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Some Symmetric Identities for Degenerate Carlitz-type (p,q)-Euler Numbers and Polynomials

Kyung-Won Hwang [1] and Cheon Seoung Ryoo [2,*]

1 Department of Mathematics, Dong-A University, Busan 604-714, Korea; khwang@dau.ac.kr
2 Department of Mathematics, Hannam University, Daejeon 34430, Korea
* Correspondence: ryoocs@hnu.kr

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Abstract: In this paper we define the degenerate Carlitz-type (p,q)-Euler polynomials by generalizing the degenerate Euler numbers and polynomials, degenerate Carlitz-type q-Euler numbers and polynomials. We also give some theorems and exact formulas, which have a connection to degenerate Carlitz-type (p,q)-Euler numbers and polynomials.

Keywords: degenerate Euler numbers and polynomials; degenerate q-Euler numbers and polynomials; degenerate Carlitz-type (p,q)-Euler numbers and polynomials

MSC: 11B68; 11S40; 11S80

1. Introduction

Many researchers have studied about the degenerate Bernoulli numbers and polynomials, degenerate Euler numbers and polynomials, degenerate Genocchi numbers and polynomials, degenerate tangent numbers and polynomials (see [1–7]). Recently, some generalizations of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials are provided (see [6,8–13]). In this paper we define the degenerate Carlitz-type (p,q)-Euler polynomials and numbers and study some theories of the degenerate Carlitz-type (p,q)-Euler numbers and polynomials.

Throughout this paper, we use the notations below: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers. We remind that the classical degenerate Euler numbers $\mathcal{E}_n(\lambda)$ and Euler polynomials $\mathcal{E}_n(x,\lambda)$, which are defined by generating functions like (1), and (2) (see [1,2])

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda)\frac{t^n}{n!}, \tag{1}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(x,\lambda)\frac{t^n}{n!}, \tag{2}$$

respectively.

Carlitz [1] introduced some theories of the degenerate Euler numbers and polynomials. We recall that well-known Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by this (see [2,7,14])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$. The numbers $S_2(n,m)$ is like this

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}.$$

We also have

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}.$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k)$$

for positive integer n, with $(x|\lambda)_0 = 1$; as we know,

$$(x|\lambda)_n = \sum_{k=0}^{n} S_1(n,k)\lambda^{n-k}x^k.$$

$(x|\lambda)_n = \lambda^n(\lambda^{-1}x|1)_n$ for $\lambda \neq 0$. Clearly $(x|0)_n = x^n$. The binomial theorem for a variable x is

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}.$$

The (p,q)-number is defined as

$$[n]_{p,q} = \frac{p^n-q^n}{p-q} = p^{n-1}+p^{n-2}q+p^{n-3}q^2+\cdots+p^2q^{n-3}+pq^{n-2}+q^{n-1}.$$

We begin by reminding the Carlitz-type (p,q)-Euler numbers and polynomials (see [9–11]).

Definition 1. *For $0 < q < p \leq 1$ and $h \in \mathbb{Z}$, the Carlitz-type (p,q)-Euler polynomials $E_{n,p,q}(x)$ and (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q}(x)$ are defined like this*

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,p,q}(x)\frac{t^n}{n!} &= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t},\\ \sum_{n=0}^{\infty} E^{(h)}_{n,p,q}(x)\frac{t^n}{n!} &= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m p^{hm} e^{[m+x]_{p,q}t},\end{aligned} \tag{3}$$

respectively (see [9–11]).

Now we make the degenerate Carlitz-type (p,q)-Euler number $\mathcal{E}_{n,p,q}(\lambda)$ and (p,q)-Euler polynomials $\mathcal{E}_{n,p,q}(x,\lambda)$. In the next section, we introduce the degenerate Carlitz-type (p,q)-Euler numbers and polynomials. We will study some their properties after introduction.

2. Degenerate Carlitz-Type (p,q)-Euler Polynomials

In this section, we define the degenerate Carlitz-type (p,q)-Euler numbers and polynomials and make some of their properties.

Definition 2. *For $0 < q < p \leq 1$, the degenerate Carlitz-type (p,q)-Euler numbers $\mathcal{E}_{n,p,q}(\lambda)$ and polynomials $\mathcal{E}_{n,p,q}(x,\lambda)$ are related to the generating functions*

$$F_{p,q}(t,\lambda) = \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(\lambda)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1+\lambda t)^{\frac{[m]_{p,q}}{\lambda}}, \tag{4}$$

and

$$F_{p,q}(t,x,\lambda) = \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x,\lambda)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1+\lambda t)^{\frac{[m+x]_{p,q}}{\lambda}}, \tag{5}$$

respectively.

Let $p = 1$ in (4) and (5), we can get the degenerate Carlitz-type q-Euler number $\mathcal{E}_{n,q}(x,\lambda)$ and q-Euler polynomials $\mathcal{E}_{n,q}(x,\lambda)$ respectively. Obviously, if $p = 1$, then we have

$$\mathcal{E}_{n,p,q}(x,\lambda) = \mathcal{E}_{n,q}(x,\lambda), \quad \mathcal{E}_{n,p,q}(\lambda) = \mathcal{E}_{n,q}(\lambda).$$

When $p = 1$, we have

$$\lim_{q\to 1} \mathcal{E}_{n,p,q}(x,\lambda) = \mathcal{E}_n(x,\lambda), \quad \lim_{q\to 1} \mathcal{E}_{n,p,q}(\lambda) = \mathcal{E}_n(\lambda).$$

We see that

$$\begin{aligned}(1+\lambda t)^{\frac{[x+y]_{p,q}}{\lambda}} &= e^{\frac{[x+y]_{p,q}}{\lambda}\log(1+\lambda t)} \\ &= \sum_{n=0}^{\infty} \left(\frac{[x+y]_{p,q}}{\lambda}\right)^n \frac{(\log(1+\lambda t))^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}[x+y]_{p,q}^m\right)\frac{t^n}{n!}.\end{aligned} \tag{6}$$

By (5), it follows that

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x,\lambda)\frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1+\lambda t)^{\frac{[m+x]_{p,q}}{\lambda}} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \\ &\quad \times \sum_{n=0}^{\infty}\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{\sum_{j=0}^{l}\binom{l}{j}(-1)^j p^{(x+m)(l-j)} q^{(x+m)j}}{(p-q)^l}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left([2]_q \sum_{l=0}^{n}\sum_{j=0}^{l} \frac{S_1(n,l)\lambda^{n-l}\binom{l}{j}(-1)^j q^{xj} p^{x(l-j)}}{(p-q)^l}\frac{1}{1+q^{j+1}p^{l-j}}\right)\frac{t^n}{n!}.\end{aligned} \tag{7}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 1. *For $0 < q < p \leq 1$ and $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}\mathcal{E}_{n,p,q}(x,\lambda) &= [2]_q \sum_{l=0}^{n}\sum_{j=0}^{l} \frac{S_1(n,l)\lambda^{n-l}\binom{l}{j}(-1)^j q^{xj} p^{x(l-j)}}{(p-q)^l} \frac{1}{1+q^{j+1}p^{l-j}} \\ &= [2]_q \sum_{m=0}^{\infty}\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}(-1)^m q^m [x+m]_{p,q}^l, \\ \mathcal{E}_{n,p,q}(\lambda) &= [2]_q \sum_{l=0}^{n}\sum_{j=0}^{l} \frac{S_1(n,l)\lambda^{n-l}\binom{l}{j}(-1)^j}{(p-q)^l} \frac{1}{1+q^{j+1}p^{l-j}} \\ &= [2]_q \sum_{m=0}^{\infty}\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}(-1)^m q^m [m]_{p,q}^l.\end{aligned}$$

We make the degenerate Carlitz-type (p,q)-Euler number $\mathcal{E}_{n,p,q}(\lambda)$. Some cases are

$$\begin{aligned}\mathcal{E}_{0,p,q}(\lambda) &= 1, \\ \mathcal{E}_{1,p,q}(\lambda) &= \frac{[2]_q}{(p-q)(1+pq)} - \frac{[2]_q}{(p-q)(1+q^2)}, \\ \mathcal{E}_{2,p,q}(\lambda) &= -\frac{[2]_q\lambda}{(p-q)(1+pq)} + \frac{[2]_q}{(p-q)^2(1+p^2q)} + \frac{[2]_q\lambda}{(p-q)(1+q^2)} \\ &\quad - \frac{2[2]_q}{(p-q)^2(1+pq^2)} + \frac{[2]_q}{(p-q)^2(1+q^3)}, \\ \mathcal{E}_{3,p,q}(\lambda) &= \frac{2[2]_q\lambda^2}{(p-q)(1+pq)} - \frac{3[2]_q\lambda}{(p-q)^2(1+p^2q)} + \frac{[2]_q}{(p-q)^3(1+p^3q)} \\ &\quad - \frac{2[2]_q\lambda^2}{(p-q)(1+q^2)} + \frac{6[2]_q\lambda}{(p-q)^2(1+pq^2)} - \frac{3[2]_q}{(p-q)^3(1+p^2q^2)} \\ &\quad - \frac{3[2]_q\lambda}{(p-q)^2(1+q^3)} + \frac{3[2]_q}{(p-q)^3(1+pq^3)} - \frac{[2]_q}{(p-q)^3(1+q^4)}.\end{aligned}$$

We use t instead of $\dfrac{e^{\lambda t}-1}{\lambda}$ in (5), we have

$$\begin{aligned}\sum_{m=0}^{\infty} E_{m,p,q}(x)\frac{t^m}{m!} &= \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x,\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n \frac{1}{n!} \\ &= \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x,\lambda)\lambda^{-n} \sum_{m=n}^{\infty} S_2(m,n)\lambda^m \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \mathcal{E}_{n,p,q}(x,\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{8}$$

Thus we have the following theorem.

Theorem 2. *For $m \in \mathbb{Z}_+$, we have*

$$E_{m,p,q}(x) = \sum_{n=0}^{m} \mathcal{E}_{n,p,q}(x,\lambda)\lambda^{m-n}S_2(m,n).$$

Use t instead of $\log(1+\lambda t)^{1/\lambda}$ in (3), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,p,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n \frac{1}{n!} \\
&= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1+\lambda t)^{\frac{[m+x]_{p,q}}{\lambda}} \\
&= \sum_{m=0}^{\infty} \mathcal{E}_{m,p,q}(x,\lambda)\frac{t^m}{m!},
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,p,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n \frac{1}{n!} \\
&= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} E_{n,p,q}(x)\lambda^{m-n} S_1(m,n)\right)\frac{t^m}{m!}.
\end{aligned} \tag{10}$$

Thus we have the below theorem from (9) and (10).

Theorem 3. *For $m \in \mathbb{Z}_+$, we have*

$$\mathcal{E}_{m,p,q}(x,\lambda) = \sum_{n=0}^{m} E_{n,p,q}(x)\lambda^{m-n} S_1(m,n).$$

We have the degenerate Carlitz-type (p,q)-Euler polynomials $\mathcal{E}_{n,p,q}(x,\lambda)$. some cases are

$$\begin{aligned}
\mathcal{E}_{0,p,q}(x,\lambda) &= 1, \\
\mathcal{E}_{1,p,q}(x,\lambda) &= \frac{[2]_q p^x}{(p-q)(1+pq)} - \frac{[2]_q q^x}{(p-q)(1+q^2)}, \\
\mathcal{E}_{2,p,q}(x,\lambda) &= -\frac{[2]_q \lambda p^x}{(p-q)(1+pq)} + \frac{[2]_q p^{2x}}{(p-q)^2(1+p^2q)} + \frac{[2]_q \lambda q^x}{(p-q)(1+q^2)} \\
&\quad - \frac{2[2]_q p^x q^x}{(p-q)^2(1+pq^2)} + \frac{[2]_q q^{2x}}{(p-q)^2(1+q^3)}, \\
\mathcal{E}_{3,p,q}(x,\lambda) &= \frac{2[2]_q \lambda^2 p^x}{(p-q)(1+pq)} - \frac{3[2]_q \lambda p^{2x}}{(p-q)^2(1+p^2q)} + \frac{[2]_q p^{3x}}{(p-q)^3(1+p^3q)} \\
&\quad - \frac{2[2]_q \lambda^2 q^x}{(p-q)(1+q^2)} + \frac{6[2]_q \lambda p^x q^x}{(p-q)^2(1+pq^2)} - \frac{3[2]_q p^{2x} q^x}{(p-q)^3(1+p^2q^2)} \\
&\quad - \frac{3[2]_q \lambda q^{2x}}{(p-q)^2(1+q^3)} + \frac{3[2]_q p^x q^{2x}}{(p-q)^3(1+pq^3)} - \frac{[2]_q q^{3x}}{(p-q)^3(1+q^4)}.
\end{aligned}$$

We introduce a (p,q)-analogue of the generalized falling factorial $(x|\lambda)_n$ with increment λ. The generalized (p,q)-falling factorial $([x]_{p,q}|\lambda)_n$ with increment λ is defined by

$$([x]_{p,q}|\lambda)_n = \prod_{k=0}^{n-1}([x]_{p,q} - \lambda k)$$

for positive integer n, where $([x]_{p,q}|\lambda)_0 = 1$.

By (4) and (5), we get

$$
\begin{aligned}
&-[2]_q(-1)^n q^n \sum_{l=0}^{\infty}(-1)^l q^l (1+\lambda t)^{\frac{[l+n]_{p,q}}{\lambda}} \\
&+[2]_q \sum_{l=0}^{\infty}(-1)^l q^l (1+\lambda t)^{\frac{[l+n]_{p,q}}{\lambda}} \\
&=[2]_q \sum_{l=0}^{n-1}(-1)^l q^l (1+\lambda t)^{\frac{[l]_{p,q}}{\lambda}}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&(-1)^{n+1} q^n \sum_{m=0}^{\infty} \mathcal{E}_{m,p,q}(n,\lambda)\frac{t^m}{m!}+\sum_{m=0}^{\infty}\mathcal{E}_{m,p,q}(\lambda)\frac{t^m}{m!} \\
&=\sum_{m=0}^{\infty}\left([2]_q \sum_{l=0}^{n-1}(-1)^l q^l ([l]_{p,q}|\lambda)_m\right)\frac{t^m}{m!}.
\end{aligned} \tag{11}
$$

By comparing the coefficients of $\frac{t^m}{m!}$ on both sides of (11), we have the following theorem.

Theorem 4. *For $n \in \mathbb{Z}_+$, we have*

$$
\sum_{l=0}^{n-1}(-1)^l q^l([l]_{p,q}|\lambda)_m=\frac{(-1)^{n+1}q^n\mathcal{E}_{m,p,q}(n,\lambda)+\mathcal{E}_{m,p,q}(\lambda)}{[2]_q}.
$$

We get that

$$
\begin{aligned}
&(1+\lambda t)^{\frac{[x+y]_{p,q}}{\lambda}} \\
&=(1+\lambda t)^{\frac{p^y[x]_{p,q}}{\lambda}}(1+\lambda t)^{\frac{q^x[y]_{p,q}}{\lambda}} \\
&=\sum_{m=0}^{\infty}(p^y[x]_{p,q}|\lambda)_m\frac{t^m}{m!}e^{\log(1+\lambda t)\frac{q^x[y]_{p,q}}{\lambda}} \\
&=\sum_{m=0}^{\infty}(p^y[x]_{p,q}|\lambda)_m\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\frac{q^x[y]_{p,q}}{\lambda}\right)^l\frac{\log(1+\lambda t)^l}{l!} \\
&=\sum_{m=0}^{\infty}(p^y[x]_{p,q}|\lambda)_m\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\frac{q^x[y]_{p,q}}{\lambda}\right)^l\sum_{k=l}^{\infty}S_1(k,l)\lambda^k\frac{t^k}{k!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(p^y[x]_{p,q}|\lambda)_{n-k}\lambda^{k-l}q^{xl}[y]_{p,q}^l S_1(k,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{12}
$$

By (5) and (12), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{E}_{n,p,q\zeta}(x,\lambda)\frac{t^n}{n!} \\
&= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m (1+\lambda t)^{\frac{[m+x]_{p,q}}{\lambda}} \\
&= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(p^m[x]_{p,q}|\lambda)_{n-k}\lambda^{k-l}q^{xl}[m]_{p,q}^{l}S_1(k,l)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left([2]_q \sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(-1)^m q^m (p^m[x]_{p,q}|\lambda)_{n-k}\lambda^{k-l}q^{xl}S_1(k,l)\right)\frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients of $\frac{t^n}{n!}$ in the above equation, we have the theorem below.

Theorem 5. *For $0 < q < p \leq 1$ and $n \in \mathbb{Z}_+$, we have*

$$
\mathcal{E}_{n,p,q}(x,\lambda) = [2]_q \sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(-1)^m q^m (p^m[x]_{p,q}|\lambda)_{n-k}\lambda^{k-l}q^{xl}S_1(k,l).
$$

3. Symmetric Properties about Degenerate Carlitz-Type (p,q)-Euler Numbers and Polynomials

In this section, we are going to get the main results of degenerate Carlitz-type (p,q)-Euler numbers and polynomials. We also make some symmetric identities for degenerate Carlitz-type (p,q)-Euler numbers and polynomials. Let w_1 and w_2 be odd positive integers. Remind that $[xy]_{p,q} = [x]_{p^y,q^y}[y]_{p,q}$ for any $x, y \in \mathbb{C}$.

By using $w_1 x + \frac{w_1 i}{w_2}$ instead of x in Definition 2, use p by p^{w_2}, use q by q^{w_2} and use λ by $\frac{\lambda}{[w_2]_{p,q}}$, respectively, we can get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left([2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,p^{w_2},q^{w_2}}\left(w_1 x+\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_{p,q}}\right)\right)\frac{t^n}{n!} \\
&= [2]_{q^{w_1}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{n=0}^{\infty}\mathcal{E}_{n,p^{w_2},q^{w_2}}\left(w_1 x+\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_{p,q}}\right)\frac{([w_2]_{p,q}t)^n}{n!} \\
&= [2]_{q^{w_1}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}[2]_{q^{w_2}}\sum_{n=0}^{\infty}(-1)^n q^{w_2 n} \\
&\quad\times\left(1+\frac{\lambda}{[w_2]_{p,q}}[w_2]_{p,q}t\right)^{\frac{[w_1 x+\frac{w_1 i}{w_2}+n]_{p^{w_2},q^{w_2}}}{\frac{\lambda}{[w_2]_{p,q}}}} \\
&= [2]_{q^{w_1}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}[2]_{q^{w_2}}\sum_{n=0}^{\infty}(-1)^n q^{w_2 n} \\
&\quad\times(1+\lambda t)^{\frac{[w_1 w_2 x+w_1 i+n w_2]_{p,q}}{\lambda}}.
\end{aligned}
$$

Since for any non-negative integer n and odd positive integer w_1, there is the unique non-negative integer r such that $n = w_1 r + j$ with $0 \leq j \leq w_1 - 1$. So this can be written as

$$\begin{aligned}
&[2]_{q^{w_1}}[2]_{q^{w_2}} \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i} \sum_{n=0}^{\infty}(-1)^n q^{w_2 n} \\
&\quad \times (1+\lambda t)^{\frac{[w_1 w_2 x + w_1 i + n w_2]_{p,q}}{\lambda}}. \\
&= [2]_{q^{w_1}}[2]_{q^{w_2}} \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i} \sum_{\substack{w_1 r + j = 0 \\ 0 \leq j \leq w_1 - 1}}^{\infty}(-1)^{w_1 r + j} q^{w_2(w_1 r + j)} \\
&\quad \times (1+\lambda t)^{\frac{[w_1 w_2 x + w_1 i + (w_1 r + j) w_2]_{p,q}}{\lambda}}. \\
&= [2]_{q^{w_1}}[2]_{q^{w_2}} \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i} \sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^{w_1 r}(-1)^j q^{w_2 w_1 r} q^{w_2 j} \\
&\quad \times (1+\lambda t)^{\frac{[w_1 w_2 x + w_1 i + w_1 w_2 r + w_2 j]_{p,q}}{\lambda}} \\
&= [2]_{q^{w_1}}[2]_{q^{w_2}} \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{w_1 i} q^{w_2 w_1 r} q^{w_2 j} \\
&\quad \times (1+\lambda t)^{\frac{[w_1 w_2 x + w_1 i + w_1 w_2 r + w_2 j]_{p,q}}{\lambda}}.
\end{aligned}$$

We have the below formula using the above formula

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left([2]_{q^{w_2}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,p^{w_2},q^{w_2}}\left(w_1 x + \frac{w_1 i}{w_2}, \frac{\lambda}{[w_2]_{p,q}}\right)\right)\frac{t^n}{n!} \\
&= [2]_{q^{w_1}}[2]_{q^{w_2}} \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{w_1 i} q^{w_2 w_1 r} q^{w_2 j} \\
&\quad \times (1+\lambda t)^{\frac{[w_1 w_2 x + w_1 i + w_1 w_2 r + w_2 j]_{p,q}}{\lambda}}.
\end{aligned} \tag{13}$$

From a similar approach, we can have that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left([2]_{q^{w_2}}[w_1]_{p,q}^n \sum_{i=0}^{w_1-1}(-1)^i q^{w_2 i}\mathcal{E}_{n,p^{w_1},q^{w_1}}\left(w_2 x + \frac{w_2 i}{w_1}, \frac{\lambda}{[w_1]_{p,q}}\right)\right)\frac{t^n}{n!} \\
&= [2]_{q^{w_1}}[2]_{q^{w_2}} \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{w_2 i} q^{w_1 w_1 r} q^{w_1 j} \\
&\quad \times (1+\lambda t)^{\frac{[w_1 w_2 x + w_2 i + w_1 w_2 r + w_1 j]_{p,q}}{\lambda}}.
\end{aligned} \tag{14}$$

Thus, we have the following theorem from (13) and (14).

Theorem 6. *Let w_1 and w_2 be odd positive integers. Then one has*

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]_{p,q}^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_{p,q}}\right)\\&=[2]_{q^{w_2}}[w_1]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{E}_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1},\frac{\lambda}{[w_1]_{p,q}}\right).\end{aligned}$$

Letting $\lambda\to 0$ in Theorem 6, we can immediately obtain the symmetric identities for Carlitz-type (p,q)-Euler polynomials (see [10])

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]_{p,q}^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\&=[2]_{q^{w_2}}[w_1]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right).\end{aligned}$$

It follows that we show some special cases of Theorem 6. Let $w_2=1$ in Theorem 6, we have the multiplication theorem for the degenerate Carlitz-type (p,q)-Euler polynomials.

Corollary 1. *Let w_1 be odd positive integer. Then*

$$\mathcal{E}_{n,p,q}(x,\lambda)=\frac{[2]_q[w_1]_{p,q}^n}{[2]_{q^{w_1}}}\sum_{j=0}^{w_1-1}(-1)^j q^j\mathcal{E}_{n,p^{w_1},q^{w_1}}\left(\frac{x+j}{w_1},\frac{\lambda}{[w_1]_{p,q}}\right). \tag{15}$$

Let $p=1$ in (15). This leads to the multiplication theorem about the degenerate Carlitz-type q-Euler polynomials

$$\mathcal{E}_{n,q}(x,\lambda)=\frac{[2]_q[w_1]_q^n}{[2]_{q^{w_1}}}\sum_{j=0}^{w_1-1}(-1)^j q^j\mathcal{E}_{n,q^{w_1}}\left(\frac{x+j}{w_1},\frac{\lambda}{[w_1]_q}\right). \tag{16}$$

Giving $q\to 1$ in (16) induce to the multiplication theorem about the degenerate Euler polynomials

$$\mathcal{E}_n(x,\lambda)=w_1^n\sum_{j=0}^{w_1-1}(-1)^j\mathcal{E}_n\left(\frac{x+i}{w_1},\frac{\lambda}{w_1}\right). \tag{17}$$

If λ approaches to 0 in (17), this leads to the multiplication theorem about the Euler polynomials(see [15])

$$E_n(x)=w_1^n\sum_{j=0}^{w_1-1}(-1)^jE_n\left(\frac{x+i}{w_1}\right).$$

Let $x=0$ in Theorem 6, then we have the following corollary.

Corollary 2. *Let w_1 and w_2 be odd positive integers. Then it has*

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]_{p,q}^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,p^{w_2},q^{w_2}}\left(\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_{p,q}}\right)\\&=[2]_{q^{w_2}}[w_1]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{E}_{n,p^{w_1},q^{w_1}}\left(\frac{w_2 j}{w_1},\frac{\lambda}{[w_1]_{p,q}}\right).\end{aligned}$$

By Theorem 3 and Corollary 2, we have the below theorem.

Theorem 7. *Let w_1 and w_2 be odd positive integers. Then*

$$\begin{aligned}
&\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}[w_2]_{p,q}^{l}[2]_{q^{w_1}} \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i} E_{l,p^{w_2},q^{w_2}}\left(\frac{w_1}{w_2}i\right)\\
&= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}[w_1]_{p,q}^{l}[2]_{q^{w_2}} \sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j} E_{l,p^{w_1},q^{w_1}}\left(\frac{w_2}{w_1}j\right).
\end{aligned}$$

We get another result by applying the addition theorem about the Carlitz-type (p,q)-Euler polynomials $E_{n,p,q}(x)$.

Theorem 8. *Let w_1 and w_2 be odd positive integers. Then we have*

$$\begin{aligned}
&\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k} S_1(n,l)\lambda^{n-l}p^{w_1w_2xk}[2]_{q^{w_1}}[w_1]_{p,q}^{k}[w_2]_{p,q}^{l-k}E^{(k)}_{l-k,p^{w_2},q^{w_2}}(w_1x)S_{l,k,p^{w_1},q^{w_1}}(w_2)\\
&= \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k} S_1(n,l)\lambda^{n-l}p^{w_1w_2xk}[2]_{q^{w_2}}[w_2]_{p,q}^{k}[w_1]_{p,q}^{l-k}E^{(k)}_{l-k,p^{w_1},q^{w_1}}(w_2x)S_{l,k,p^{w_2},q^{w_2}}(w_1),
\end{aligned}$$

where $S_{l,k,p,q}(w_1)=\sum_{i=0}^{w_1-1}(-1)^i q^{(l-k+1)i}[i]_{p,q}^{k}$ is called as the (p,q)-sums of powers.

Proof. From (3), Theorems 3 and 6, we have

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_{p,q}}\right)\\
&= [2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{l=0}^{n} E_{l,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\left(\frac{\lambda}{[w_2]_{p,q}}\right)^{n-l}S_1(n,l)\\
&= [2]_{q^{w_1}}\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}[w_2]_{p,q}^{l}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{k=0}^{l} q^{w_1(l-k)i}p^{w_1w_2xk}\\
&\qquad\times E^{(k)}_{l-k,p^{w_2},q^{w_2}}(w_1x)\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^{k}[i]_{p^{w_1},q^{w_1}}^{k}\\
&= [2]_{q^{w_1}}\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\sum_{k=0}^{l}\binom{l}{k}p^{w_1w_2xk}[w_1]_{p,q}^{k}[w_2]_{p,q}^{l-k}p^{w_1w_2xl}E^{(k)}_{l-k,p^{w_2},q^{w_2}}(w_1x)\\
&\qquad\times\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}q^{(l-k)w_1 i}[i]_{p^{w_1},q^{w_1}}^{k}.
\end{aligned}$$

Therefore, we induce that

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_{p,q}}\right)\\
&= \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}S_1(n,l)\lambda^{n-l}p^{w_1w_2xk}[2]_{q^{w_1}}[w_1]_{p,q}^{k}[w_2]_{p,q}^{l-k}p^{w_1w_2xl} \qquad (18)\\
&\qquad\times E^{(k)}_{l-k,p^{w_2},q^{w_2}}(w_1x)S_{l,k,p^{w_1},q^{w_1}}(w_2),
\end{aligned}$$

and

$$[2]_{q^{w_2}}[w_1]^n_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{E}_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1},\frac{\lambda}{[w_1]_{p,q}}\right)$$
$$=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}S_1(n,l)\lambda^{n-l}p^{w_1w_2xk}[2]_{q^{w_2}}[w_2]^k_{p,q}[w_1]^{l-k}_{p,q} \tag{19}$$
$$\times E^{(k)}_{l-k,p^{w_1},q^{w_1}}(w_2x)S_{l,k,p^{w_2},q^{w_2}}(w_1).$$

By (18) and (19), we make the desired symmetric identity. □

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