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Generalized Fixed-Point Results for Almost (α, F_σ)-Contractions with Applications to Fredholm Integral Inclusions

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Abstract: The purpose of this article is to define almost (α, F_σ)-contractions and establish some generalized fixed-point results for a new class of contractive conditions in the setting of complete metric spaces. In application, we apply our fixed-point theorem to prove the existence theorem for Fredholm integral inclusions $\varpi(t) \in \left[f(t) + \int_0^1 K(t,s,x(s))\vartheta s\right],\ t \in [0,1]$ where $f \in C[0,1]$ is a given real-valued function and $K : [0,1] \times [0,1] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is a given multivalued operator, where K_{cv} represents the family of nonempty compact and convex subsets of $\mathbb{R}$ and $\varpi \in C[0,1]$ is the unknown function. We also provide a non-trivial example to show the significance of our main result.

Keywords: almost (α, F_σ)-contractions; multivalued mappings; fixed point; complete metric space; Fredholm integral inclusions

MSC: primary 47H10; 46S40; secondary 54H25

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1. Introduction

In nonlinear analysis, the theory of fixed points plays one of the important parts and has many applications in computing sciences, physical sciences, and engineering. In 1922, Stefan Banach [1] established a prominent fixed-point result for contractive mapping in complete metric space (Ω, ϑ). Berinde [2] gave the notion of almost contraction and extended Banach's contraction principle.

Definition 1 ([2])**.** *A mapping $\mathcal{Z} : \Omega \to \Omega$ is called an almost contraction if $\exists\, \lambda \in [0,1)$ and some $L \geq 0$ such that*

$$\vartheta(\mathcal{Z}\varpi, \mathcal{Z}\omega) \leq \lambda\vartheta(\varpi, \omega) + L\vartheta(\omega, \mathcal{Z}\varpi) \tag{1}$$

$\forall \varpi, \omega \in \Omega$.

Samet et al. [3] defined the concept of α-admissible mappings as follows:

Definition 2 ([3])**.** *Let $\mathcal{Z} : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to [0, +\infty)$. We say that $\mathcal{Z}$ is a α-admissible mapping if*

$$\varpi, \omega \in \Omega, \quad \alpha(\varpi, \omega) \geq 1 \quad \Longrightarrow \quad \alpha(\mathcal{Z}\varpi, \mathcal{Z}\omega) \geq 1. \tag{2}$$

In 2012, Wardowski [4] introduced a new class of contractions called F-contraction and proved a fixed-point result as a generalization of the Banach contraction principle.

Let $\digamma$ be the collection of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) F is strictly increasing;
(F_2) for all $\{\varpi_n\} \subseteq R^+$, $\lim_{n\to\infty} \varpi_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(\varpi_n) = -\infty$;

(F_3) $\exists\, 0 < r < 1$ so that $\lim_{\varpi \to 0^+} \varpi^r F(\varpi) = 0$.

Definition 3 ([4])**.** *A mapping $\mathcal{Z} : \Omega \to \Omega$ is said to be a F-contraction if there exists $\tau > 0$ such that*

$$\vartheta(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0 \Longrightarrow \tau + F\big(\vartheta(\mathcal{Z}\varpi, \mathcal{Z}\omega)\big) \leq F\big(\vartheta(\varpi, \omega)\big) \tag{3}$$

$\forall \varpi, \omega \in \Omega$. We denote by Δ_F, the set of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying (F_1)–(F_3) and continuous from the right. For more details in the direction of F-contractions, we refer the readers to [5–10].

On the other hand, Nadler [11] initiated the notion of multivalued contraction and extended the Banach contraction principle from single-valued mapping to multivalued mapping.

Definition 4 ([11])**.** *A point $\varpi \in \Omega$ is called a fixed point of the multivalued mapping $\mathcal{Z}$: $\Omega \to 2^\Omega$ if $\varpi \in \mathcal{Z}\varpi$.*

For $A, B \in C(\Omega)$, let $H : C(\Omega) \times C(\Omega) \to [0, \infty)$ be defined by

$$H(A, B) = \max\{\sup_{\varpi \in A} \vartheta(\varpi, B), \sup_{\omega \in B} \vartheta(\omega, A)\}$$

where $\vartheta(\varpi, A) = \inf\{\vartheta(\varpi, \omega) : \omega \in A\}$. Such H is called the generalized Hausdorff–Pompieu metric induced by the metric ϑ and 2^Ω, $CL(\Omega)$ and $CB(\Omega)$ indicate the class of all nonempty, closed, and closed and bounded subsets of Ω, respectively.

Definition 5 ([11])**.** *A mapping $\mathcal{Z}$: $\Omega \to CB(\Omega)$ is said to be a multivalued contraction if $\exists\, 0 \leq \lambda < 1$ such that*

$$H(\mathcal{Z}\varpi, \mathcal{Z}\omega) \leq \lambda \vartheta(\varpi, \omega) \tag{4}$$

$\forall \varpi, \omega \in \Omega$.

Berinde et al. [12] introduced the notion of almost multivalued contraction as follows:

Definition 6 ([12])**.** *Let K a nonempty subset of Ω. A mapping $\mathcal{Z}$: $K \to CB(\Omega)$ is said to be an almost multivalued contraction if $\exists\, 0 \leq \lambda < 1$ and some $L \geq 0$ such that*

$$H(\mathcal{Z}\varpi, \mathcal{Z}\omega) \leq \lambda \vartheta(\varpi, \omega) + L\vartheta(\omega, \mathcal{Z}\varpi) \tag{5}$$

$\forall \varpi, \omega \in \Omega$.

Theorem 1 ([12])**.** *Let (Ω, ϑ) be a complete metric space and $\mathcal{Z}$: $\Omega \to CB(\Omega)$ an almost multivalued contraction, then $\mathcal{Z}$ has a fixed point.*

In 1994, Constantin [13] introduced a new family of continuous functions $\sigma : \mathbb{R}^{+5} \to \mathbb{R}^+$ satisfying the following assertions:

(ϱ_1) $\sigma(1,1,1,2,0), \sigma(1,1,1,0,2), \sigma(1,1,1,1,1) \in (0,1]$,

(ϱ_2) σ is sub-homogeneous, i.e., for all $(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) \in (\mathbb{R}^+)^5$ and $\alpha \geq 0$, we have $\sigma(\alpha\varpi_1, \alpha\varpi_2, \alpha\varpi_3, \alpha\varpi_4, \alpha\varpi_5) \leq \alpha\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5)$;

(ϱ_3) σ is a non-decreasing function, i.e., for $\varpi_i, \omega_i \in \mathbb{R}^+$, $\varpi_i \leq \omega_i$, $i = 1, ..., 5$, we have

$$\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) \leq \sigma(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$$

and if $\varpi_i, \omega_i \in \mathbb{R}^+$, $i = 1, ..., 4$, then $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, 0) \leq \sigma(\omega_1, \omega_2, \omega_3, \omega_4, 0)$ and $\sigma(\varpi_1, \varpi_2, \varpi_3, 0, \varpi_4) \leq \sigma(\omega_1, \omega_2, \omega_3, 0, \omega_4)$

and obtained a random fixed-point theorem for multivalued mappings. Following the lines in [13], we denote, by $\mathcal{S}$, the set of all above continuous functions. Isik [14] used the above family of functions and established multivalued fixed-point theorem in complete metric space. For more details in the direction of multivalued generalization, we refer the reader to (see [15–22]).

The theory of multivalued mappings has applications in control theory, convex optimization, differential equations, and economics. In recent years, the study of fixed point for multivalued mappings has gone beyond mere generalization of the single-valued case. Such studies have also been applied to prove the existence of equilibria in the context of game theory, and one such example is that of the famous Nash equilibrium. Thus, the correlation of symmetry is inherent in the study of multivalued fixed-point theory.

In the present paper, we define the notion of almost (α, F_σ)-contraction by considering the concept of α-admissibility, F-contraction, almost multivalued contraction, and the above set of continuous functions $\sigma : \mathbb{R}^{+5} \to \mathbb{R}^+$ to obtain generalized fixed-point results for a new class of contractive conditions in the context of complete metric spaces.

The following lemmas of Isik [14] are needed in the sequel.

Lemma 1 ([14])**.** *If $\sigma \in \mathcal{S}$ and $\varpi, \omega \in \mathbb{R}^+$ are such that*

$$\varpi < \max\left\{\sigma\left(\omega,\omega,\varpi,\omega+\varpi,0\right),\sigma\left(\omega,\omega,\varpi,0,\omega+\varpi\right),\sigma\left(\omega,\varpi,\omega,\omega+\varpi,0\right),\sigma\left(\omega,\varpi,\omega,0,\omega+\varpi\right)\right\},$$

then $\varpi < \omega$.

Lemma 2 ([14])**.** *Let (Ω, ϑ) be a metric space and $A, B \in CL(\Omega)$ with $H(A, B) > 0$. Then, $\forall\, h > 1$ and $a \in A$, $\exists\, b = b(a) \in B$ so that $\vartheta(a, b) < hH(A, B)$.*

2. Results

Definition 7. *A multivalued mapping $\mathcal{Z} : \Omega \to CB(\Omega)$ is said to be an almost (α, F_σ)-contraction, if $\exists$ $\alpha : \Omega \times \Omega \to [0, \infty)$, $F_\sigma \in \Delta F$, $\sigma \in \mathcal{S}$, $L \geq 0$ and $\tau > 0$ so that*

$$2\tau + F_\sigma(\alpha(\varpi,\omega)H(\mathcal{Z}\varpi,\mathcal{Z}\omega)) \leq F_\sigma\left(\sigma\left(\begin{array}{c}\vartheta(\varpi,\omega),\vartheta(\varpi,\mathcal{Z}\varpi),\vartheta(\omega,\mathcal{Z}\omega),\\ \vartheta(\varpi,\mathcal{Z}\omega),\vartheta(\omega,\mathcal{Z}\varpi)\end{array}\right)\right) + L\vartheta(\omega,\mathcal{Z}\varpi) \tag{6}$$

$\forall \varpi, \omega \in \Omega$ with $H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$.

Theorem 2. *Let (Ω, ϑ) be a complete metric space and $\mathcal{Z} : \Omega \to CB(\Omega)$ be an almost (α, F_σ)-contraction such that these assertions hold:*

(i) *$\mathcal{Z}$ is an α-admissible mapping,*
(ii) *$\exists\, \varpi_0 \in \Omega$ and $\varpi_1 \in \mathcal{Z}\varpi_0$ with $\alpha(\varpi_0, \varpi_1) \geq 1$,*
(iii) *for any $\{\varpi_n\}$ in Ω so that $\varpi_n \to \varpi$ and $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$, $\forall\, n \in \mathbb{N}$, we have $\alpha(\varpi_n, \varpi) \geq 1$, $\forall\, n \in \mathbb{N}$.*

Then $\exists \varpi^* \in \Omega$ such that $\varpi^* \in \mathcal{Z}\varpi^*$.

Proof. By hypothesis (ii), there exist $\varpi_0 \in \Omega$ and $\varpi_1 \in \mathcal{Z}\varpi_0$ with $\alpha(\varpi_0, \varpi_1) \geq 1$. If $\varpi_1 \in \mathcal{Z}\varpi_1$, then ϖ_1 is a fixed point of $\mathcal{Z}$ and so the proof is finished. Thus, we suppose that $\varpi_1 \notin \mathcal{Z}\varpi_1$. Then $\vartheta(\varpi_1, \mathcal{Z}\varpi_1) > 0$ and hence $H(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1) > 0$. From (6), we get

$$\begin{aligned} 2\tau + F_\sigma(\vartheta(\varpi_1, \mathcal{Z}\varpi_1)) &\leq 2\tau + F_\sigma(H(\mathcal{Z}\varpi_0, T\varpi_1)) \\ &\leq 2\tau + F_\sigma(\alpha(\varpi_0, \varpi_1) H(\mathcal{Z}\varpi_0, T\varpi_1)) \\ &\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_0, \mathcal{Z}\varpi_0), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \\ \vartheta(\varpi_0, \mathcal{Z}\varpi_1), \vartheta(\varpi_1, \mathcal{Z}\varpi_0) \end{array}\right)\right) + L\vartheta(\varpi_1, \mathcal{Z}\varpi_0) \\ &\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \\ \vartheta(\varpi_0, \mathcal{Z}\varpi_1), 0 \end{array}\right)\right) \end{aligned}$$

and so

$$\vartheta(\varpi_1, \mathcal{Z}\varpi_1) < \sigma\left(\begin{array}{c} \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \\ \vartheta(\varpi_0, \mathcal{Z}\varpi_1), 0 \end{array}\right)$$

Then Lemma 1 shows that $\vartheta(\varpi_1, \mathcal{Z}\varpi_1) < \vartheta(\varpi_0, \varpi_1)$. Thus, we obtain

$$\begin{aligned} 2\tau + F_\sigma(\vartheta(\varpi_1, \mathcal{Z}\varpi_1)) &\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \\ \vartheta(\varpi_0, \mathcal{Z}\varpi_1), 0 \end{array}\right)\right) \\ &< F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_0, \varpi_1), \vartheta(\varpi_0, \varpi_1), \\ 2\vartheta(\varpi_0, \varpi_1), 0 \end{array}\right)\right) \\ &\leq F_\sigma(\vartheta(\varpi_0, \varpi_1)\sigma(1, 1, 1, 2, 0)) \\ &\leq F_\sigma(\vartheta(\varpi_0, \varpi_1)) \end{aligned}$$

Thus

$$2\tau + F_\sigma(\vartheta(\varpi_1, \mathcal{Z}\varpi_1)) \leq F_\sigma(\vartheta(\varpi_0, \varpi_1)) \tag{7}$$

Since $F_\sigma \in \Delta\mathcal{F}$, so $\exists\, l > 1$ such that

$$F_\sigma(lH(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1)) < F_\sigma(H(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1)) + \tau. \tag{8}$$

Next as

$$\vartheta(\varpi_1, \mathcal{Z}\varpi_1) \leq H(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1) < lH(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1) \tag{9}$$

by Lemma 2, there exists $\varpi_2 \in \mathcal{Z}\varpi_1$ (obviously, $\varpi_2 \neq \varpi_1$) such that

$$\vartheta(\varpi_1, \varpi_2) \leq \vartheta(\varpi_1, \mathcal{Z}\varpi_1). \tag{10}$$

Thus, by (8)–(10), we have

$$F_\sigma(\vartheta(\varpi_1, \varpi_2)) \leq F_\sigma(lH(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1)) < F_\sigma(H(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1)) + \tau \tag{11}$$

which implies by (7) that

$$\begin{aligned} 2\tau + F_\sigma(\vartheta(\varpi_1, \varpi_2)) &\leq 2\tau + F_\sigma(H(\mathcal{Z}\varpi_0, \mathcal{Z}\varpi_1)) + \tau \\ &\leq F_\sigma(\vartheta(\varpi_0, \varpi_1)) + \tau \end{aligned}$$

Thus, we have

$$\tau + F_\sigma(\vartheta(\varpi_1, \varpi_2)) \leq F_\sigma(\vartheta(\varpi_0, \varpi_1)). \tag{12}$$

Since $\alpha(\varpi_0, \varpi_1) \geq 1$. So by the α-admissibility of $\mathcal{Z}$ and (6), we have

$$\begin{aligned}
2\tau + F_\sigma\left(\vartheta\left(\varpi_2, \mathcal{Z}\varpi_2\right)\right) &\leq 2\tau + F_\sigma\left(H\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right)\right) \\
&\leq 2\tau + F_\sigma\left(\alpha(\varpi_1, \varpi_2) H\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right)\right) \\
&\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \vartheta(\varpi_2, \mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1, \mathcal{Z}\varpi_2), \vartheta(\varpi_2, \mathcal{Z}\varpi_1) \end{array}\right)\right) + L\vartheta(\varpi_2, \mathcal{Z}\varpi_1) \\
&\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_2, \mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1, \mathcal{Z}\varpi_2), 0 \end{array}\right)\right)
\end{aligned}$$

and so

$$\vartheta\left(\varpi_2, \mathcal{Z}\varpi_2\right) < \sigma\left(\begin{array}{c} \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_2, \mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1, \mathcal{Z}\varpi_2), 0 \end{array}\right).$$

Then Lemma 1 gives that $\vartheta\left(\varpi_2, \mathcal{Z}\varpi_2\right) < \vartheta(\varpi_1, \varpi_2)$. Thus, we obtain

$$\begin{aligned}
2\tau + F_\sigma\left(\vartheta\left(\varpi_2, \mathcal{Z}\varpi_2\right)\right) &\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \vartheta(\varpi_2, \mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1, \mathcal{Z}\varpi_2), \vartheta(\varpi_2, \mathcal{Z}\varpi_1) \end{array}\right)\right) \\
&\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \varpi_2), \\ 2\vartheta(\varpi_1, \varpi_2), 0 \end{array}\right)\right) \\
&\leq F_\sigma\left(\vartheta(\varpi_1, \varpi_2)\sigma\left(1, 1, 1, 2, 0\right)\right) \\
&\leq F_\sigma\left(\vartheta(\varpi_1, \varpi_2)\right).
\end{aligned}$$

Thus, we get

$$2\tau + F_\sigma\left(\vartheta\left(\varpi_2, \mathcal{Z}\varpi_2\right)\right) \leq F_\sigma\left(\vartheta(\varpi_1, \varpi_2)\right) \tag{13}$$

Since $F_\sigma \in \Delta\digamma$, so $\exists\, l > 1$ such that

$$F_\sigma\left(lH\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right)\right) < F_\sigma\left(H\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right)\right) + \tau. \tag{14}$$

Next, as

$$\vartheta\left(\varpi_2, \mathcal{Z}\varpi_2\right) \leq H\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right) < lH\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right) \tag{15}$$

by Lemma 1, there exists $\varpi_3 \in \mathcal{Z}\varpi_2$ (obviously, $\varpi_3 \neq \varpi_2$) such that

$$\vartheta(\varpi_2, \varpi_3) \leq \vartheta\left(\varpi_2, \mathcal{Z}\varpi_2\right). \tag{16}$$

Thus, by (14)–(16), we have

$$F_\sigma\left(\vartheta(\varpi_2, \varpi_3)\right) \leq F_\sigma\left(lH\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right)\right) < F_\sigma\left(H\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right)\right) + \tau \tag{17}$$

which implies by (13) that

$$\begin{aligned}
2\tau + F_\sigma\left(\vartheta(\varpi_2, \varpi_3)\right) &\leq 2\tau + F_\sigma\left(H\left(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2\right)\right) + \tau \\
&\leq F_\sigma\left(\vartheta(\varpi_1, \varpi_2)\right) + \tau.
\end{aligned}$$

Thus, we have

$$\tau + F_\sigma\left(\vartheta(\varpi_2, \varpi_3)\right) \leq F_\sigma(\vartheta(\varpi_1, \varpi_2)). \tag{18}$$

Thus, pursuing these lines, we obtain $\{\varpi_n\}$ in Ω so that $\varpi_{n+1} \in \mathcal{Z}\varpi_n$ and $\alpha(\varpi_n, \varpi_{n+1}) \geq 1, \forall\, n \in \mathbb{N}$. Furthermore

$$\tau + F_\sigma(\vartheta(\varpi_n, \varpi_{n+1})) \leq F_\sigma(\vartheta(\varpi_{n-1}, \varpi_n)) \tag{19}$$

$\forall\ n \in \mathbb{N}$.Therefore by (19), we have

$$\begin{aligned} F_\sigma\left(\vartheta(\varpi_n,\varpi_{n+1})\right) &\leq F_\sigma\left(\vartheta(\varpi_{n-1},\varpi_n)\right)-\tau \leq F_\sigma\left(\vartheta(\varpi_{n-2},\varpi_{n-1})\right)-2\tau \\ &\leq \ldots \leq F_\sigma\left(\vartheta(\varpi_0,\varpi_1)\right)-n\tau. \end{aligned} \tag{20}$$

Letting $n \to \infty$, we have $\lim\limits_{n\to\infty} F_\sigma\left(\vartheta(\varpi_n,\varpi_{n+1})\right) = -\infty$ that jointly with (F_2) gives

$$\lim_{n\to\infty} \vartheta(\varpi_n,\varpi_{n+1}) = 0.$$

Thus, from (F_3), $\exists\ r \in (0,1)$ so that

$$\lim_{n\to\infty} [\vartheta(\varpi_n,\varpi_{n+1})]^r F_\sigma\left(\vartheta(\varpi_n,\varpi_{n+1})\right) = 0. \tag{21}$$

By (20) and (21), we obtain

$$\begin{aligned} & [\vartheta(\varpi_n,\varpi_{n+1})]^r F_\sigma\left(\vartheta(\varpi_n,\varpi_{n+1})\right) - [\vartheta(\varpi_n,\varpi_{n+1})]^r F_\sigma\left(\vartheta(\varpi_0,\varpi_1)\right) \\ \leq\ & [\vartheta(\varpi_n,\varpi_{n+1})]^r [F_\sigma\left(\vartheta(\varpi_0,\varpi_1)\right) - n\tau] - [\vartheta(\varpi_n,\varpi_{n+1})]^r F_\sigma\left(\vartheta(\varpi_0,\varpi_1)\right) \\ \leq\ & -n\tau[\vartheta(\varpi_n,\varpi_{n+1})]^r \leq 0. \end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} n[\vartheta(\varpi_n,\varpi_{n+1})]^r = 0. \tag{22}$$

Thus, $\lim\limits_{n\to\infty} n^{\frac{1}{r}}\vartheta(\varpi_n,\varpi_{n+1}) = 0$, which implies that $\sum_{n=1}^{\infty} \vartheta(\varpi_n,\varpi_{n+1})$ converges. Hence the sequence $\{\varpi_n\}$ is Cauchy in Ω. As (Ω,ϑ) is complete, so $\exists\ \varpi^* \in \Omega$ such that

$$\lim_{n\to\infty} \varpi_n = \varpi^*. \tag{23}$$

Now, we prove that $\varpi^* \in \mathcal{Z}\varpi^*$. By condition (iii), we have $\alpha(\varpi_n,\varpi^*) \geq 1, \forall n \in \mathbb{N}$. Assume on the contrary that $\varpi^* \notin \mathcal{Z}\varpi^*$, then $\exists\ n_0 \in \mathbb{N}$ and $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ so that $\vartheta(\varpi_{n_k+1},\mathcal{Z}\varpi^*) > 0, \forall\ n_k \geq n_0$. Now, using (3.1) with $\varpi = \varpi_{n_k+1}$ and $\omega = \varpi^*$, we have

$$\begin{aligned} 2\tau + F_\sigma(\vartheta(\varpi_{n_k+1},\mathcal{Z}\varpi^*)) &\leq 2\tau + F_\sigma(H(\mathcal{Z}\varpi_{n_k},\mathcal{Z}\varpi^*)) \\ &\leq 2\tau + F_\sigma(\alpha(\varpi_{n_k},\varpi^*)H(\mathcal{Z}\varpi_{n_k},\mathcal{Z}\varpi^*)) \\ &\leq F_\sigma\left(\sigma\left(\begin{array}{c} \vartheta(\varpi_{n_k},\varpi^*),\vartheta(\varpi_{n_k},\mathcal{Z}\varpi_{n_k}),\vartheta(\varpi^*,\mathcal{Z}\varpi^*), \\ \vartheta(\varpi_{n_k},\mathcal{Z}\varpi^*),\vartheta(\varpi^*,\mathcal{Z}\varpi_{n_k}) \end{array}\right)\right) \end{aligned}$$

By (F_1), we get

$$\vartheta(\varpi_{n_k+1},\mathcal{Z}\varpi^*) < \sigma\left(\begin{array}{c} \vartheta(\varpi_{n_k},\varpi^*),\vartheta(\varpi_{n_k},\varpi_{n_k+1}),\vartheta(\varpi^*,\mathcal{Z}\varpi^*), \\ \vartheta(\varpi_{n_k},\mathcal{Z}\varpi^*),\vartheta(\varpi^*,\varpi_{n_k+1}) \end{array}\right)$$

Taking $n \to \infty$, we get

$$\vartheta(\varpi^*,\mathcal{Z}\varpi^*) \leq \sigma\left(0,0,\vartheta(\varpi^*,\mathcal{Z}\varpi^*),\vartheta(\varpi^*,\mathcal{Z}\varpi^*),0\right)$$

which implies by Lemma 1 that

$$0 < \vartheta(\varpi^*,\mathcal{Z}\varpi^*) < 0$$

which is a contradiction. Hence $\vartheta(\varpi^*,\mathcal{Z}\varpi^*) = 0$. Thus, by the closedness of $\mathcal{Z}\varpi^*$, we deduce that $\varpi^* \in \mathcal{Z}\varpi^*$. Hence $\varpi^* \in \mathcal{Z}\varpi^*$. $\square$

3. Consequences

Now we give a result of Banach-type F_σ-contraction [1] in this way.

Corollary 1. *Let $\mathcal{Z} : \Omega \to CB(\Omega)$. Suppose that $\exists \tau > 0$ and $F_\sigma \in \Delta\digamma$ such that*

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi, \mathcal{Z}\omega)) \leq F_\sigma\left(\vartheta(\varpi, \omega)\right)$$

$\forall\, \varpi, \omega \in \Omega$ with $H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$.Then $\exists \varpi^* \in \Omega$ such that $\varpi^* \in \mathcal{Z}\varpi^*$.

Proof. Considering $\sigma \in \mathcal{S}$ given by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1$ and $L = 0$ in Theorem 2. □

Now we give a result of Kannan-type F-contraction [23] in this way.

Corollary 2. *Let $\mathcal{Z} : \Omega \to CB(\Omega)$. Suppose that $\exists \tau > 0$ and $F_\sigma \in \Delta\digamma$ such that*

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi, \mathcal{Z}\omega)) \leq F_\sigma\left(\vartheta(\varpi, \mathcal{Z}\varpi) + \vartheta(\omega, \mathcal{Z}\omega)\right)$$

$\forall\, \varpi, \omega \in \Omega$ with $H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$.Then $\exists \varpi^* \in \Omega$ such that $\varpi^* \in \mathcal{Z}\varpi^*$.

Proof. Considering $\sigma \in \mathcal{S}$ given by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_2 + \varpi_3$ and $L = 0$ in Theorem 2. □

Now we give a result of Chatterjea-type F-contraction [24] in this way.

Corollary 3. *Let $\mathcal{Z} : \Omega \to CB(\Omega)$. Suppose that $\exists \tau > 0$ and $F_\sigma \in \Delta\digamma$ such that*

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi, \mathcal{Z}\omega)) \leq F_\sigma\left(\vartheta(\varpi, \mathcal{Z}\omega) + \vartheta(\omega, \mathcal{Z}\varpi)\right)$$

$\forall\, \varpi, \omega \in \Omega$ with $H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$.Then $\exists \varpi^* \in \Omega$ such that $\varpi^* \in \mathcal{Z}\varpi^*$.

Proof. Considering $\sigma \in \mathcal{S}$ given by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_4 + \varpi_5$ and $L = 0$ in Theorem 2. □

Now we give a result of Hardy–Roger-type F-contraction [25] in this way.

Corollary 4 ([9])**.** *Let $\mathcal{Z} : \Omega \to CB(\Omega)$. Suppose that $\exists \tau > 0$ and $F_\sigma \in \Delta\digamma$ and non-negative real numbers $\beta_1, \beta_2, \beta_3, \beta_4$ and β_5 with $\beta_1 + \beta_2 + \beta_3 + \beta_4 + 2\beta_5 \leq 1$ such that*

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi, \mathcal{Z}\omega)) \leq F_\sigma\left(\begin{array}{c} \beta_1\vartheta(\varpi, \omega) + \beta_2\vartheta(\varpi, \mathcal{Z}\varpi) + \beta_3\vartheta(\omega, \mathcal{Z}\omega) \\ +\beta_4\vartheta(\varpi, \mathcal{Z}\omega) + \beta_5\vartheta(\omega, \mathcal{Z}\varpi) \end{array}\right)$$

$\forall\, \varpi, \omega \in \Omega$ with $H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$.Then $\exists \varpi^* \in \Omega$ such that $\varpi^* \in \mathcal{Z}\varpi^*$.

Proof. Considering $\sigma \in \mathcal{S}$ given by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \beta_1\varpi_1 + \beta_2\varpi_2 + \beta_3\varpi_3 + \beta_4\varpi_4 + \beta_5\varpi_5$ and $L = 0$ in Theorem 2. □

Now we give a result of Ćirić-type F-contraction [26] in this way.

Corollary 5. *Let $\mathcal{Z} : \Omega \to CB(\Omega)$. Suppose that $\exists \tau > 0$ and $F_\sigma \in \Delta\digamma$ such that*

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi, \mathcal{Z}\omega)) \leq F_\sigma\left(\max\left\{\begin{array}{c} \vartheta(\varpi, \omega), \vartheta(\varpi, \mathcal{Z}\varpi), \vartheta(\omega, \mathcal{Z}\omega), \\ \frac{\vartheta(\varpi, \mathcal{Z}\omega) + \vartheta(\omega, \mathcal{Z}\varpi)}{2} \end{array}\right\}\right)$$

$\forall\, \varpi, \omega \in \Omega$ with $H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$.Then $\exists \varpi^* \in \Omega$ such that $\varpi^* \in \mathcal{Z}\varpi^*$.

Proof. Considering $\sigma \in \mathcal{S}$ given by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \max\left\{\varpi_1, \varpi_2, \varpi_3, \frac{\varpi_4+\varpi_5}{2}\right\}$ and $L = 0$ in Theorem 2. □

The next result is also a Ćirić-type F-contraction [27].

Corollary 6. *Let $\mathcal{Z} : \Omega \to CB(\Omega)$. Suppose that $\exists \tau > 0$ and $F_\sigma \in \Delta \digamma$ such that*

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi, \mathcal{Z}\omega)) \leq F_\sigma\left(\max\left\{\begin{array}{c} \vartheta(\varpi, \omega), \vartheta(\varpi, \mathcal{Z}\varpi), \vartheta(\omega, \mathcal{Z}\omega), \\ \vartheta(\varpi, \mathcal{Z}\omega), \vartheta(\omega, \mathcal{Z}\varpi) \end{array}\right\}\right) \tag{24}$$

$\forall\, \varpi, \omega \in \Omega$ with $H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$.Then $\exists \varpi^* \in \Omega$ such that $\varpi^* \in \mathcal{Z}\varpi^*$.

Proof. Considering $\sigma \in \mathcal{S}$ given by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \max\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$ in Theorem 2. □

Example 1. *Let $\Omega = \mathbb{N} \cup \{0\}$ be endowed with the usual metric*

$$\vartheta(\varpi, \omega) = |\varpi - \omega|$$

$\forall \varpi, \omega \in \Omega$. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(\varpi, \omega) = \begin{cases} 2, \text{ if } \varpi, \omega \in \{0, 1\} \\ \frac{1}{2}, \text{ if } \quad \varpi, \omega > 1 \\ 0, \text{ otherwise} \end{cases}$$

and $\mathcal{Z} : \Omega \to CB(\Omega)$ by

$$\mathcal{Z}\varpi = \begin{cases} \{0, 1\}, \text{ if } \varpi = 0, 1 \\ \{\varpi - 1, \varpi\}, \text{ if } \quad \varpi > 1. \end{cases}$$

We declare that $\mathcal{Z}$ is an almost (α, F_σ)-contraction with $F_\sigma : \mathbb{R}^+ \to \mathbb{R}$ defined by $F_\sigma(t) = t + \ln t$, $\forall\, t \in \mathbb{R}^+$, $\tau = \frac{1}{2}$, $\sigma : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1$ and $L = 0$. For that, we need to show that

$$\frac{H(\mathcal{Z}\varpi, \mathcal{Z}\omega)}{\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5)} e^{H(\mathcal{Z}\varpi, \mathcal{Z}\omega) - \sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5)} \leq e^{-\tau}$$

$\forall\, \varpi, \omega \in \Omega$ with $\alpha(\varpi, \omega) H(\mathcal{Z}\varpi, \mathcal{Z}\omega) > 0$. Now we discuss these cases:

Case 1. If $\varpi, \omega \in \{0, 1\}$.

Case 2. If $\varpi, \omega > 1$, with $\varpi \neq \omega$. Then we have

$$\frac{1}{2} e^{-\frac{1}{2}|\varpi - \omega|} < e^{-\frac{1}{2}}$$

Case 3. If ϖ or $\omega \in \{0, 1\}$ and ϖ or ω, with $\varpi \neq \omega$. Then $\alpha(\varpi, \omega) H(\mathcal{Z}\varpi, \mathcal{Z}\omega) = 0$. Then the contractive condition is satisfied trivially. Thus, $\mathcal{Z}$ is an almost (α, F_σ)-contraction. For $\varpi_0 = 1$, we have $\varpi_1 = 0 \in \mathcal{Z}\varpi_0$ such that $\alpha(\varpi_0, \varpi_1) > 1$. Furthermore, it is simple to show that $\mathcal{Z}$ is strict α-admissible and for $\{\varpi_n\} \subseteq \Omega$ so that $\varpi_n \to \varpi$ as $n \to \infty$ and $\alpha(\varpi_n, \varpi_{n+1}) > 1$, $\forall$ $n \in \mathbb{N}$, we get $\alpha(\varpi_n, \varpi) > 1$, $\forall\, n \in \mathbb{N}$. Therefore, by Theorem 2, $\mathcal{Z}$ has a fixed point in Ω.

4. Applications

Fixed-point results for multivalued mappings in ordered Banach spaces are extensively explored and have a variety of applications in differential and integral inclusions (see [19,21,28]). In the present section, we apply the established theorems to obtain the existence of solutions for a recognized Fredholm integral inclusion

$$\varpi(t) \in \left[f(t) + \int_0^1 K(t,s,x(s))\vartheta s \right], \quad t \in [0,1]. \tag{25}$$

Consider the metric ϑ on $C[0,1]$ defined by

$$\vartheta(\varpi, \omega) = (\max_{t\in[0,1]} |\varpi(t) - \omega(t)|) = \max_{t\in[0,1]} |\varpi(t) - \omega(t)| \tag{26}$$

$\forall\ \varpi, \omega \in C[0,1]$. Then $(C[0,1], \vartheta)$ is a complete metric space.

We will suppose the following conditions:

(A_1) for each $\varpi \in C[0,1]$, $K : [0,1] \times [0,1] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is such that $K(t,s,\varpi(s))$ is lower semi-continuous in $[0,1] \times [0,1]$,

(A_2) there exists some continuous function $l : [0,1] \times [0,1] \to [0,+\infty)$ such that

$$|k_\varpi(t,s) - k_\omega(t,s)| \le l(t,s) \left\{ \begin{array}{c} \max\{|\varpi(s) - \omega(s)|, |\varpi(s) - K(t,s,\varpi(s))|, \\ |\omega(s) - K(t,s,\omega(s))|, |\varpi(s) - K(t,s,\omega(s))|, |\omega(s) - K(t,s,\varpi(s))|\} \end{array} \right\}$$

$\forall\ t,s \in [0,1], \varpi, \omega \in C[0,1]$.

(A_3) $\exists\ \tau > 0$ such that

$$\sup_{t\in[0,1]} \int_0^1 l(t,s)\vartheta s \le e^{-2\tau}.$$

Theorem 3. *With assertions (A_1)–(A_3), the integral inclusion (25) has a solution in* $C[0,1]$.

Proof. Let $\Omega = C[0,1]$. Define the multivalued mapping $\mathcal{Z} : \Omega \to CB(\Omega)$ by

$$\mathcal{Z}\varpi = \left\{ \omega \in \Omega : \omega(t) \in f(t) + \int_0^1 K(t,s,\varpi(s))\vartheta s, \ \ t \in [0,1] \right\}.$$

It is simple and direct that the set of solutions of integral inclusion (24) synchronizes with the set of fixed points of $\mathcal{Z}$. Thus, we must show that with the stated conditions, $\mathcal{Z}$ has at least one fixed point in Ω. For it, we shall examine that the conditions of Corollary 6 satisfied. □

Let $\varpi \in \Omega$. For the multivalued operator $K_\varpi(t,s) : [0,1] \times [0,1] \to K_{cv}(\mathbb{R})$, it acts in accordance with the Michael selection result that $\exists\ k_\varpi(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ such that $k_\varpi(t,s) \in K_\varpi(t,s)$ $\forall t,s \in [0,1]$. This follows that $f(t) + \int_0^1 k_\varpi(t,s)\vartheta s \in \mathcal{Z}\varpi$. Thus, $\mathcal{Z}\varpi \neq \emptyset$. It is an obvious matter to prove that $\mathcal{Z}\varpi$ is closed, and so specific aspects are excluded (see also [28]). Moreover, since f is continuous on $[0,1]$ and $K_\varpi(t,s)$ is continuous on $[0,1] \times [0,1]$, their ranges are bounded. It follows that $\mathcal{Z}\varpi$ is also bounded. Hence $\mathcal{Z}\varpi \in CB(\Omega)$.

We now analyze that (24) holds for $\mathcal{Z}$ on Ω with some $F_\sigma \in \Delta F$ and $\tau > 0$ i.e.,

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2)) \le F_\sigma \left(\max \left\{ \begin{array}{c} \vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \vartheta(\varpi_2, \mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1, \mathcal{Z}\varpi_2), \vartheta(\varpi_2, \mathcal{Z}\varpi_1) \end{array} \right\} \right) \tag{27}$$

for $\varpi_1, \varpi_2 \in \Omega$. Let $\omega_1 \in \mathcal{Z}\varpi_1$ be arbitrary such that

$$\omega_1(t) \in f(t) + \int_0^1 K(t,s,\varpi_1(s))\vartheta s$$

for $t \in [0,1]$ holds. It implies that $\forall\, t,s \in [0,1]$, $\exists\, k_{\varpi_1}(t,s) \in K_{\varpi_1}(t,s) = K(t,s,\varpi_1(s))$ such that

$$\omega_1(t) = f(t) + \int_0^1 k_{\varpi_1}(t,s)\vartheta s$$

for $t \in [0,1]$. For all $\varpi_1, \varpi_2 \in \Omega$, it follows from (A_2) that

$$H(K(t,s,\varpi_1) - K(t,s,\varpi_2) \leq l(t,s) \left\{ \begin{array}{c} \max\{|\varpi_1(s) - \varpi_2(s)|, |\varpi_1(s) - K(t,s,\varpi_1(s))|, \\ |\varpi_2(s) - K(t,s,\varpi_2(s))|, |\varpi_1(s) - K(t,s,\varpi_2(s))|, \\ |\varpi_2(s) - K(t,s,\varpi_1(s))|\} \end{array} \right\}.$$

This implies that $\exists\, z(t,s) \in K_{\varpi_2}(t,s)$ such that

$$|k_{\varpi_1}(t,s) - z(t,s)| \leq l(t,s) \left\{ \begin{array}{c} \max\{|\varpi_1(s) - \varpi_2(s)|, |\varpi_1(s) - K(t,s,\varpi_1(s))|, \\ |\varpi_2(s) - K(t,s,\varpi_2(s))|, |\varpi_1(s) - K(t,s,\varpi_2(s))|, \\ |\varpi_2(s) - K(t,s,\varpi_1(s))|\} \end{array} \right\}.$$

$\forall\, t,s \in [0,1]$.

Now, we can deal with the multivalued mapping U defined by

$$U(t,s) = K_{\varpi_2}(t,s) \cap \{u \in \mathbb{R} : |k_{\varpi_1}(t,s) - u| \leq l(t,s)|\varpi_1(s) - \varpi_2(s)|\}.$$

Hence, by (A_1), U is lower semi-continuous, it implies that $\exists\, k_{\varpi_2}(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ such that $k_{\varpi_2}(t,s) \in U(t,s)$ for $t,s \in [0,1]$. Then $\omega_2(t) = f(t) + \int_0^1 k_{\varpi_1}(t,s)\vartheta s$ satisfies that

$$\omega_2(t) \in f(t) + \int_0^1 K(t,s,\varpi_2(s))\vartheta s, \quad t \in [0,1].$$

$t \in [0,1]$. That is $\omega_2 \in \mathcal{Z}\varpi_2$ and

$$\begin{aligned} |\omega_1(t) - \omega_2(t)| &\leq \int_0^1 |k_{\varpi_1}(t,s) - k_{\varpi_2}(t,s)|\, \vartheta\, s \\ &\leq \int_0^1 l(t,s)|\varpi_1(s) - \varpi_2(s)|\vartheta s \\ &\leq \max_{t\in[0,1]} \left(\int_0^1 l(t,s) \left\{ \begin{array}{c} \max\{|\varpi_1(s) - \varpi_2(s)|, |\varpi_1(s) - K(t,s,\varpi_1(s))|, \\ |\varpi_2(s) - K(t,s,\varpi_2(s))|, |\varpi_1(s) - K(t,s,\varpi_2(s))|, \\ |\varpi_2(s) - K(t,s,\varpi_1(s))|\} \end{array} \right\} \vartheta s \right) \\ &\leq e^{-2\tau} \max \left\{ \begin{array}{c} \vartheta(\varpi_1,\varpi_2), \vartheta(\varpi_1,\mathcal{Z}\varpi_1), \vartheta(\varpi_2,\mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1,\mathcal{Z}\varpi_2), \vartheta(\varpi_2,\mathcal{Z}\varpi_1) \end{array} \right\} \end{aligned}$$

for all $t,s \in [0,1]$. Hence, we have

$$\vartheta(\omega_1,\omega_2) \leq e^{-2\tau} \max \left\{ \begin{array}{c} \vartheta(\varpi_1,\varpi_2), \vartheta(\varpi_1,\mathcal{Z}\varpi_1), \vartheta(\varpi_2,\mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1,\mathcal{Z}\varpi_2), \vartheta(\varpi_2,\mathcal{Z}\varpi_1) \end{array} \right\}$$

Changing the task of ϖ_1 and ϖ_2, we get

$$H(\mathcal{Z}\varpi_1,\mathcal{Z}\varpi_2) \leq e^{-2\tau} \max \left\{ \begin{array}{c} \vartheta(\varpi_1,\varpi_2), \vartheta(\varpi_1,\mathcal{Z}\varpi_1), \vartheta(\varpi_2,\mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1,\mathcal{Z}\varpi_2), \vartheta(\varpi_2,\mathcal{Z}\varpi_1) \end{array} \right\}$$

Taking natural log on both sides, we have

$$2\tau + \ln\left(H(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2)\right) \leq \ln\left(\max\left\{\begin{array}{c}\vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \vartheta(\varpi_2, \mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1, \mathcal{Z}\varpi_2), \vartheta(\varpi_2, \mathcal{Z}\varpi_1)\end{array}\right\}\right)$$

Taking $F_\sigma \in \Delta\mathcal{F}$ defined by $F_\sigma(t) = \ln(t)$ for $t > 0$, we have

$$2\tau + F_\sigma(H(\mathcal{Z}\varpi_1, \mathcal{Z}\varpi_2)) \leq F_\sigma\left(\max\left\{\begin{array}{c}\vartheta(\varpi_1, \varpi_2), \vartheta(\varpi_1, \mathcal{Z}\varpi_1), \vartheta(\varpi_2, \mathcal{Z}\varpi_2), \\ \vartheta(\varpi_1, \mathcal{Z}\varpi_2), \vartheta(\varpi_2, \mathcal{Z}\varpi_1)\end{array}\right\}\right).$$

All other conditions of Theorem 6 immediately follow by the hypothesis of taking the function $\sigma \in \mathcal{S}$ given by $\sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \max\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$ and the given integral inclusion (25) has a solution.

5. Conclusions

In this article, we have defined almost (α, F_σ)-contractions to establish new fixed-point results for a new class of contractive conditions in the context of complete metric spaces. The given results extended and improved the well-known results of Banach, Kannan, Chatterjea, Hardy–Rogers, and Ćirić by means of this new class of contractions. As an application of our main results, the existence of a solution for a certain Fredholm integral inclusion is also investigated. Our results are new and significantly contribute to the existing literature in fixed-point theory.

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