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Commuting Graphs, $\mathcal{C}(G,X)$ in Symmetric Groups $\mathrm{Sym}(n)$ and Its Connectivity

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Abstract: A commuting graph is a graph denoted by $\mathcal{C}(G,X)$ where G is any group and X, a subset of a group G, is a set of vertices for $\mathcal{C}(G,X)$. Two distinct vertices, $x, y \in X$, will be connected by an edge if the commutativity property is satisfied or $xy = yx$. This study presents results for the connectivity of $\mathcal{C}(G,X)$ when G is a symmetric group of degree n, $\mathrm{Sym}(n)$, and X is a conjugacy class of elements of order three in G.

Keywords: commuting graph; symmetric group; order 3 elements

1. Introduction

An element g of the symmetric group of degree n, $\mathrm{Sym}(n)$, can be written as a product of mutually disjoint cycles in the form

$$g = g_1 g_2 \dots g_r. \tag{1}$$

Generally, the cycle type of g is defined to be

$$c_1^{m_1} c_2^{m_2} \dots c_s^{m_s}$$

if in the cycle decomposition of g (Equation (1)), there are m_i cycles of length c_i for $1 \leq i \leq s$. Each m_i and c_i will be positive integers and $c_1 < c_2 < \dots < c_s$, an increasing sequence. We note that fixed points as 1-cycles must not be excluded in presenting the cycle type of any permutations of $\mathrm{Sym}(n)$.

Suppose an arbitrary element $a \in G$ has cycle type ${e_1}^{f_1} \dots {e_m}^{f_m}$. For some $1 \leq h_i \leq f_i$ and $1 \leq h_j \leq f_j$, we define a graph, denoted by Γ, whose vertex set is $V(\Gamma) = \{1, \dots, m\}$ and $i, j \in V(\Gamma)$ are linked by an edge if and only if $e_i h_i = e_j h_j$. Clearly, the cycle type of the element $a \in G$ determines the graph Γ. An edge $(i,j) \in E(\Gamma)$ is called an exact edge if $h_i = f_i$ and $h_j = f_j$. A special edge (i,j) with source i of Γ satisfies $e_j f_j = e_i$ and $b(i) = e_i$ where

$$b(i) := \frac{e_i}{\mathrm{lcm}\{d : d \mid e_i, d \leq f_i\}} \tag{2}$$

for $1 \leq i \leq m$.

Given this notation, it is worth noting the definition of commuting graph and two related results in [1]. These two theorems are extremely useful to prove the connectivity of commuting graphs $\mathcal{C}(G,X)$ in symmetric groups for elements of order three described in Section 2.

Definition 1. *Let G be any group and X a subset of a group G.* ***Commuting graph*** *denoted by $\mathcal{C}(G, X)$, is a graph whose set of vertices is X and two distinct vertices, $x, y \in X$, will be connected by an edge if the commutativity property is satisfied or $xy = yx$.*

Theorem 1. *[1] Let $G = Sym(n)$, $a \in G$ be of cycle type e^f, and $X = a^G$. Then, $\mathcal{C}(G, X)$ is connected if and only if $b(1) = 1$, or $e \leq 3$ and $f = 1$.*

Theorem 2. *[1] Let $G = Sym(n)$, $a \in G$ be of cycle type ${e_1}^{f_1} \dots {e_m}^{f_m}$ with $m > 1$. Let $X = a^G$. Then, $\mathcal{C}(G, X)$ is connected if and only if the following hold:*

1. *Γ is connected.*
2. $\gcd \{b(i) : 1 \leq i \leq m\} = 1$.
3. *Γ has at least one edge which is not exact.*
4. *The vertex set of Γ is* ***not*** *of the form $E \cup Y$, with $E \cap Y = \emptyset$ and $E, Y \neq \emptyset$, such that the following hold:*
   - *(a)* *for all $i, j \in E$ with $i \neq j$, (i, j) is an exact edge;*
   - *(b)* *there exists a vertex $y \in Y$ such that for all $i \in E$, (i, y) is a special edge with source y;*
   - *(c)* *no vertex of E is joined to a vertex of $Y \setminus \{y\}$; and*
   - *(d)* $\gcd \{b(i) : i \in Y\} = e_y$.

For instance, having G as the symmetric group of degree 7, Sym(7) and $X = (1, 2, 3)^G$, the commuting graph $\mathcal{C}(G, X)$ is connected with diameter 3. It shows an absolute contrast to the case having G as the symmetric group of degree 6, Sym(6) and $X = (1, 2, 3)^G$, where $\mathcal{C}(G, X)$ is disconnected even though both cases deal with the set of elements of cycle type 3^1 as the vertex set. More specific examples of connected and disconnected commuting graphs can be referred from the results in Table 1 of [2] and Table 1 of [3]. Evidently, there are certain cases where $\mathcal{C}(G, X)$ is not connected or disconnected, which further motivate us to categorize them. Consequently, some questions arise when $\mathcal{C}(G, X)$ is not connected: How many connected components are there? What is the size or diameter of the connected components?

Commuting graphs have been studied for a large varieties of groups and subset of the groups [2,4]. Woodcock [5] analyzed commuting graphs $\mathcal{C}(G, X)$ for symmetric group, Sym(n) and having the set of nontrivial elements of the group as the vertex set. It is proven there that $\mathcal{C}(G, X)$ is disconnected for two particular cases, either n or $n - 1$ is a prime, where n is the degree for the symmetric group. The multiple connected components are also identified alongside with their diameters. Researchers also give attention to the case where X is a conjugacy class of involutions, the commuting involution graphs [6–8]. Indeed, conjugacy classes of non-involution have a great impact in several other area of mathematics, see for instance in subspace arrangements [9], in rational geometry [10] or in algebraic geometry [11,12].

This paper consists of three sections. Section 2 provides three cases where $\mathcal{C}(G, X)$ will be disconnected when the vertex set of the graph is the conjugacy class of elements of order three in Sym(n): (1) when $n = 3r$ and $r = 2$; (2) when $n = 3r + 1$ or $n = 3r + 2$ and $r \geq 1$; and (3) when $n = 3r + q$; $r = 1$ or $r = 2$ and $q = 3$, where r is the number of 3-cycles in elements of X. In Section 3, we attempt to give some information about the number of connected components of some specific disconnected graphs of symmetric groups and also the size of each connected components. As a result, we also obtain the isomorphism of every connected component of disconnected $\mathcal{C}(G, X)$ with the complete graph of the same order as the size of the component.

Throughout we assume $G = \mathrm{Sym}(\Omega) = \mathrm{Sym}(n)$ acts on the set $\Omega = \{1, \dots, n\}$ in the usual manner. Write, without loss of generality, an element $t \in G$ as follows:

$$t = (1, 2, 3)(4, 5, 6)(7, 8, 9) \dots (3r - 2, 3r - 1, 3r).$$

Thus, t has order 3 and is of cycle type $1^{n-3r}3^r$. Set $X = t^G$, the G-conjugacy class of t. Evidently, the centralizer of t in G is

$$C_G(t) \cong (3^r : \mathrm{Sym}(r)) \times \mathrm{Sym}(n-3r).$$

2. Connectedness of the Commuting Graph

By using Theorems 1 and 2 where appropriate, we can now put these to good use to partially prove the following result (note that this observation is based on the results in Table 1 of [2] and Table 1 of [3]):

Theorem 3. *Let $G = Sym(n)$ and $t \in G$ be of cycle type 3^r with $r \geq 1$. Let $X = t^G$. Then, $\mathcal{C}(G, X)$ is disconnected if and only if one of the following holds:*

1. *$n = 3r$ and $r = 2$;*
2. *$n = 3r + 1$ or $n = 3r + 2$ and $r \geq 1$; and*
3. *$n = 3r + q$; $r = 1$ or $r = 2$ and $q = 3$.*

Proof of Theorem 3. The case $r = 1$ ($n = 3$) is trivial. The latter statement of Theorem 1 shows that $\mathcal{C}(G, X)$ is connected for $G = \mathrm{Sym}(3)$ and $t = (1, 2, 3)$.

Now, suppose $r = 2$ ($n = 6$). This means $G = \mathrm{Sym}(6)$ and $t \in G$ is of cycle type 3^2. Applying Equation (2) to get $b(i)$ where $i = 1, e_i = e_1 = 3, f_i = f_1 = 2$ and $d = 1$, we get $b(1) = 3$. Thus, by Theorem 1, $\mathcal{C}(G, X)$ is disconnected.

Now, suppose $r \geq 3$ ($n \geq 9$). This means $f_1 \geq 3$ and hence d is either 1 or 3, since d is a divisor of 3. Thus, $\mathrm{lcm}\{1, 3\} = 3$. Therefore, $b(1) = 1$ and again by Theorem 1, $\mathcal{C}(G, X)$ is connected. Hence, for the case $n = 3r$, $\mathcal{C}(G, X)$ is disconnected if and only if $r = 2$.

We can prove the second part by employing Theorem 2. However, here we can also prove it by using a more direct approach. Note that t is of cycle type $1^{n-3r}3^r$, $r \geq 1$. If $n = 3r + 1$, then t leaves only the point $3r + 1$ fixed. Therefore, any elements of G which belong to the same conjugacy class as t, named X, and satisfy the commutativity property with t also fix $3r + 1$. By induction, every element of $X = t^G$ in the connected component of t will fix $3r + 1$. Thus, $\mathcal{C}(G, X)$ is disconnected.

If $n = 3r + 2$, then t fixes the points $3r + 1$ and $3r + 2$. Thus, by using the same argument as before, any elements in $X = t^G$ which commutes with t will also fix $3r + 1$ and $3r + 2$. Again, by induction, every element of G which belongs to the same conjugacy class as t and is in the connected component of t fixes $3r + 1$ and $3r + 2$. Thus, $\mathcal{C}(G, X)$ is disconnected.

Generally, here we have $n = 3r + q$; $r \geq 1$ and $q \geq 3$. We need to show that $\mathcal{C}(G, X)$ is only connected for the cases $r = 1$ and $q \geq 4$, $r = 2$ and $q \geq 4$, and, $r \geq 3$ and $q \geq 3$ or otherwise $\mathcal{C}(G, X)$ is disconnected.

First, we prove that $\mathcal{C}(G, X)$ is connected when $n = 3r + q$; $r \geq 3$ and $q \geq 3$. We do this by observing each condition in Theorem 2:

Condition 1. *We claim that Γ is connected. Thus, there exists at least one edge which joins two vertices. The only possible edge is $(1, 2)$ since there exists only $e_1 = 1$ and $e_2 = 3$ (note that $(2, 1)$ is the same edge as $(1, 2)$ as they both connect the same vertices, (1 and 2). In this case, $f_1 = q \geq 3$ and $f_2 = r \geq 3$. Therefore, we always get this equality:*

$$1h_1 = 3h_2, \textit{ for some } 1 \leq h_1 \leq f_1 \textit{ and } 1 \leq h_2 \leq f_2$$

and hence $(1, 2)$ is an edge. Thus, Γ is connected and Theorem 2 (1) is satisfied.

Condition 2. *We require the formula in Equation (2) to get $b(i)$. In this case, we have $1 \leq i \leq 2$ where $e_1 = 1$ and $e_2 = 3$. Note again that $f_1 = q \geq 3$ and $f_2 = r \geq 3$. Calculations for $b(1)$ and $b(2)$ are as follows:*

$$\begin{aligned}
b(1) &= \frac{e_1}{\mathrm{lcm}\{d : d \mid e_1, d \leq f_1\}}, f_1 \geq\ 3 \\
&= \frac{1}{\mathrm{lcm}\{d : d \mid 1, d \leq f_1\}}, f_1 \geq\ 3 \\
&= \frac{1}{\mathrm{lcm}\{1\}} \\
&= 1 \\
b(2) &= \frac{e_2}{\mathrm{lcm}\{d : d \mid e_2, d \leq f_2\}}, f_2 \geq\ 3 \\
&= \frac{3}{\mathrm{lcm}\{d : d \mid 3, d \leq f_2\}}, f_2 \geq\ 3 \\
&= \frac{3}{\mathrm{lcm}\{1,3\}} \\
&= \frac{3}{3} \\
&= 1
\end{aligned}$$

Furthermore; gcd $\{b(i) : 1 \leq i \leq 2\} =$ gcd $\{b(1), b(2)\} =$ gcd $\{1, 1\} = 1$. Hence, Theorem 2 (2) is satisfied.

Condition 3. *Note again $(1, 2)$ is the only edge of Γ. Now, by employing the definition given in the Introduction, $(1, 2)$ is an exact edge if $h_1 = f_1$ and $h_2 = f_2$ are the only possibility to get the following equality:*

$$1h_1 = 3h_2. \tag{3}$$

Indeed, we can get the equality in (3) with many choices of h_1 and h_2 (but the minimum is $1(3) = 3(1)$). Hence, $(1, 2)$ is not an exact edge and Theorem 2 (3) is satisfied.

Condition 4. *To observe this condition, let us first say that the set of vertices of Γ is of the form $E \cup Y$, provided that there is no intersection between E and Y, and E and Y are not empty sets. Without loss of generality, set the vertex $1 \in E$ and vertex $2 \in Y$.*

(a) *Since there is only one vertex in E, we can join any two vertices neither by an exact edge nor by a non-exact edge.*

(b) *Note that $(1, 2)$ is the only edge of Γ and it is not a special edge with source 2 since $b(2) = 1 \neq e_2$. Thus, Theorem 2 $(4)(b)$ is not satisfied.*

This is a contradiction (we still get the contradiction here even if we assume the vertex $2 \in E$ and vertex $1 \in Y$ since $(1, 2)$ is also not a special edge with source 1 because $e_2 f_2 = 3f_2 \geq 9 \neq e_1$). Therefore, this verifies that $V(\Gamma) \neq E \cup Y$ and, hence, Theorem 2 (4) is satisfied.

Thus, by Theorem 2, $\mathcal{C}(G, X)$ is connected when $n = 3r + q$; $r \geq 3$ and $q \geq 3$. We can use the same way to prove that $\mathcal{C}(G, X)$ is also connected for the cases $r = 1$ and $q \geq 4$, and, $r = 2$ and $q \geq 4$.

Note that $\mathcal{C}(G, X)$ is disconnected when $n = 3r + q$; $r = 1$ and $q = 3$ since Theorem 2 (3) is not satisfied. This is because $(1, 2)$ as the only edge in Γ is an exact edge.

$\mathcal{C}(G, X)$ is also disconnected when $n = 3r + q$; $r = 2$ and $q = 3$ because Theorem 2 (4) is not satisfied. This means, without loss of generality by putting the vertex $1 \in E$ and vertex $2 \in Y$, the vertex set of Γ can be written as union of E and Y, no intersection between E and Y, and both are not empty sets and it fulfills all four conditions in Theorem 2 (4). Conditions (a) and (c) are met since there is only one vertex in each E and Y. Condition (b) is satisfied because $(1,2)$ is a special edge with source 2 whilst Condition (d) is also satisfied since gcd $\{b(i) : i \in Y\} = \text{gcd } \{b(2)\} = \text{gcd } \{3\} = 3 = e_2$. $\square$

3. Disconnected Commuting Graph and Its Connected Components

In Section 2, there are certain cases where $\mathcal{C}(G,X)$ is disconnected. Consequently, when $\mathcal{C}(G,X)$ is not connected, we are concerned with the number of connected components and also the size or diameter of the connected components. In this paper, a connected component is denoted by C_i where i ranges from 1 to the number of connected components in some disconnected $\mathcal{C}(G,X)$ and r is the number of 3-cycles in an element $t \in G$. Two examples of the calculations are as follows:

Example 1. *Let $t = (1,2,3) \in G$ where $G = Sym(6)$. Note that t has cycle type $1^3 3^1$ ($r = 1$). Since $|C_G(t)| = 18$, then $|X| = |t^G| = 40$ where X is the set of all elements of G with cycle type $1^3 3^1$. Suppose $C_1, \ldots, C_{10}$ are the connected components of $\mathcal{C}(G,X)$. Then,*

$$\begin{aligned}
C_1 &= \{(1,2,3),(1,3,2),(4,5,6),(4,6,5)\}\\
C_2 &= \{(1,2,4),(1,4,2),(3,5,6),(3,6,5)\}\\
C_3 &= \{(1,2,5),(1,5,2),(3,4,6),(3,6,4)\}\\
C_4 &= \{(1,2,6),(1,6,2),(3,4,5),(3,5,4)\}\\
C_5 &= \{(1,3,4),(1,4,3),(2,5,6),(2,6,5)\}\\
C_6 &= \{(1,3,5),(1,5,3),(2,4,6),(2,6,4)\}\\
C_7 &= \{(1,3,6),(1,6,3),(2,4,5),(2,5,4)\}\\
C_8 &= \{(1,4,5),(1,5,4),(2,3,6),(2,6,3)\}\\
C_9 &= \{(1,4,6),(1,6,4),(2,3,5),(2,5,3)\}\\
C_{10} &= \{(1,5,6),(1,6,5)(2,3,4),(2,4,3)\}
\end{aligned}$$

Therefore, when $n = 6$ and $r = 1$, $\mathcal{C}(G,X)$ consists of ten connected components each of size four.

Proposition 1. *Let $G = Sym(6)$ and $t = (1,2,3) \in G$. Then, every connected component of disconnected $\mathcal{C}(G,X)$ is isomorphic to the complete graph of order 4, K_4, as shown in Figure 1.*

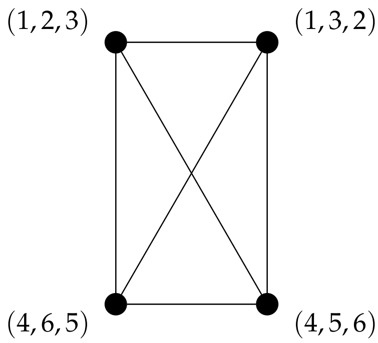

Figure 1. Connected component C_1 of disconnected $\mathcal{C}(G,X)$ when $G = \text{Sym}(6)$ and $X = (1,2,3)^G$.

Example 2. *Let $t = (1,2,3)(4,5,6) \in G$ where $G = Sym(6)$. Note that t has cycle type 3^2 ($r = 2$). Since $|C_G(t)| = 18$, $|X| = |t^G| = 40$ where X is the set of all elements of G with cycle type 3^2. Suppose $C_1, \ldots, C_{10}$ are the connected components of $\mathcal{C}(G,X)$. Then,*

$$\begin{aligned}
C_1 &= \{(1,2,3)(4,5,6),(1,3,2)(4,5,6),(1,2,3)(4,6,5),(1,3,2)(4,6,5)\}\\
C_2 &= \{(1,5,2)(3,4,6),(1,2,5)(3,4,6),(1,5,2)(3,6,4),(1,2,5)(3,6,4)\}\\
C_3 &= \{(1,2,4)(3,6,5),(1,4,2)(3,6,5),(1,2,4)(3,5,6),(1,4,2)(3,5,6)\}\\
C_4 &= \{(1,2,6)(3,5,4),(1,6,2)(3,5,4),(1,2,6)(3,4,5),(1,6,2)(3,4,5)\}\\
C_5 &= \{(1,6,5)(2,4,3),(1,5,6)(2,4,3),(1,6,5)(2,3,4),(1,5,6)(2,3,4)\}
\end{aligned}$$

$$C_6 = \{(1,3,6)(2,5,4),(1,6,3)(2,5,4),(1,3,6)(2,4,5),(1,6,3)(2,4,5)\}$$
$$C_7 = \{(1,5,4)(2,6,3),(1,4,5)(2,6,3),(1,5,4)(2,3,6),(1,4,5)(2,3,6)\}$$
$$C_8 = \{(1,4,6)(2,5,3),(1,6,4)(2,5,3),(1,4,6)(2,3,5),(1,6,4)(2,3,5)\}$$
$$C_9 = \{(1,5,3)(2,4,6),(1,3,5)(2,4,6),(1,5,3)(2,6,4),(1,3,5)(2,6,4)\}$$
$$C_{10} = \{(1,4,3)(2,6,5),(1,3,4)(2,6,5),(1,4,3)(2,5,6),(1,3,4)(2,5,6)\}$$

Therefore, when $n = 6$ and $r = 2$, $\mathcal{C}(G,X)$ consists of ten connected components each of size four.

Proposition 2. *Let $G = Sym(6)$ and $t = (1,2,3)(4,5,6) \in G$. Then, every connected component of disconnected $\mathcal{C}(G,X)$ is isomorphic to the complete graph of order 4, K_4, as shown in Figure 2.*

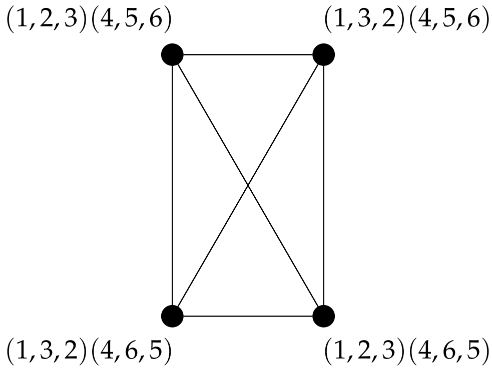

Figure 2. Connected component C_1 of disconnected $\mathcal{C}(G,X)$ when $G = \text{Sym}(6)$ and $X = (1,2,3)(4,5,6)^G$.

Based on the calculations carried out by hand and also with the help of MAGMA [13], we give a general formula of size and number of connected components for some disconnected $\mathcal{C}(G,X)$.

Theorem 4. *If t has cycle type 3^r and $n = 3r+1$ or $n = 3r+2$, then the size of each connected components is 2^r. Moreover, the number of connected components is $\frac{|X|}{2^r}$.*

Proof of Theorem 4. We need to count the number of permutations in $X = t^G$ (where G is either $\text{Sym}(3r+1)$ or $\text{Sym}(3r+2)$),which commute with each other and are in the same connected component. For each element in X, we have r 3-cycles and each 3-cycle has only two possibilities, either being inverted or left alone. This gives 2^r possible permutations. Consequently, it follows immediately that in this case the number of connected components is $\frac{|X|}{2^r}$. □

In view of Propositions 1 and 2 and Theorem 4, we have the following theorem.

Theorem 5. *Every connected component of disconnected $\mathcal{C}(G,X)$ in Theorem 3 is isomorphic to the complete graph of order 2^r, K_{2^r}.*

Proof of Theorem 5. All elements in each connected component commute with each other and consequently they are of distance 1 from one vertex to another vertex. In other words, one vertex is adjacent to another vertex in this component and this shows that the resulting graph is a complete graph of order 2^r since the size of each connected components is 2^r. □

Table 1 gives information about the number of connected components of some specific disconnected graphs of symmetric groups and also the size of each connected components.

Table 1. Number of connected components, C_i and its size of some disconnected $\mathcal{C}(G, X)$.

t	n	$\vert C_G(t)\vert$	$\vert X\vert$	No. of C_i	Size of Each C_i
$(1,2,3)$	4	3	8	4	2
	5	6	20	10	2
	6	18	40	10	4
$(1,2,3)(4,5,6)$	6	18	40	10	4
	7	18	280	70	4
	8	36	1120	280	4
	9	108	3360	280	12
$(1,2,3)(4,5,6)(7,8,9)$	10	162	22,400	2800	8
	11	324	123,200	15,400	8
$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	13	1944	3,203,200	200,200	16
	14	3888	22,422,400	1,401,400	16

Clearly, if the commuting graph $\mathcal{C}(G, X)$ is disconnected when G is a symmetric group of degree n, $\mathrm{Sym}(n)$, and X is a conjugacy class of elements of order three in G, the connected components have the same order, the same size and even the same edge connectivity, in other words they are isomorphics. The results obtained can be extended to find the adjacency matrices of each connected components of $\mathcal{C}(G, X)$, their eigen values and hence giving the input for the graph energy.

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Abbreviations

The following abbreviations are used in this manuscript:

$\mathcal{C}(G, X)$	Commuting graph
$C_G(t)$	Centralizer of an element t in group G
C_i	i-th connected components
$\mathrm{Sym}(n)$	Symmetric group of degree n
lcm	Lowest common multiple
gcd	Greatest common divisor

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