Sasaki-Einstein 7-Manifolds, Orlik Polynomials and Homology

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Abstract: In this article, we give ten examples of 2-connected seven dimensional Sasaki-Einstein manifolds for which the third homology group is completely determined. Using the Boyer-Galicki construction of links over particular Kähler-Einstein orbifolds, we apply a valid case of Orlik's conjecture to the links so that one is able to explicitly determine the entire third integral homology group. We give ten such new examples, all of which have the third Betti number satisfy $10 \leq b_3(L_f) \leq 20$.

1. Introduction

A rich source of constructing Sasaki-Einstein (SE) metrics of positive Ricci curvature pioneered by Boyer and Galicki in Reference [1] is via links of isolated hypersurface singularities defined by weighted homogenous polynomials. These smooth manifolds have been used to show the existence of SE metrics on many types of manifolds such as exotic spheres [2], rational homology spheres ([3,4]) and connected sums of $S^2 \times S^3$ [1] (see Reference [5] for more comprehensive survey.) SE manifolds are also extremely important in relation to the AdS/CFT Correspondence which is a conjecture that, in certain environments, relates Sasaki-Einstein geometries to particular superconformal theories. (See for example, Reference [6] for recent progress in the relationship between SE geomtries and the AdS/CFT conjecture.) In general it is very difficult to determine the diffeomorphism or even homeomorphism type of a given link so determining any such geometric or topological data about the link is always helpful. Along these lines, for a given link of dimension $2n - 1$, Milnor and Orlik [7] determined a formula for the $n - 1$ Betti number of the link and later on Orlik conjectured a formula [8] (or see section two) for the torsion in $n - 1$ integral homology group. This conjecture due to Orlik regarding the torsion in integral homology of links is known to hold in certain cases. Both of these formulas have been instrumental in extracting some topological data on certain SE manifolds arising as links. For example, based on work of Cheltsov [9], Boyer gave fourteen examples [10] of SE 7-manifolds arising from links of isolated hypersurface singularities for which the third integral homology group is completely determined. He used Brieskorn-Pham polynomials and Orlik polynomials (see Section 1), both of which are cases in which the aforementioned conjecture holds. Inspired by these examples, the main motivation for this article is to find other examples of SE 7-manifolds arising as links generated by Brieskorn-Pham polynomials or Orlik polynomials so that one can explicitly calculate the third integral homology group.

In general, there are obstructions to finding SE metrics (e.g., Bishop obstruction and Lichnerowicz obstruction [11]) so it is worth finding as many examples as possible of manifolds which due admit SE metrics. Indeed, the main result of the paper is a list of ten examples (see Section 2) of SE links defined by Orlik polynomials. Because of this, we are then able to calculate the torsion in the third integral

homology group explicitly. In Section 2, we review the necessary background and in Section 3 we give the table of ten examples together with the third Betti number and explicit forms of H_3.

2. Background

Define the weighted $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$ by

$$(z_0, ..., z_n) \longmapsto (\lambda^{w_0} z_0, ..., \lambda^{w_n} z_n)$$

where w_i are the weights which are positive integers and $\lambda \in \mathbb{C}^*$. We use the standard notation $\mathbf{w} = (w_0, ..., w_n)$ to denote a weight vector. In addition, we assume

$$gcd(w_0, ..., w_n) = 1.$$

Definition 1. *A polynomial $f \in \mathbb{C}[z_0, ..., z_n]$ is weighted homogenous if it satisfies*

$$f(\lambda^{w_0} z_0, ..., \lambda^{w_n} z_n) = \lambda^d f(z_0, ..., z_n)$$

for any $\lambda \in \mathbb{C}^*$ and the positive integer d is the degree of f.

The link L_f of an isolated hypersurface singularity defined by a weighted homogenous polynomial f with isolated singularity only at the origin is given by

$$L_f = C_f \cap S^{2n+1}$$

where C_f is the weighted affine cone defined by $f = 0$ in $\mathbb{C}^{n+1}$. By Milnor [12], L_f is a smooth $n - 2$ connected manifold of dimension $2n - 1$.

Recall a Fano orbifold Z is an orbifold for which the orbifold anticanonical bundle is ample.

Theorem 1 ([1]). *The link L_f as defined above admits as Sasaki-Einstein structure if and only if the Fano orbifold $\mathcal{Z}_f$ admits a Kähler-Einstein orbifold metric of scalar curvature $4n(n + 1)$*

Note that one simply needs to rescale a Kähler-Einstein metric of positive scalar curvature to get the desired scalar curvature in the statement of the theorem. We can think of the weighted hypersurface $\mathcal{Z}_f$ as the quotient space of the link L_f by the locally free circle action where this circle action comes from the weighted Sasakian structure on the link L_f. In fact this whole process is summarized in the commutative diagram [1]

$$
\begin{array}{ccc}
L_f & \longrightarrow & S^{2n+1} \\
\downarrow{\scriptstyle \pi} & & \downarrow \\
\mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w})
\end{array}
$$

where $S_{\mathbf{w}}^{2n+1}$ denotes the unit sphere with a weighted Sasakian structure, $\mathbb{P}(\mathbf{w})$ is weighted projective space coming from the quotient of $S_{\mathbf{w}}^{2n+1}$ by a weighted circle action generated from the weighted Sasakian structure. The top horizontal arrow is a Sasakian embedding and the bottom arrow is Kähler embedding. Moreover the vertical arrows are orbifold Riemannian submersions.

Thus, a mechanism for constructing 2-connected Sasaki-Einstein 7-manifolds boils down to finding orbifold Fano Kähler-Einstein hypersurfaces in weighted projective 4-space $\mathbb{P}(\mathbf{w})$. Johnson and Kollár in Reference [13] construct 4442 Fano orbifolds and of this list, 1936 of these are known to admit orbifold Kähler-Einstein metrics. Therefore, by the above construction we state a theorem of Boyer, Galicki and Nakamaye:

Theorem 2 ([3]). *There exists 1936 2-connected Sasaki-Einstein 7-manifolds realized as links of isolated hypersurface singularities defined by weighted homogenous polynomials.*

In Reference [3], the authors were able to determine many from the list of 1936 which yield rational homology 7-spheres and they also determined the order of $H_3(L_f, \mathbb{Z})$. In this paper, we identify ten links of isolated hypersurface singularities which can be given by so called Orlik polynomials, thus allowing us to calculate the third integral homology group explicitly. First, we need to define some quantities [7]:

$$u_i = \frac{d}{gcd(d, w_i)}, \qquad v_i = \frac{w_i}{gcd(d, w_i)}$$

Let L_f denote a link of an isolated hypersurface singularity defined by a weighted homogenous polynomial. The formula for the Betti number $b_{n-1}(L_f)$ is given by:

$$b_{n-1}(L_f) = \sum (-1)^{n+1-s} \frac{u_{i_1}, \ldots u_{i_s}}{v_{i_1} \ldots v_{i_s} \text{lcm}(u_{i_1}, \ldots, u_{i_s})}.$$

Here the sum is over all possible 2^{n+1} subsets $\{i_1, \ldots, i_s\}$ of $\{0, \ldots n\}$.

For the torsion data, Orlik conjectured [8] that for a given link L_f of dimension $2n - 1$ one has

$$H_{n-1}(L_f, \mathbb{Z})_{tor} = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r} \tag{1}$$

We should now review how the d_i data are given, using the presentation given in Reference [5]. Given an index set $\{i_1, i_2, \ldots, i_s\}$, define I to be the set of all of the 2^s subsets and let us designate J to be all of the proper subsets. For each possible subset, we must define (inductively) a pair of numbers $c_{i_1, \ldots, i_s}$ and $k_{i_1, \ldots, i_s}$. For each ordered subset $\{i_1, \ldots, i_s\} \subset \{0, 1, 2, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_s$ one defines the set of 2^s positive integers, beginning with $c_\emptyset = gcd(u_0, \ldots, u_n)$:

$$c_{i_1, \ldots, i_s} = \frac{gcd(u_0, \ldots, \widehat{u}_{i_1}, \ldots, \widehat{u}_{i_s}, \ldots, u_n)}{\prod_J c_{j_1, \ldots, j_t}}.$$

Now, to get the $k's$:

$$k_{i_1, \ldots, i_s} = \epsilon_{n-s+1} \kappa_{i_1, \ldots, i_s} = \epsilon_{n-s+1} \sum_I (-1)^{s-t} \frac{u_{j_1} \cdots u_{j_t}}{v_{j_1} \cdots v_{j_t} \text{lcm}(u_{j_1}, \ldots, u_{j_t})}$$

where

$$\epsilon_{n-s+1} = \begin{cases} 0, & \text{if } n - s + 1 \text{ is even} \\ 1, & \text{if } n - s + 1 \text{ is odd.} \end{cases}$$

Then for each $1 \leq j \leq r = \lfloor max\{k_{i_1, \ldots, i_s}\} \rfloor$ we put

$$d_j = \prod_{k_{i_1, \ldots, i_s} \geq j} c_{i_1, \ldots, i_s}.$$

Though the full conjecture is still open 45 years later, it is known to hold in certain cases. If the link is given by either of the polynomials below

$$z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}, \quad z_0^{a_0} + z_0 z_1^{a_1} + \cdots + z_{n-1} z_n^{a_n} \tag{2}$$

then the conjecture holds [8,14]. The first type of polynomial is called Brieskorn-Pham and the second one is called Orlik. We will discuss these a bit more in the next section.

The formulas for the Betti numbers and torsion would indeed be quite tedious to compute by hand, especially when the degree and the weights are large. Fortunately, Evan Thomas developed

a program written in C which computes the Betti numbers and the numbers d_i, which generates the torsion in $H_{n-1}(L_f, \mathbb{Z})$. Hence if the link is generated by a Brieskorn-Pham polynomial or an Orlik polynomial, then one explicitly knows the torsion in H_{n-1}. This program was also used extensively in References [5,10,15]. I would like to thank Evan Thomas for giving me permission to use the program and to make it available. See the Appendix A.

3. Examples

The paper of Johnson and Kollár [13] lists (see appendix for link to list) Kähler-Einstein and Tiger of Fano orbifolds in weighted projective space $\mathbb{P}(\mathbf{w})$ gives the weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ with $w_0 \leq w_1 \leq w_2 \leq w_3 \leq w_4$ (which can always be done after an affine change of coordinates) and it indicates if the weighted hypersurface admits an orbifold Kähler-Einstein structure. The degree d is given by $d = (w_0 + \cdots + w_4) - 1$. It is easy to identify whether or not Kähler-Einstein orbifolds on the list come from Brieskorn-Pham polynomials since for a given weight vector $\mathbf{w}$ on the list, the exponents of the Brieskorn-Pham polynomial would have to be $a_i = d/w_i$ for $i = 0, ..., 4$ and therefore one can do a computer search to see if one gets integer results for the exponents. But are there any coming from Orlik polynomials? To get some Orlik examples, one must search among the weighted hypersurfaces in the list of 1936 Kähler-Einstein orbifolds and see if the given weights can be represented by Orlik polynomials. This is more difficult than in the Brieskorn-Pham case since the constraints, given in 3.1, are more complicated. The search was done within the range $9 \leq w_0 \leq 11$ where there are 436 Fano orbifolds. Of this lot, 149 Fano orbifolds are known to admit an orbifold Fano Kähler-Einstein structure. Therefore, for a given weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ one needs to see if there exists exponents a_i, in the Orlik polynomials satisfying

$$d = a_0 w_0 = w_0 + w_1 a_1 = w_1 + w_2 a_2 = w_2 + w_3 a_3 = w_3 + w_4 a_4. \tag{3}$$

The ten examples were found by hand, checking many different weights against the given conditions. Once they were found, the computer program developed by Evan Thomas was implemented to determine the Betti number and the torsion data. We now give the table of ten examples. We list the weights, the quasihomogenous polynomial generating the link, the degree and finally the third homology group. It is not claimed that this list is exhaustive. There may very well be more examples using these methods.

(75,10,163,331,247)	$z_0^{11}+z_0z_1^{75}+z_1z_2^5+z_2z_3^2+z_3z_4^2$	825	$\mathbb{Z}^{10} \oplus \mathbb{Z}_{55} \oplus (\mathbb{Z}_5)^4$
(62,124,155,9,85)	$z_0^7+z_0z_1^3+z_1z_2^2+z_2z_3^{31}+z_3z_4^5$	434	$\mathbb{Z}^{12} \oplus \mathbb{Z}_{14} \oplus (\mathbb{Z}_2)^2$
(9,174,467,277,649)	$z_0^{175}+z_0z_1^9+z_1z_2^3+z_2z_3^4+z_3z_4^2$	1575	$\mathbb{Z}^{12} \oplus \mathbb{Z}_{525} \oplus (\mathbb{Z}_3)^2$
(87,348,145,11,193)	$z_0^9+z_0z_1^2+z_1z_2^3+z_2z_3^{58}+z_3z_4^4$	783	$\mathbb{Z}^{12} \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_3$
(100,350,9,113,229)	$z_0^8+z_0z_1^2+z_1z_2^{50}+z_2z_3^7+z_3z_4^3$	800	$\mathbb{Z}^{14} \oplus \mathbb{Z}_{400}$
(9,291,488,181,787)	$z_0^{195}+z_0z_1^6+z_1z_2^3+z_2z_3^7+z_3z_4^2$	1755	$\mathbb{Z}^{14} \oplus \mathbb{Z}_{585} \oplus \mathbb{Z}_3$
(10,164,333,71,253)	$z_0^{83}+z_0z_1^5+z_1z_2^2+z_2z_3^7+z_3z_4^3$	830	$\mathbb{Z}^{14} \oplus \mathbb{Z}_{166}$
(10,540,275,163,103)	$z_0^{109}+z_0z_1^2+z_1z_2^2+z_2z_3^5+z_3z_4^9$	1090	$\mathbb{Z}^{16} \oplus \mathbb{Z}_{218} \oplus \mathbb{Z}_2$
(32,144,11,103,31)	$z_0^{10}+z_0z_1^2+z_1z_2^{16}+z_2z_3^3+z_3z_4^7$	320	$\mathbb{Z}^{18} \oplus \mathbb{Z}_{160}$
(45,36,27,11,107)	$z_0^5+z_0z_1^5+z_1z_2^7+z_2z_3^{18}+z_3z_4^2$	225	$\mathbb{Z}^{20} \oplus \mathbb{Z}_5$

4. Conclusions

Because Sasaki-Einstein manifolds of positive Ricci curvature play such an important role in the AdS/CFT conjecture in string theory, it is of utmost importance to have as many examples as possible of Sasaki-Einstein manifolds especially in dimensions five and seven. In this paper, ten new examples of seven dimensional 2-connected Sasaki-Einstein manifolds were constructed by constructing links using Orlik polynomials over particular Kähler-Einstein Fano orbifolds. The third homology group was explicitly calculated using a conjectural formula which is known to be true for Orlik polynomials.

It is likely there are more examples using this approach but is difficult to detect them without a more systematic approach.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

(a) The Johnson-Kollár list of hypersurfaces in weighted projective 4-space $\mathbb{P}(\mathbf{w})$ admitting Kähler-Einstein orbifold metrics is available at https://web.math.princeton.edu/~jmjohnso/delpezzo/KEandTiger.txt. It lists the weights followed by data on wether or not it is known if the hypersurface admits a Kähler-Einstein orbifold metric.

(b) The code developed by Evan Thomas to compute the homology of links is available at https://blogs.swarthmore.edu/gomez/wp-content/uploads/2016/07/evans.c.

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