-\alpha)I + \alpha T$ with $0 < \alpha < \min\{1, \frac{\lambda}{c^2}\}$, it is easy from (3.15) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Define a net $\{u_t\}$ by $u_t = (1-t)Au_t + tf(u_t)$. So,

$$u_t - x_n = t(f(u_t) - x_n) + (1-t)(Au_t - x_n). \tag{19}$$

It follows that

$$\begin{aligned}
\|u_t - x_n\|^2 &\leq 2t\langle f(u_t) - x_n, j(u_t - x_n)\rangle + (1-t)^2\|Au_t - x_n\|^2 \\
&\leq 2t\langle f(u_t) - x_n, j(u_t - x_n)\rangle + (1-t)^2[\|Au_t - Ax_n\| + \|Ax_n - x_n\|]^2 \\
&\leq 2t\langle f(u_t) - x_n, j(u_t - x_n)\rangle + (1-t)^2[\|u_t - x_n\| + \|Ax_n - x_n\|]^2 \\
&= (1-t)^2[\|u_t - x_n\|^2 + \|Ax_n - x_n\|^2 + 2\|u_t - x_n\|\|Ax_n - x_n\|] \\
&\quad + 2t\langle f(u_t) - x_n, j(u_t - x_n)\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\|u_t - x_n\|^2 &\leq \|Ax_n - x_n\|(2\|u_t - x_n\| + \|Ax_n - x_n\|) + 2t\|u_t - x_n\|^2 \\
&\quad + 2t\langle f(u_t) - u_t, j(u_t - x_n)\rangle + (1-t)^2\|u_t - x_n\|^2 \\
&= (1+t^2)\|u_t - x_n\|^2 + 2t\langle f(u_t) - u_t, j(u_t - x_n)\rangle \\
&\quad + \|Ax_n - x_n\|(2\|u_t - x_n\| + \|Ax_n - x_n\|).
\end{aligned}$$

It follows that

$$\langle u_t - f(u_t), j(u_t - x_n)\rangle \leq \frac{t}{2}\|u_t - x_n\|^2 + \frac{1}{2t}(2\|u_t - x_n\| + \|Ax_n - x_n\|)\|Ax_n - x_n\|. \tag{20}$$

Letting $n \to \infty$ in (20), from (17), we have

$$\overline{\lim_{n\to\infty}}\langle u_t - f(u_t), j(u_t - x_n)\rangle \leq \frac{tM_0}{2} \tag{21}$$

where M_0 is a constant such that $\|u_t - x_n\|^2 \leq M_0, \forall n \geq 0, \forall t \in (0,1)$. By Lemma 6, $u_t \to x^* \in \Omega$, which solves $\langle (I-f)x^*, j(x^* - x)\rangle \leq 0,\ \forall x \in \Omega$. Letting $t \to 0^+$ in (21), we deduce

$$\overline{\lim_{n\to\infty}}\langle x^* - f(x^*), j(x^* - x_n)\rangle \leq 0.$$

Putting $u^\dagger = x^*$ in (10), we obtain

$$\begin{aligned}
\|x_n - x^*\|^2 &\leq \alpha_n\langle f(x^*) - x^*, j(x_n - x^*)\rangle + [1 - (1-k)\alpha_n - \gamma_n](\|x_{n-1} - x^*\|^2 + \|x_n - x^*\|^2)/2 \\
&\quad + \beta_n\|Vx^* - x^*\|\|x_n - x^*\| + \gamma_n\|x_n - x^*\|^2.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}\|x_n - x^*\|^2 \le& \frac{2\alpha_n}{1+(1-k)\alpha_n - \gamma_n}\langle f(x^*) - x^*, j(x_n - x^*)\rangle + \frac{1-(1-k)\alpha_n - \gamma_n}{1+(1-k)\alpha_n - \gamma_n}\|x_{n-1} - x^*\|^2 \\ &+ \frac{2\beta_n}{1+(1-k)\alpha_n - \gamma_n}\|Vx^* - x^*\|\|x_n - x^*\| \\ =& (1-v_n)\|x_{n-1} - x^*\|^2 + v_n\varrho_n,\end{aligned} \tag{22}$$

where $v_n = \dfrac{2(1-k)\alpha_n}{1+(1-k)\alpha_n - \gamma_n}$ and

$$\varrho_n = \frac{\beta_n}{(1-k)\alpha_n}\|Vx^* - x^*\|\|x_n - x^*\| + \frac{1}{1-k}\langle f(x^*) - x^*, j(x_n - x^*)\rangle.$$

Now, observe that

$$(1-k)\alpha_n = \frac{2(1-k)\alpha_n}{2} \le \frac{2(1-k)\alpha_n}{1-\gamma_n + (1-k)\alpha_n} = v_n.$$

Observe that $\overline{\lim}_{n\to\infty}\varrho_n \le 0$. With the help of Lemma 7, we get $x_n \to x^*$. Moreover, putting $u^\dagger = x^*$ and $q = y^* = J_{\varsigma_2}^{M_2}(I - \varsigma_2 A_2)x^*$ in (15), we obtain

$$\lim_{n\to\infty}\|x_n - y_n - (x^* - y^*)\| = 0.$$

Note that

$$\|y_n - y^*\| \le \|x_n - y_n - (x^* - y^*)\| + \|x^* - x_n\|.$$

So, it follows that $y_n \to y^*$ as $n \to \infty$. Consequently, (x^*, y^*) is a solution of (1) by Lemma 2. □

Corollary 1. *Let X be a uniformly convex and 2-uniformly smooth Banach space and $\emptyset \ne C \subset X$ a closed convex set. Let $M : C \to 2^X$ be an m-accretive operator and $A : C \to X$ be a ζ-inverse-strongly accretive operator. Let $f : C \to C$ be a contraction with coefficient $k \in [0,1)$. Let $V : C \to C$ be a nonexpansive operator and $T : C \to C$ be a λ-strict pseudocontraction with $\Omega := F(T) \cap F(Q) \ne \emptyset$, where the operator $Q = J_{\varsigma_1}^M(I - \varsigma_1 A)J_{\varsigma_2}^M(I - \varsigma_2 A)$ and $0 < \varsigma_i < \frac{\zeta}{c^2}(i = 1,2)$. Assume that the sequences $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0,1)$ satisfy*

(i) $\alpha_n + \delta_n + \beta_n + \gamma_n = 1 (\forall n \ge 1)$;
(ii) $\alpha_n \to 0$ *and* $\dfrac{\beta_n}{\alpha_n} \to 0$;
(iii) $\gamma_n \to 1$;
(iv) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Given $x_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} y_n = J_{\varsigma_2}^M(x_n - \varsigma_2 A x_n), \\ x_n = \delta_n x_{n-1} + \alpha_n f(x_{n-1}) + \beta_n V x_{n-1} + \gamma_n[\mu S x_n + (1-\mu)J_{\varsigma_1}^M(y_n - \varsigma_1 A y_n)], \ \forall n \ge 1, \end{cases}$$

where $Sx = (1-\alpha)x + \alpha Tx$, $\forall x \in C$ with $0 < \alpha < \min\{1, \frac{\lambda}{c^2}\}$ and $\mu \in (0,1)$. Then $x_n \to x^*$, $y_n \to y^*$ and

(a) (x^*, y^*) *solves the GSVI (2);*
(b) x^* *solves the variational inequality:* $\langle (I - f)x^*, j(u - x^*)\rangle \ge 0, \ \forall u \in \Omega$.

Corollary 2. *Let H be a Hilbert space and $\emptyset \ne C \subset H$ a closed convex set. Let $M : C \to 2^H$ be a maximal monotone operator and $A : C \to H$ be a ζ-inverse-strongly monotone operator. Let $f : C \to C$ be a contraction* *with coefficient $k \in [0,1)$. Let $V : C \to C$ be a nonexpansive operator and $T : C \to C$ be a λ-strict pseudocontraction with $\Omega := F(T) \cap F(Q) \neq \emptyset$, where the operator $Q = J^M_{\varsigma_1}(I - \varsigma_1 A)J^M_{\varsigma_2}(I - \varsigma_2 A)$ and $0 < \varsigma_i < \frac{\zeta}{c^2}(i = 1,2)$. Assume that the sequences $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0,1)$ satisfy*

(i) $\alpha_n + \delta_n + \beta_n + \gamma_n = 1 (\forall n \geq 1)$;
(ii) $\alpha_n \to 0$ *and* $\frac{\beta_n}{\alpha_n} \to 0$;
(iii) $\gamma_n \to 1$;
(iv) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Given $x_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} y_n = J^M_{\varsigma_2}(x_n - \varsigma_2 A x_n), \\ x_n = \delta_n x_{n-1} + \alpha_n f(x_{n-1}) + \beta_n V x_{n-1} + \gamma_n[\mu S x_n + (1-\mu) J^M_{\varsigma_1}(y_n - \varsigma_1 A y_n)], \ \forall n \geq 1, \end{cases}$$

where $Sx = (1-\alpha)x + \alpha Tx$, $\forall x \in C$ with $0 < \alpha < \min\{1, \frac{\lambda}{c^2}\}$ and $\mu \in (0,1)$. Then $x_n \to x^*$, $y_n \to y^*$ and

(a) (x^*, y^*) *solves the GSVI (2);*
(b) x^* *solves the variational inequality:* $\langle (I - f)x^*, j(u - x^*) \rangle \geq 0, \ \forall u \in \Omega$.

4. Conclusions

In this paper, we consider the GSVI (1) with the hierarchical variational inequality (HVI) constraint for a strict pseudocontraction in a uniformly convex and 2-uniformly smooth Banach space. By utilizing the equivalence between the GSVI (1) and the fixed point problem, we construct an implicit composite viscosity approximation method for solving the GSVI (1) with the HVI constraint for strict pseudocontractions. We prove the strong convergence of the proposed algorithm to a solution of the GSVI (1) with the HVI constraint for strict pseudocontraction under very mild conditions. Note that our algorithm (4) is an implicit manner. This brings us a natural question: could we construct an explicit algorithm with strong convergence?

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