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Nonlinear Caputo Fractional Derivative with Nonlocal Riemann-Liouville Fractional Integral Condition Via Fixed Point Theorems

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Abstract: In this paper, we study and investigate an interesting Caputo fractional derivative and Riemann–Liouville integral boundary value problem (BVP): ${}^{c}D_{0^+}^{q}u(t) = f(t,u(t)), t \in [0,T]$, $u^{(k)}(0) = \xi_k,\ u(T) = \sum_{i=1}^{m} \beta_i {}_{RL}I_{0^+}^{p_i} u(\eta_i)$, where $n-1 < q < n$, $n \geq 2$, $m,n \in \mathbb{N}$, $\xi_k, \beta_i \in \mathbb{R}$, $k = 0,1,\ldots,n-2$, $i = 1,2,\ldots,m$, and ${}^{c}D_{0^+}^{q}$ is the Caputo fractional derivatives, $f : [0,T] \times C([0,T],E) \to E$, where E is the Banach space. The space E is chosen as an arbitrary Banach space; it can also be $\mathbb{R}$ (with the absolute value) or $C([0,T],\mathbb{R})$ with the supremum-norm. ${}_{RL}I_{0^+}^{p_i}$ is the Riemann–Liouville fractional integral of order $p_i > 0$, $\eta_i \in (0,T)$, and $\sum_{i=1}^{m} \beta_i \eta_i^{p_i+n-1} \dfrac{\Gamma(n)}{\Gamma(n+p_i)} \neq T^{n-1}$. Via the fixed point theorems of Krasnoselskii and Darbo, the authors study the existence of solutions to this problem. An example is included to illustrate the applicability of their results.

Keywords: Caputo fractional derivative; existence of a solution; fixed point theorem; integral boundary value problems

1. Introduction

The fractional differential equations have an important role in numerous fields of study carried out by mathematicians, physicists, and engineers. They have used it basically to develop mathematical modeling, many physical applications, and engineering disciplines such as viscoelasticity, control, porous media, phenomena in electromagnetics etc. (see [1–3]). The major differences between the fractional order differential operator and classical calculus is its nonlocal behavior, that is the feature future state based on the fractional differential operator depends on its current and past states. More details on the fundamental concepts of fractional calculus, fractional differential equations, and fractional integral equations can be found in books like A. A.Kilbas, H. M Srivastava and J. J. Trujillo [1], K. S Miller and B. Ross [2], and J. Banas and K. Goebel [4]. Fractional integro-differential equations involving the Caputo–Fabrizio derivative have been studied by many researchers from different points of view (see, for example, [5–8], and the references therein). The qualitative theory of differential equations has significant applications, and the existence of solutions and of positive solutions of fractional differential equations, which respect the initial and boundary value, have also received considerable attention. In order to study such a type of problem, different kinds of techniques, such as fixed point theorems [9–11], the fixed point index [10,12], the upper and lower solution method [13], coincidence theory [14], etc., are in vogue. For instance, in [15–17], the authors investigated the existence of solutions of initial value problems.

$$\begin{aligned} &{}^cD^{\alpha}_{0+}u(t) = f(t, u(t), {}^cD^{\beta}_{0+}u(t)), \quad t \in (0,1], \\ &u^{(k)}(0) = \eta_k, \quad k = 0, 1, ..., n-1, \end{aligned} \tag{1}$$

where $n-1 < \beta < \alpha < n, \quad (n \in \mathbb{N})$ are the real numbers and ${}^cD^{\alpha}_{0+}$ and ${}^cD^{\beta}_{0+}$ are the Caputo fractional derivatives of order α, β, and $f \in C([0,1] \times \mathbb{R})$.

In [18], the authors investigated the existence of solutions of the following boundary value problems:

$$ {}^cD^{\alpha}_{0+}y(t) = -f(t, y(t), {}^cD^{\alpha}_{0+}y(t)), \ t \in (0,1], \ 1 < \alpha < 2 \tag{2}$$

$$ ay(0) - by'(0) = 0, \quad y(1) = \int_0^1 k(s)g(t, y(s))ds + \mu, \tag{3}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative order α, E is the Banach space, $f : [0,1] \times C([0,1], E) \times E \to E$, $g : [0,1] \times C([0,1], E) \to E$, $k \in C([0,1], E)$, $k \neq 0$.

In [19], the authors investigated the existence and uniqueness of solutions of the nonlocal fractional integral condition.

$$ {}_{RL}D^{q}_{0+}x(t) = f(t, x(t)), \ t \in [0, T], \tag{4}$$

$$ x(0) = 0, \quad x(T) = \sum_{i=1}^{n} \alpha_i {}_HI^{p_i}_{0+}x(\eta_i), \tag{5}$$

where $1 < q \leq 2$, ${}_{RL}D^{q}_{0+}$ is the Riemann–Liouville fractional derivative of order q, ${}_HI^{p_i}_{0+}$ is the Hadamard fractional integral of order $p_i > 0$, $\eta_i \in (0, T)$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, and $\alpha_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ are real constants such that $\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{p_i}} \neq T^{q-1}$.

Inspired by the above papers in [15–19], the objective of this paper is to derive the existence solution of the fractional differential equations and nonlocal fractional integral conditions:

$$\begin{aligned} &{}^cD^{q}_{0+}u(t) = f(t, u(t)), t \in [0, T] \\ &u^{(k)}(0) = \xi_k, \quad u(T) = \sum_{i=1}^{m} \beta_i {}_{RL}I^{p_i}_{0+}u(\eta_i), \end{aligned} \tag{6}$$

where $n-1 < q < n, n \geq 2$, $m, n \in \mathbb{N}$, $\xi_k, \beta_i \in \mathbb{R}$, $k = 0, 1, \ldots, n-2$, $i = 1, 2, \ldots, m$, and ${}^cD^{q}_{0+}$ is the Caputo fractional derivatives, $f : [0, T] \times C([0, T], E) \to E$, and ${}_{RL}I^{p_i}$ is the Riemann–Liouville fractional integral of order $p_i > 0$, $\eta_i \in (0, T)$, and $\sum_{i=1}^{m} \beta_i \eta_i^{p_i+n-1} \frac{\Gamma(n)}{\Gamma(n+p_i)} \neq T^{n-1}$.

The results obtained in the present paper were based on the fixed point theorems of Krasnoselskii and Darbo. Further, we provide some examples to show the applicability of our results. The next part of the paper is organized in the following order: We recall some notations, definitions, and preliminary facts about fractional differential calculus and Kuratowski's measure of noncompactness (Kuratowski MNC), as well as some known results in Section 2. In Section 3, based on Kransnoselskii's fixed point theorem and Darbo's fixed point theorem together and the idea of the measure of noncompactness, the main result is formulated and proven. We also show an example of the main results.

2. Background Materials

In this section, we recall some basic notations, definitions, and lemmas regarding fractional differential equations in order to obtain our main results. See [1,3,4,17,20,21], and the references therein. Denote by $C([0,T],\mathbb{R})$ the space of all continuous functions from $[0,T]$ into $\mathbb{R}$. Endowed with the norm:

$$\|u\|_\infty := \sup\{|u(t)|: t \in [0,T]\},\ u \in C([0,T],\mathbb{R}),$$

this space is a Banach space. Let $(E, \|\cdot\|)$ be a Banach space. We also denote:

$$C^n([0,T],E) := \{u \in C([0,T],E) : u^{(k)} \in C([0,T],E),\ 0 \le k \le n\}.$$

Equipped with the norm $\|u\|_{C^n} := \sum_{k=0}^{n} \|u^{(k)}\|_C$ for $u \in C^n([0,T],E)$, this space is a Banach space, as well. Here, $\|u\|_C := \sup_{0<t<T} \|u(t)\|$. For measurable functions $g : [0,T] \to \mathbb{R}$, define the norm $\|g\|_{L^p([0,T],\mathbb{R})} = \left(\int_{[0,T]} |g(t)|^p dt\right)^{\frac{1}{p}}$, $1 \le p < \infty$. We also denote by $L^p([0,T],\mathbb{R})$ the Banach space of all Lebesgue measurable functions g for which $\|g\|_{L^p([0,T],\mathbb{R})} < \infty$.

Definition 1 ([1,3])**.** *Let $u : (0,\infty) \to \mathbb{R}$ be a function and $q > 0$. The Riemann–Liouville fractional integral of orders q of u is defined by:*

$$_{RL}I^q_{0^+}u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}u(s)ds$$

provided that the integral exists. The Caputo fractional derivative of order q of u is defined by:

$$^cD^q_{0^+}u(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1}u^{(n)}(s)ds$$

provided that the right side is point-wise defined on $(0,\infty)$, where n is the smallest integer greater than or equal to q and Γ denotes the gamma function. If $q = n$, then $^cD^q_{0^+}u(t) = u^{(n)}(t)$.

Lemma 1 ([1,3])**.** *Let $n-1 < q < n$. If $u \in C^n([a,b])$, then:*

$$_{RL}I^q_{0^+}(^cD^q_{0^+}u)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + C_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, where n is the smallest integer greater than or equal to q.

For a given set V of functions $v : [0,T] \to E$, let us denote $V(t) = \{v(t) : v \in V\}$, $t \in [0,T]$, and $V([0,T]) = \{v(t) : v \in V, t \in [0,T]\}$. Next, we provide the definition of the measure of noncompactness and some auxiliary results; see for more details [11,13,15] and the references therein.

Definition 2. *Let E be a Banach space and Ω_E the collection of subsets of E. The Kuratowski MNC is the map $\alpha : \Omega_E \to [0,\infty]$ defined by $\alpha(X) = \inf\{d > 0 : X \subseteq \bigcup_{i=1}^{n} X_i \text{ and } diam(X_i) \le d\}$, where:*

$$diam(X_i) = \sup\{\| x - y \|: x, y \in X_i\}.$$

We also adopt some techniques from the Kuratowski MNC and the theorem of Arzela–Ascoli in Lemma 3.

Lemma 2 ([4,20,22])**.** *Let E be a Banach space. X and Y are bounded sets,*
(a) $\alpha(X) = 0 \leftrightarrow \bar{X}$ *is compact (X is relatively compact), where* $\bar{X}$ *denotes the closure of X,*
(b) *nonsingularity:* α *is equal to zero on every element set,*
(c) $\alpha(X) = \alpha(\bar{X}) = \alpha(convX)$*, where convX is the convex hull of X,*
(d) *monotonicity:* $X \subset Y \to \alpha(X) \subset \alpha(Y)$*,*
(e) *algebraic semi-additively:* $\alpha(X+Y) \leq \alpha(X) + \alpha(Y)$*, where* $X+Y = \{x+y : x \in X, y \in Y\}$*,*
(f) *semi-homogenicity:* $\alpha(\lambda X) =\mid \lambda \mid \alpha(X)$*,* $\lambda \in \mathbb{R}$*, where* $\lambda X = \{\lambda x : x \in X\}$*,*
(g) *semi-additivity:* $\alpha(X \cup Y) = \max\{\alpha(X), \alpha(Y)\}$*.*
(h) *invariance under translation:* $\alpha(X + x_0) = \alpha(X)$ *for any* $x_0 \in E$*.*

Lemma 3 (Ascoli–Arzela theorem)**.** *If a family* $F = \{f(t)\}$ *in* $C([0,T],\mathbb{R})$ *is uniformly bounded and equicontinuous on* $[0,T]$*, then F has a uniformly-convergent subsequence* $\{f_n(t)\}_{n=1}^{\infty}$*. If a family* $F = \{f(t)\}$ *in* $C([0,T],E)$ *is uniformly bounded and equicontinuous on* $[0,T]$ *and for any* $t^* \in [0,T]$*,* $\{f(t^*)\}$ *is relatively compact in the Banach space E, then F has a uniformly-convergent subsequence* $\{f_n(t)\}_{n=1}^{\infty}$*.*

Theorem 1 (Krasnoselskii's fixed point theorem [21])**.** *Let N be a bounded, closed, convex, and nonempty subset of Banach space E. Let* $A_1, A_2 : E \to E$ *be operators with the following properties:*
(a) $A_1x + A_2y \in N$ *whenever* $x, y \in N$*;*
(b) A_1 *is continuous, and* A_1N *is a compact subset of E;*
(c) A_2 *is a contraction mapping (i.e.,* $\|A_2x - A_2y\| \leq k\|x-y\|$ *for some* $k \in (0,1)$ *and for all* $x, y \in N$*).*
Then, there exist $z \in N$ *such that* $z = A_1z + A_2z$*.*

Theorem 2 (Darbo's fixed point theorem [23])**.** *Let E be a Banach space, and let N be a bounded, closed, convex, and nonempty subset of E. Suppose a continuous mapping* $A : N \to N$ *is such that for all closed subsets M of N,*

$$\alpha(A(M)) \leq k\alpha(M),$$

where $0 \leq k < 1$*. Then, A has a fixed point in N.*

3. Main Result

In this section, we consider the existence of solutions of the nonlocal Riemann–Liouville fractional integral condition and Caputo nonlinear fractional differential Equation (6).

Definition 3. *A function* $u \in C([0,T],E)$ *is said to be a solution of (6) if u satisfies the equation* ${}^cD_{0^+}^q u(t) = f(t,u(t))$ *on* $[0,T]$*, and the fractional integral conditions* $u^{(k)}(0) = \xi_k$*,* $u(T) = \sum_{i=1}^{m} \beta_i \, {}_{RL}I_{0^+}^{p_i} u(\eta_i)$*.*

We prove the following lemma to establish the existence of a solution to Problem (6).

Lemma 4. *Let the function h belong to* $C([0,T],E)$*. Suppose that the function* $u \in C([0,T],E)$ *is a solution of the following boundary value problem (BVP):*

$$\begin{aligned} {}^cD_{0^+}^q u(t) &= h(t), \quad t \in [0,T], \\ u^{(k)}(0) &= \xi_k, \quad u(T) = \sum_{i=1}^{m} \beta_i \, {}_{RL}I_{0^+}^{p_i} u(\eta_i), \end{aligned}$$

where $n-1<q<n, n\geq 2,\ m,n\in\mathbb{N}, \xi_k,\ \beta_i\in\mathbb{R}, \eta\in(0,T), k=0,1,\ldots,n-2,\ i=1,2,\ldots,m$. Here, ${}^cD^q_{0^+}$ denotes the Caputo fractional derivatives, and ${}_{RL}I^{p_i}_{0^+}$ is the Riemann–Liouville non-local fractional integral of order $p_i>0$. Assume that:

$$\Lambda := T^{n-1} - \sum_{i=1}^{m}\beta_i\eta_i^{p_i+n-1}\frac{\Gamma(n)}{\Gamma(n+p_i)} \neq 0.$$

Then, the solution of the above BVP has a unique solution given by:

$$\begin{aligned} u(t) &= {}_{RL}I^q_{0^+}h(t) + \frac{t^{n-1}}{\Lambda}\Big[\sum_{i=1}^{m}\beta_i {}_{RL}I^{p_i+q}_{0^+}h(\eta_i) + \sum_{i=1}^{m}\sum_{k=0}^{n-2}\beta_i\frac{\xi_k\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta^{p_i+k} \\ &\quad - {}_{RL}I^q_{0^+}h(T) - \sum_{k=0}^{n-2}\frac{\xi_k T^k}{k!}\Big] + \sum_{k=0}^{n-2}\frac{\xi_k t^k}{k!}. \end{aligned}$$

Proof. From Lemma (1), we get, for certain constant vectors $c_0,\ldots,c_{n-1}$ belonging to E,

$$u(t) = {}_{RL}I_{0^+}{}^q h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1}t^{n-1}.$$

From the first condition in BVP, we see,

$$c_0=\xi_0,\ c_1=\xi_1,\ c_2=\frac{\xi_2}{2!},\ldots,c_{n-2}=\frac{\xi_{n-2}}{(n-2)!},$$

and so,

$$u(t) = {}_{RL}I^q_{0^+}h(t) + \sum_{k=0}^{n-2}\frac{\xi_k t^k}{k!} + c_{n-1}t^{n-1}.$$

The substitution $T=t$ yields,

$$u(T) = {}_{RL}I^q_{0^+}h(T) + c_{n-1}T^{n-1} + \sum_{k=0}^{n-2}\frac{\xi_k T^k}{k!},$$

and applying the operator ${}_{RL}I^{p_i}_{0^+}$ results in:

$${}_{RL}I^{p_i}_{0^+}u(t) = ({}_{RL}I^{p_i+q}_{0^+}h)(t) + c_{n-1}\frac{\Gamma(n)}{\Gamma(p_i+n)}t^{p_i+n-1} + \sum_{k=0}^{n-2}\frac{\xi_k\Gamma(k+1)t^{p_i+k}}{k!\Gamma(p_i+k+1)}.$$

By employing the second boundary value condition, we infer:

$$\begin{aligned} {}_{RL}I^q_{0^+}h(T) + c_{n-1}T^{n-1} + \sum_{k=0}^{n-2}\frac{\xi_k T^k}{k!} &= \sum_{i=1}^{m}\beta_i\Big\{({}_{RL}I^{p_i+q}_{0^+}h(\eta_i)) + c_{n-1}\frac{\Gamma(n)}{\Gamma(p_i+n)}\eta_i^{p_i+n-1} \\ &\quad + \sum_{k=0}^{n-2}\frac{\xi_k\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta_i^{p_i+k}\Big\}. \end{aligned}$$

As a consequence, we get,

$$\begin{aligned} c_{n-1} &= \frac{1}{\Lambda}\Big[\sum_{i=1}^{m}\beta_i({}_{RL}I^{p_i+q}_{0^+}h)(\eta_i) + \sum_{i=1}^{m}\sum_{k=0}^{n-2}\beta_i\frac{\xi_k\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta_i^{p_i+k} \\ &\quad - {}_{RL}I^q_{0^+}h(T) - \sum_{k=0}^{n-2}\frac{\xi_k T^k}{k!}\Big]. \end{aligned}$$

Hence, the result:

$$
\begin{aligned}
u(t) &= {}_{RL}I^{q}_{0^+}h(t) + \frac{t^{n-1}}{\Lambda}\Big[\sum_{i=1}^{m}\beta_{i\,RL}I^{p_i+q}_{0^+}h(\eta_i) + \sum_{i=1}^{m}\sum_{k=0}^{n-2}\beta_i\frac{\xi_k\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta_i^{p_i+k} \\
&\quad - {}_{RL}I^{q}_{0^+}h(T) - \sum_{k=0}^{n-2}\frac{\xi_k T^k}{k!}\Big] + \sum_{k=0}^{n-2}\frac{\xi_k t^k}{k!}
\end{aligned}
$$

follows. □

Let E be the real vector space of all real-valued continuous functions defined on $[0,T]$, that is $E = C([0,T],\mathbb{R})$. Equipped with the supremum norm $\|u\|_\infty := \sup_{t\in[0,T]}|u(t)|$, $u \in E$, the space E is a Banach space. Define the non-linear operator $A : E \to E$ as follows:

$$
\begin{aligned}
(Au)(t) &= {}_{RL}I^{q}_{0^+}f(s,u(s))(t) + \frac{t^{n-1}}{\Lambda}\Big[- {}_{RL}I^{q}_{0^+}f(s,u(s))(T) + \sum_{i=1}^{m}\beta_{i\,RL}I^{p_i+q}_{0^+}f(s,u(s))(\eta_i) \\
&\quad + \sum_{i=1}^{m}\sum_{k=0}^{n-2}\beta_i\frac{\xi_k\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta^{p_i+k}\Big] + \sum_{k=0}^{n-2}\frac{\xi_k}{k!}(t^k - \frac{T^k t^{n-1}}{\Lambda}).
\end{aligned}
$$

Then, the operator A has a fixed point if and only if Problem (6) possesses a solution. In the next theorem, we present the existence of solutions for Problem (6) via the fixed point theorems of Krasnoselskii and Darbo.

3.1. Existence Result Via Krasnoselskii's Fixed Point Theorem

We begin with an existence result via the Krasnoselskii's fixed point theorem.

Theorem 3. *Let the function $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function, which is Lipschitz continuous in the second variable, that is there exists a finite constant L such that $|f(t,u) - f(t,v)| \le L|u-v|$ for all $t \in [0,T]$ and $u, v \in \mathbb{R}$. Suppose that there exists a continuous function $\varphi : [0,T] \to \mathbb{R}^+$ such that $|f(t,u)| \le \varphi(t)$, for all $(t,u) \in [0,T]\times\mathbb{R}$. Then, the boundary value problem (6) has at least one solution provided that $\gamma := L\left(\frac{T^{n+q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\frac{|\beta_i|\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}\right)$.*

Proof. Besides $\|\varphi(t)\|_\infty = \sup_{t\in[0,T]}|\varphi(t)|$, we write $\phi = \frac{T^q}{\Gamma(q+1)} + \frac{T^{n+q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\frac{|\beta_i|\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}$, $M = \frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=0}^{n-2}|\beta_i|\frac{|\xi_k|\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta_i^{p_i+k} + \left(1 + \frac{T^{n-1}}{|\Lambda|}\right)\sum_{k=0}^{n-2}\frac{|\xi_k|T^k}{k!}$, and choose $\rho \ge \|\varphi\|_\infty\phi + M$. Let $B_\rho = \{u \in E : \|u\| \le \rho\}$ be the ball of radius $\rho \ge \|\varphi\|_\infty\phi + M$ centered at the origin in E. In addition, introduce the operator A_1 and A_2 on $E = C([0,T],\mathbb{R})$ by:

$$
\begin{aligned}
A_1u(t) &= {}_{RL}I^{q}_{0^+}f(s,u(s))(t) \\
A_2u(t) &= -\frac{t^{n-1}}{\Lambda}{}_{RL}I^{q}_{0^+}f(s,u(s))(T) + \frac{t^{n-1}}{\Lambda}\sum_{i=1}^{m}\beta_{i\,RL}I^{p_i+q}_{0^+}f(s,u(s))(\eta_i) \\
&\quad + \frac{t^{n-1}}{\Lambda}\sum_{i=1}^{m}\sum_{k=0}^{n-2}\beta_i\frac{\xi_k\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta_i^{p_i+k} + \sum_{k=0}^{n-2}\frac{\xi_k}{k!}(t^k - \frac{t^{n-1}}{\Lambda}T^k).
\end{aligned}
$$

For any $u, v \in B_\rho$, we get:

$$
\begin{aligned}
|(A_1u)(t)+(A_2v)(t)| &\le \sup_{t\in[0,T]}\Big\{ {}_{RL}I^q_{0^+}|f(s,u(s))|(t)+\frac{t^{n-1}}{|\Lambda|}{}_{RL}I^q_{0^+}|f(s,v(s))|(T)\\
&\quad+\frac{t^{n-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|_{RL}I^{p_i+q}|f(s,v(s))|(\eta_i)\\
&\quad+\frac{t^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=0}^{n-2}|\beta_i|\frac{|\xi_k|\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta_i^{p_i+k}+\sum_{k=0}^{n-2}\frac{|\xi_k|}{k!}(t^k+\frac{t^{n-1}}{|\Lambda|}T^k)\Big\}\\
&\le \|\varphi\|\Big\{\frac{T^q}{\Gamma(q+1)}+\frac{T^{n+q-1}}{|\Lambda|\Gamma(q+1)}\\
&\quad+\frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\frac{|\beta_i|\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}\Big\}\\
&\quad+\frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=0}^{n-2}\frac{|\beta_i\xi_k|\Gamma(k+1)\eta_i^{p_i+k}}{k!\Gamma(p_i+k+1)}+(1+\frac{T^{n-1}}{|\Lambda|})\sum_{k=0}^{n-2}\frac{|\xi_k|T^k}{k!}\\
&= \|\varphi\|\phi+M\le\rho.
\end{aligned}
$$

These inequalities show that $A_1u+A_2v\in B_\rho$. In order to prove that A_2 is a contraction, we take $u,v\in E$ and get,

$$
\begin{aligned}
|(A_2u)(t)-(A_2v)(t)| &\le \frac{T^{n-1}}{|\Lambda|}\,{}_{RL}I^q_{0^+}|f(s,u(s))-f(s,v(s))|(T)\\
&\quad+\frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|({}_{RL}I^{p_i+q}_{0^+}|f(s,u(s))-f(s,v(s))|(\eta_i))\\
&\le L|u-v|\Big(\frac{T^{n+q-1}}{|\Lambda|\Gamma(q+1)}+\frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\frac{|\beta_i|\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}\Big)\\
&\le \gamma\|u-v\|.
\end{aligned}
$$

This implies that $\|A_2u-A_2v\|\le\gamma\|u-v\|$. Hence, A_2 is a contraction. Therefore, the operator A_1 is continuous by the continuity of f. Since for $u\in E$, we have $\|A_1u\|\le\|\varphi\|\frac{T^q}{\Gamma(q+1)}$, the operator A_1 is uniformly bounded on B_ρ. Next, we show that the operator A_1 is compact.

We define $\sup_{(t,u)\in[0,T]\times B_\rho}|f(t,u)|=\theta<\infty$, and for any $0<\tau_1<\tau_2<T$, we get:

$$
\begin{aligned}
|A_1u(\tau_2)-A_1u(\tau_1)| &= \frac{1}{\Gamma(q)}\Big|\int_0^{\tau_1}\big[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\big]f(s,u(s))ds\\
&\quad+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}f(s,u(s))ds\Big|\\
&\le \frac{\theta}{\Gamma(q+1)}[\tau_2^q-\tau_1^q].
\end{aligned}
$$

A consequence of these inequalities is that $\{A_1u:u\in B_\rho\}$ is a uniformly-bounded and equicontinuous set in E. Thus, by the Arzela–Ascoli theorem, the operator A_1 is compact on B_ρ. A combination of this property of the operator A_1 with the inclusion property $A_1B_\rho+A_2B_\rho\subset B_\rho$ implies, by Krasnoselskii's theorem, that the problem (6) has at least one solution on $[0,T]$. □

Example 1. *Consider the fractional boundary value problems:*

$$
\begin{cases}
{}^cD^{\frac{3}{2}}_{0^+}u(t)=\Big(\frac{|u(t)|}{|u(t)|+1}\Big)\frac{\cos^2(3t)}{(e^{2t}+3)^2}+\frac{\sqrt{3}t}{2}, \quad for \quad each \quad t\in[0,3],\\
u(0)=\pi,\ \ u(3)={}_{RL}I^{\frac{2}{5}}_{0^+}(1)+\sqrt{2}{}_{RL}I^{\frac{3}{5}}_{0^+}(2)+\sqrt{3}{}_{RL}I^{\frac{4}{5}}_{0^+}(\frac{5}{2}).
\end{cases}
\tag{7}
$$

By comparing the system (7) and (6), we obtain the following values: $q = \frac{3}{2}$, $m = 3$, $T = 3$, $n = 2$, $\beta_1 = 1$, $\beta_2 = \sqrt{2}$, $\beta_3 = \sqrt{3}$, $p_1 = \frac{2}{5}$, $p_2 = \frac{3}{5}$, $p_3 = \frac{4}{5}$, $\eta_1 = 1$, $\eta_2 = 2$, $\eta_3 = \frac{5}{2}$.

Here, $f(t,u) = \left(\frac{u(t)}{u(t)+1}\right)\frac{\cos^2(3t)}{(e^{2t}+3)^2} + \frac{\sqrt{3}t}{2}$.

As, $\|f(t,u) - f(t,v)\| \leq \frac{1}{16}\|u - v\|$, therefore the condition of the Theorem 3 is satisfied with $L = \frac{1}{16}$. Furthermore, we have $\gamma \approx 0.380034 < 1$ and:

$$|f(t,u(t))| = |\left(\frac{u(t)}{u(t)+1}\right)\frac{\cos^2(3t)}{(e^{2t}+3)^2} + \frac{\sqrt{3}t}{2}| \leq \frac{1}{(e^{2t}+3)^2} + \frac{\sqrt{3}t}{2}.$$

Hence, the system (7) has at least one solution on $[0,3]$.

3.2. Existence Result via Darbo's Fixed Point Theorem

In order to prove our main result, we assume the following hypotheses are satisfied:

Hypothesis 1 (H1). *$f : [0,T] \times E \to E$ be a continuous function.*

Hypothesis 2 (H2). *There exists a constant $L > 0$ such that $\|f(t,u) - f(t,v)\| \leq L\|u - v\|$ for each $t \in [0,T]$ and $u,v \in E$.*

Now, we prove our existence result for the problem (6) by Kuratowski MNC and Darbo's fixed point theorem.

Theorem 4. *Suppose that (H1)–(H2) hold. If:*

$$\phi L < 1$$

where $\phi = \frac{T^q}{\Gamma(q+1)} + \frac{T^{n+q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}\frac{|\beta_i|\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}$, *then the problem (6) has at least one solution on $[0,T]$.*

Proof. A solution to the boundary value problem (6) can be considered as a fixed point of the operator $A : E \to E$, defined by:

$$\begin{aligned}(Au)(t) \quad = \quad & {}_{RL}I_{0^+}^{q} f(s,u(s))(t) + \frac{t^{n-1}}{\Lambda}\Big[-{}_{RL}I_{0^+}^{q} f(s,u(s))(T) + \sum_{i=1}^{m}\beta_i {}_{RL}I^{p_i+q} f(s,u(s))(\eta_i) \\ & + \sum_{i=1}^{m}\sum_{k=0}^{n-2}\beta_i\frac{\xi_k\Gamma(k+1)}{k!\Gamma(p_i+k+1)}\eta_i^{p_i+k}\Big] + \sum_{k=0}^{n-2}\frac{\xi_k}{k!}(t^k - \frac{T^k t^{n-1}}{\Lambda}).\end{aligned}$$

Step 1: A is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in E, when $n \to \infty$. If $t \in [0,T]$, we get:

$$\begin{aligned}|(Au_n)(t) - (Au)(t)| \quad \leq \quad & {}_{RL}I_{0^+}^{q}|f(s,u_n(s)) - f(s,u(s))|(t) \\ & + \frac{T^{n-1}}{|\Lambda|}{}_{RL}I_{0^+}^{q}|f(s,u_n(s)) - f(s,u(s))|(T) \\ & + \frac{T^{n-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|{}_{RL}I_{0^+}^{p_i+q}|f(s,u_n(s)) - f(s,u(s))|(\eta_i) \\ \leq \quad & L\phi\|u_n - u\|\end{aligned}$$

which implies:

$$\|Au_n - Au\| \leq L\phi\|u_n - u\|,$$

so that $\|Au_n - Au\| \to 0$, if $n \to \infty$.

It follows that the operator A is continuous.

Define $B_r = \{u \in C([0,T],E) : \|u\| \leq r\}$, where $r = \frac{N\phi+M}{1-L\phi}$, and let $\sup_{t\in[0,T]} |f(t,0)| = N < \infty$.

Obviously, the set B_r is a closed, bounded, convex subset of the Banach space $C([0,T],E)$.

Step 2: $A(B_r) \subset B_r$.

Let u belong to B_r. In order to prove that $Au \in B_r$, it suffices to show that $|Au(t)| \leq r$ for $t \in [0,T]$. However, for $t \in [0,T]$, we have:

$$\begin{aligned} |Au(t)| \leq & \sup_{t\in[0,T]} \Big\{ {}_{RL}I^{q}_{0^+}|f(s,u(s))|(t) + \frac{t^{n-1}}{|\Lambda|}\Big[\sum_{i=1}^{m} |\beta_i| ({}_{RL}I^{p_i+q}_{0^+}|f(s,u(s))|(\eta_i) \\ & + \sum_{i=1}^{m}\sum_{k=0}^{n-2} |\beta_i| \frac{|\xi_k|\Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} + {}_{RL}I^{q}_{0^+}|f(s,u(s))|(T) + \sum_{k=0}^{n-2} \frac{|\xi_k| T^k}{k!} \Big] \\ & + \sum_{k=0}^{n-2} \frac{|\xi_k| t^k}{k!} \Big\}. \end{aligned}$$

Thus, we get:

$$\begin{aligned} |Au(t)| \leq & \sup_{t\in[0,T]} \Big\{ {}_{RL}I^{q}_{0^+}(|f(s,u(s)) - f(s,0)| + |f(s,0)|)(t) \\ & + \frac{T^{n-1}}{|\Lambda|}\Big[\sum_{i=1}^{m} |\beta_i| ({}_{RL}I^{p_i+q}|f(s,u(s)) - f(s,0)| + |f(s,0)|(\eta_i) \\ & + \sum_{i=1}^{m}\sum_{k=0}^{n-2} |\beta_i| \frac{|\xi_k|\Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} + {}_{RL}I^{q}_{0^+}|(f(s,u(s)) - f(s,0) + |f(s,0)|)(T) \\ & + \sum_{k=0}^{n-2} \frac{|\xi_k| T^k}{k!} \Big] + \sum_{k=0}^{n-2} \frac{|\xi_k| T^k}{k!} \Big\} \end{aligned}$$

and therefore, we see:

$$\begin{aligned} |Au(t)| \leq & (L\|u\| + N)_{RL}I^{q}_{0^+}(t) + (L\|u\| + N)\frac{T^{n-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i|_{RL}I^{p_i+q}_{0^+}(\eta_i) \\ & + \frac{T^{n-1}}{|\Lambda|}(L\|u\| + N)_{RL}I^{q}_{0^+}(T) + \frac{T^{n-1}}{|\Lambda|} \sum_{i=1}^{m}\sum_{k=0}^{n-2} \frac{|\beta_i||\xi_k|\Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} \\ & + (\frac{T^{n-1}}{|\Lambda|} + 1) \sum_{k=0}^{n-2} |\xi_k| \frac{T^k}{k!}. \end{aligned}$$

This implies,

$$\begin{aligned} |Au(t)| & \leq (Lr + N)\phi + \frac{T^{n-1}}{|\Lambda|} \sum_{i=1}^{m}\sum_{k=0}^{n-2} \frac{|\beta_i||\xi_k|\Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} + (\frac{T^{n-1}}{|\Lambda|} + 1) \sum_{k=0}^{n-2} |\xi_k| \frac{T^k}{k!} \\ & \leq r, \end{aligned}$$

where,

$$r \geq \frac{N\phi + \frac{T^{n-1}}{|\Lambda|} \sum_{i=1}^{m}\sum_{k=0}^{n-2} \frac{|\beta_i||\xi_k|\Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} + (\frac{T^{n-1}}{|\Lambda|} + 1) \sum_{k=0}^{n-2} |\xi_k| \frac{T^k}{k!}}{1 - L\phi}.$$

Thus, we get $A(B_r) \subset B_r$.

Step 3: $A(B_r)$ is uniformly bounded and equicontinuous.

From Step 2, we get $A(B_r) = \{Au : u \in B_r\} \subset B_r$. Hence, for each $u \in B_r$, we get $\|Au\| \leq r$, which means that $A(B_r)$ is uniformly bounded. Let $\tau_1, \tau_2 \in [0, T], \tau_1 < \tau_2$, define $\sup_{(t,u)\in[0,T]\times B_r} |f(t, u)| \leq \theta < \infty$, and choose $u \in B_r$. Then, we obtain,

$$\begin{aligned}
|(Au)(\tau_2) - (Au)(\tau_1)| \leq & \Bigg| \int_0^{\tau_1} \left[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\right] f(s, x(s)) \\
& + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} f(s, x(s)) ds \\
& + \frac{\tau_2^{n-1} - \tau_1^{n-1}}{\Lambda} \Big[\sum_{i=1}^{m} \beta_{i\,RL} I_{0^+}^{p_i+q} f(s, u(s))(\eta_i) \\
& + \sum_{i=1}^{m} \sum_{k=0}^{n-2} \frac{\beta_i \xi_k \Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} + {}_{RL}I_{0^+}^{q} f(s, u(s))(T) \\
& + \sum_{k=0}^{n-2} \frac{\xi_k T^k}{k!} \Big] + \sum_{k=0}^{n-2} \frac{\xi_k}{k!} (\tau_2^k - \tau_1^k) \Bigg|.
\end{aligned}$$

This implies,

$$\begin{aligned}
|(Au)(\tau_2) - (Au)(\tau_1)| \leq & \frac{\theta}{\Gamma(q+1)} (\tau_2^q - \tau_1^q) + \frac{\tau_2^{n-1} - \tau_1^{n-1}}{|\Lambda|} \Bigg[\theta \sum_{i=1}^{m} |\beta_i| \frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)} \\
& + \frac{\theta T^q}{\Gamma(q+1)} + \sum_{i=1}^{m} \sum_{k=0}^{n-2} |\beta_i| \frac{|\xi_k| \Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} \\
& + \sum_{k=0}^{n-2} \frac{|\xi_k| T^k}{k!} \Bigg] + \sum_{k=0}^{n-2} \frac{|\xi_k|}{k!} (\tau_2^k - \tau_1^k).
\end{aligned}$$

As $\tau_2 \to \tau_1$, the right-hand side tends to zero. Thus, $A(B_r)$ is equicontinuous and uniformly bounded. Hence, from the Arzela–Ascoli theorem, it follows that the set AB_r is relatively compact in B_r.

Step 4: The operator $A : B_r \to B_r$ is a strict set contraction. For a subset $V \subset B_r$ and $t \in [0, T]$, we have:

$$\begin{aligned}
\alpha(AV(t)) &= \alpha((Au)(t), u \in V) \\
&\leq \alpha\Bigg({}_{RL}I_{0^+}^{q} f(s, u(s))(t) + \frac{t^{n-1}}{\Lambda} \Bigg[\sum_{i=1}^{m} \beta_{i\,RL} I_{0^+}^{p_i+q} f(s, u(s))(\eta_i) \\
&\quad + \sum_{i=1}^{m} \sum_{k=0}^{n-2} \beta_i \frac{\xi_k \Gamma(k+1)}{k!\Gamma(p_i+k+1)} \eta_i^{p_i+k} + {}_{RL}I_{0^+}^{q} f(s, u(s))(T) \\
&\quad - \sum_{k=0}^{n-2} \frac{\xi_k T^k}{k!} \Bigg] + \sum_{k=0}^{n-2} \frac{\xi_k t^k}{k!}, u \in V \Bigg)
\end{aligned}$$

Lemma (2) together with the Kuratowski measure of noncompactness implies that for each $t \in [0, T]$,

$$\begin{aligned}
\alpha(AV(t)) \leq & \alpha\Bigg(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds + \frac{t^{n-1}}{\Lambda\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, u(s)) ds \\
& + \frac{t^{n-1}}{\Lambda} \sum_{i=1}^{m} \frac{\beta_i}{\Gamma(p_i+q)} \int_0^{\eta_i} (\eta_i - s)^{p_i+q-1} f(s, u(s)) ds, u \in V \Bigg).
\end{aligned}$$

It follows that,

$$\begin{aligned}
\alpha(AV(t)) &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\{\alpha(f(s,u(s))), u \in V\}ds \\
&\quad + \frac{t^{n-1}}{\Lambda\Gamma(q)}\int_0^T (T-s)^{q-1}\{\alpha(f(s,u(s))), u \in V\}ds \\
&\quad + \frac{t^{n-1}}{\Lambda\Gamma(p_i+q)}\sum_{i=1}^{m}|\beta_i|\int_0^{n_i} (\eta_i-s)^{p_i+q-1}\{\alpha(f(s,u(s))), u \in V\}ds \\
&\leq \Bigg(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{t^{n-1}}{\Lambda\Gamma(q)}\int_0^T (T-s)^{q-1}ds \\
&\quad + \frac{t^{n-1}}{\Lambda\Gamma(p_i+q)}\sum_{i=1}^{m}|\beta_i|\int_0^{\eta_i} (\eta_i-s)^{p_i+q-1}ds\Bigg)\{\alpha(f(s,u(s))), u \in V\} \\
&\leq \phi L\{\alpha(u(s), u \in V)\}, s \in [0,T] \\
&\leq \phi L\alpha(V(s)).
\end{aligned}$$

Hence, we obtain,

$$\alpha_c(AV) \leq \phi L\alpha_c(V).$$

Therefore, the operator A is a set contraction. By Darbo's fixed point theorem, the operator A has a fixed point, which is a solution to the problem (6). □

Example 2. *Consider the fractional boundary value problems:*

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}}u(t) = \left(\frac{|u(t)|}{|u(t)|+1}\right)\frac{\sin^4(t)}{(10^t+9)^2} + \frac{t^2}{5} + \frac{\sqrt{7}}{3}, \quad for \quad each \quad t \in [0,\pi], \\ u(0) = u'(0) = -1,\ u(\pi) = \frac{3}{2}{}_{RL}I_{0^+}^{\frac{1}{2}}(\frac{1}{2}) + \frac{3}{4}{}_{RL}I_{0^+}^{\frac{1}{3}}(1) + \frac{3}{5}{}_{RL}I_{0^+}^{\frac{1}{4}}(2). \end{cases} \tag{8}$$

By comparing the systems (8) *and* (6)*, we obtain the following values:* $q = \frac{5}{2}$, $m = 3$, $T = \pi$, $n = 3$, $\beta_1 = \frac{3}{2}$, $\beta_2 = \frac{3}{4}$, $\beta_3 = \frac{3}{5}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{4}$, $\eta_1 = \frac{1}{2}$, $\eta_2 = 1$, $\eta_3 = 2$, *and* $f(t,u) = \left(\frac{|u(t)|}{|u(t)|+1}\right)\frac{\sin^4(t)}{(10^t+9)^2} + \frac{t^2}{5} + \frac{\sqrt{7}}{3}$*. Since* $\|f(t,u) - f(t,v)\| \leq \frac{1}{10^2}\|u-v\|$*, the condition* $L\phi < 1$ *is satisfied with* $L = \frac{1}{100}$*. Further, it is found that* $\phi \approx 47.656474$ *and* $L\phi \approx 0.47656474 < 1$*. Hence, the system* (8) *has at least one solution on* $[0,\pi]$*.*

4. Conclusions

In this paper, we prove the existences of solutions of nonlinear Caputo fractional derivative with nonlocal Riemann-Liouville fractional integral condition. We obtained our results based on fractional calculus, Krasnoselskii's and Darbo's fixed point theorems respectively. Finally some examples are provided to illustrate our result.

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