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Double-Quantitative Generalized Multi-Granulation Set-Pair Dominance Rough Sets in Incomplete Ordered Information System

Zhan-ao Xue [1,2,*], Min Zhang [1,2,*], Yong-xiang Li [1,2], Li-ping Zhao [1,2] and Bing-xin Sun [1,2]

1 College of Computer and Information Engineering, Henan Normal University, Xinxiang 453007, China; 17634298195@163.com (Y.-x.L.); zhaoliping_20@163.com (L.-p.Z.); sun928081744@163.com (B.-x.S.)

2 Engineering Lab of Henan Province for Intelligence Business & Internet of Things, Henan Normal University, Xinxiang 453007, China

* Correspondence: xuezhanao@163.com (Z.-a.X.); zhang_min95@163.com (M.Z.)

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Abstract: Since the rough sets theory based on the double quantification method was proposed, it has attracted wide attention in decision-making. This paper studies the decision-making approach in Incomplete Ordered Information System (IOIS). Firstly, to better extract the effective information in IOIS, combined with the advantages of set-pair dominance relation and generalized multi-granulation, the generalized multi-granulation set-pair dominance variable precision rough sets (GM-SPD-VPRS) and the generalized multi-granulation set-pair dominance graded rough sets (GM-SPD-GRS) are proposed. Moreover, we discuss their related properties. Secondly, considering the GM-SPD-VPRS and the GM-SPD-GRS describe information from relative view and absolute view, respectively, we further combine the two rough sets to obtain six double-quantitative generalized multi-granulation set-pair dominance rough sets (GM-SPD-RS) models. Among them, the first two models fuse the approximation operators of two rough sets, and investigate the extreme cases of optimistic and pessimistic. The last four models combine the two rough sets by the logical disjunction operator and the logical conjunction operator. Then, we discuss relevant properties and derive the corresponding decision rules. According to the decision rules, an associated algorithm is constructed for one of the models to calculate the rough regions. Finally, we validate the effectiveness of these models with a medical example. The results indicate that the model is effective for dealing with practical problems.

Keywords: double quantification; set-pair dominance relation; generalized multi-granulation; variable precision rough sets; graded rough sets

1. Introduction

The acceleration of the information era makes it possible to acquire and process diversified feature data. However, in reality, due to the limitations of data acquisition technology and errors in data measurement, the acquired information contains incomplete and inaccurate data in most cases, thus increasing the requirements for data analysis tools. As we know, the uncertain knowledge hidden in these systems is of great significance in decision-making. In reality, there are many problems that require the consideration both relative and absolute information. For example, in the comprehensive evaluation of graduate students, a school assesses students' scientific research ability according to the quality and quantity of articles published. Inspired by the above, we consider decision-making in IOIS with double-quantitative RS theory.

Rough sets (RS) theory is an effective mathematical tool proposed by Pawlak in 1982 to address uncertainties and imprecisions [1]. It has been successfully applied in feature selection [2], safety monitoring data classification [3], decision making [4–6], information fusion [7], uncertainty analysis [8,9], medical diagnosis [10], and other fields [11–14]. Classical RS require strict inclusion relations between their equivalent classes and sets, and because there is no fault-tolerant mechanism, they are limited. Accordingly, scholars have proposed a series of extended RS models [15–61]. Yao et al. [15] presented a decision theory rough sets (DTRS) model by introducing Bayes risk decision theory into RS. In 1993, Ziarko [16] introduced parameter β into the RS model and proposed a variable precision rough sets (VPRS) model that reflects the relative quantitative information of approximation space and has a certain fault-tolerant ability. Considering the importance of overlapping information of equivalence classes and sets, Yao et al. [17] put forward graded rough sets (GRS) in 1996. GRS can allow certain errors and can reflect the absolute quantization information of approximation space. From the perspective of information quantization, it is of great significance to combine absolute quantization information with relative quantization information [18–29]. Zhang et al. [18] performed a comparative study of VPRS and GRS and put forward two double-quantitative RS models, which enriched rough sets theory. Li et al. [19] studied two double-quantitative DTRS models and verified the validities of the models through a medical diagnosis case. Moreover, Fan et al. [20] introduced fuzzy sets into the double-quantitative problem, constructed upper and lower approximation sets based on logical operators, which can solve the pattern recognition problem in big data. Since RS model based on equivalence relation is suitable only for discrete data, discrete preprocessing of data will cause some information to be lost and reduce classification accuracy. To address the abovementioned problem, literature [21] explored a distance-based double-quantitative rough fuzzy sets model that can settle the problem of information loss. Guo et al. [22,23] studied double-quantitative RS theory under fuzzy relation and presented a local logic disjunctive double-quantitative RS model based on local RS, which provided an effective approach for decision-making. In addition, dominance relation often plays a considerable role in practical applications, and it was necessary to establish the ordered information system (OIS) [24–26] through the dominance relation. Some scholars have investigated various double-quantitative RS models based on logical combinations of VPRS and GRS in OIS [27–29]. The above studies rarely deal with decision-making problem in multi-granulation environment. However, in many cases, multi-granulation is needed to describe the concept of the target.

From the granular computing perspective, Qian et al. [30,31] first proposed the multi-granulation rough sets model (MRS) and extended the optimistic multi-granulation rough sets and pessimistic multi-granulation rough sets, which attracted extensive attention from scholars [32,33]. Subsequently, Xu et al. [34–36] explored a generalized multi-granulation rough sets (GMRS) model. Then, they introduced generalized multi-granulation into the DTRS and GRS, provided two RS models, and compared them with GMRS before demonstrating the advantages of the model through examples. Literature [37] established three double-quantitative DTRS in multi-granulation approximate space, but there are still some problems, such as double-quantitative multi-granulation DTRS under dynamic granulation and the practical applications.

The abovementioned RS models are all under the complete information system (IS), but in reality, the IS encountered are often incomplete, such as data integration [38], data mining [39], fault diagnosis [40], uncertainty measurement [41], and many others [42–44]. Therefore, a large number of studies on Incomplete Information System (IIS) have emerged [45–61]. Kryszkiewicz [45] described an RS model by introducing the tolerance relation in IIS. Furthermore, Stefanowski et al. [48] defined the similarity relation in IIS. In the light of granular computing, Zhai et al. [49,50] introduced the tolerance relation into VPRS, combined with the multi-granulation, and gave the optimistic and pessimistic variable precision MRS model under the IIS. Then, they expanded the single granulation RS model based on the extended dominance relation and the limited dominance relation. Yao et al. made a series of studies [51–53] about MRS in IIS. Based on maximal consistent block, a new optimistic MRS and a new pessimistic MRS in IIS were established. Then, they investigated the granular space reductions in the variable precision MRS model. Lin and Xu [54] constructed optimistic MRS and pessimistic MRS in interval-valued information system. As the ranking of values plays a significant role in many practical applications, Scholars have extended the method to various types of IOIS [55–59]. Huang [58] introduced the idea of set-pair analysis and proposed a set-pair dominance degree RS model, which can overcome the shortcomings of the dominance relation in existing IOIS. Literature [60] mainly investigated attribute reduction methods in IOIS. However, the above researches have not applied the rough sets theory based on the double quantification approach to the decision-making in IOIS. To settle the problem, we extend the double quantification rough sets model. The paper studies decision-making in IIS, and provide better decision results for decision makers without preprocessing the data. The works of this paper provide a method for decision-making research of IIS and expand its application scope.

The main contributions of this article are shown below.

(1) To better extract effective information in IOIS, we combine the set-pair dominance relation and generalized multi-granulation. In consideration that VPRS can reflect the relative quantization information of approximation space, we propose generalized multi-granulation set-pair dominance variable precision rough sets (GM-SPD-VPRS), and the related properties are discussed.

(2) Based on the proposed GM-SPD-VPRS, considering the GRS reflect absolute quantization information of approximation space, we defined generalized multi-granulation set-pair dominance graded rough sets (GM-SPD-GRS). In addition, we give the corresponding properties.

(3) To better reflect the relative and absolute quantization information in IOIS, the lower and upper approximation sets of GM-SPD-VPRS and GM-SPD-GRS are fused and combined with logical operators, and five double-quantitative generalized multi-granulation set-pair dominance relation rough sets (DQGM-SPD-RS$^{\mathrm{I}}$ to DQGM-SPD-RS$^{\mathrm{V}}$) are obtained.

(4) Taking DQGM-SPD-RS$^{\mathrm{I}}$ model as an example, the Algorithm 1 is given. In addition, the validity of the novel models is proved by a medical diagnosis case. It can also guide people to select a suitable model.

The rest of this paper is organized below. In Section 2, some basic concepts related to RS, IOIS, and set-pair dominance RS are recalled. In Section 3, GM-SPD-VPRS and GM-SPD-GRS are proposed in IOIS, and the related properties are given and proven. In Section 4, DQGM-SPD-RS$^{\mathrm{I}}$, DQGM-SPD-RS$^{\mathrm{II}}$, DQGM-SPD-RS$^{\mathrm{III}}$, DQGM-SPD-RS$^{\mathrm{IV}}$, and DQGM-SPD-RS$^{\mathrm{V}}$ are proposed to reflect the relative and absolute quantization information better in IOIS. Moreover, taking into account extreme circumstances, DQGM-SPD-RS$^{\mathrm{I}}$ and DQGM-SPD-RS$^{\mathrm{II}}$ are discussed with optimistic and pessimistic extremes, respectively. Subsequently, an algorithm to calculate rough regions of DQGM-SPD-RS$^{\mathrm{I}}$ is constructed. In Section 5, the validity of the novel models is illustrated through a medical example. Finally, this paper concludes a brief summary and looks forward to the future research in Section 6.

2. Preliminaries

Definition 1 (VPRS). [16] *Let (U, R) be an approximate space, where U is a set of non-empty finite objects, for $\forall X \subseteq U$, the lower and upper approximations of X with precision $1-\beta$ are defined as follows:*

$$\begin{aligned} \underline{R}_{\beta}(X) &= \left\{x \in U \middle| c([x]_R, X) \leq \beta\right\}; \\ \overline{R}_{\beta}(X) &= \left\{x \in U \middle| c([x]_R, X) < 1-\beta\right\}. \end{aligned}$$

where $c([x]_R, X) = 1 - \left|[x]_R \cap X\right| / \left|[x]_R\right|$ denotes the error classification rate. As can be seen from the above, VPRS improve the fault-tolerant ability of RS by giving a less strict definition of inclusion relation in the above approximations of RS. Generally, X is called definable if $\underline{R}_{\beta}(X) = \overline{R}_{\beta}(X)$; otherwise, X is rough if $\underline{R}_{\beta}(X) \neq \overline{R}_{\beta}(X)$. Specially, when $\beta = 0$, VPRS degenerate into RS, that is, $\underline{R}_{\beta}(X) = \underline{R}(X)$ and $\overline{R}_{\beta}(X) = \overline{R}(X)$.

Based on approximation sets of VPRS, the rough regions of X with precision $1-\beta$ are defined below:

$$\begin{aligned} POS_{\beta}(X) &= \underline{R}_{\beta}(X); \\ BND_{\beta}(X) &= \overline{R}_{\beta}(X) - \underline{R}_{\beta}(X); \\ NEG_{\beta}(X) &= U - \overline{R}_{\beta}(X). \end{aligned}$$

Definition 2 (GRS). [9] *Let* (U, R) *be an approximate space, and assume* $K \in N$, *N is the set of natural number; the lower and upper approximations of X with respect to the grade K can be defined as follows:*

$$\underline{R}_K(X) = \left\{x \in U \middle| \left|[x]_R\right| - \left|[x]_R \cap X\right| \leq K\right\};$$
$$\overline{R}_K(X) = \left\{x \in U \middle| [x]_R \cap X > K\right\}.$$

The pair $(\underline{R}_K(X), \overline{R}_K(X))$ *is called the lower and upper approximation sets of GRS, where* $\underline{R}_K(X)$ *is a set of equivalent classes with at most K elements that do not belong to X, and* $\overline{R}_K(X)$ *is a set with more than K equivalent classes belonging to X, which can reflect the absolute quantitative information in approximate space. Generally, X is called definable if* $\underline{R}_K(X) = \overline{R}_K(X)$; *otherwise, X is rough if* $\underline{R}_K(X) \neq \overline{R}_K(X)$. *In particular, when* $K = 0$, *the GRS model degenerates into the RS model, that is,* $\underline{R}_K(X) = \underline{R}(X)$ *and* $\overline{R}_K(X) = \overline{R}(X)$. *Moreover, it should be noted that* $\underline{R}_K(X)\overline{R}_K(X)$ *and* $\overline{R}_K(X)\underline{R}_K(X)$ *in general.*

Similarly, the positive region, the upper boundary region, the lower boundary region, and the negative region of X with respect to graded K can be defined:

$$POS_K(X) = \underline{R}_K(X) \cap \overline{R}_K(X);$$
$$UBND_K(X) = \overline{R}_K(X) - \underline{R}_K(X);$$
$$LBND_K(X) = \underline{R}_K(X) - \overline{R}_K(X);$$
$$NEG_K(X) = \sim (\underline{R}_K(X) \cup \overline{R}_K(X)).$$

Definition 3 (approximation accuracy). [1] *Let (U, R) be an approximate space, the approximation accuracy of X under the equivalence relation R can be defined:*

$$\alpha_R(X) = \frac{\left|\underline{R}\right|}{\left|\overline{R}\right|}$$

Approximation accuracy describes the ratio of correct decisions in possible decisions when using knowledge R to classify the objects.

Definition 4 (OIS). [24,25] *Let* $\mathrm{IS} = (U, AT, V, f)$ *be a complete information system, AT is non-empty finite attribute set, and* $V = \cup_{a \in AT} V_a$ *is a set of all attribute values, where* V_a *is the value range of attribute a and has a partial order relation and* $f : U \times AT \to V$ *is an information function. For* $\forall A \subseteq AT$, $\forall X \in U$, *there is a general dominance relation:*

$$R_A^{\leq} = \left\{(x, y) \in U \times U \middle| f(x, a) \leq f(y, a)\right\},$$

Then, $R_A^{\leq}$ *is called dominance relation under IS, and the IS satisfies this dominance relation is called ordered information systems (OIS). According to the dominance relation, the dominance classes of the object can be calculated as* $[x]_A^{\leq} = \left\{y \in U \middle| (x, y) \in R_A^{\leq}\right\}$. *The dominance classes of the object represent all object sets whose values are superior to those of the attribute set A.*

Definition 5 (Set-pair dominance degree). [58] *Let* $\mathrm{IOIS} = (U, AT, VT, f)$, *for* $\forall A \subseteq AT$, $\forall X \in U$, *the set-pair dominance degree of x and y is denoted as:*

$$S_A(x, y) = S_1 + S_2 p + S_3 q,$$

where $S_1 = \left|M(x, y)\right|/n$ *is the strong dominance degree of x and y,* $S_2 = \left|N(x, y)\right|/n$ *is the weak dominance degree of x and y, and* $S_3 = \left|K(x, y)\right|/n$ *is the disadvantage degree of x and y. In addition:*

$$M(x,y) = \{a \in A \big| (f_a(x) \le f_a(y)) \vee (f_a(x) = * \wedge f_a(y) = \max v_a) \vee (f_a(y) = * \wedge f_a(x) = \min v_a)\},$$
$$N(x,y) = \{a \in A \big| (f_a(x) = * \wedge f_a(y) = *) \vee (f_a(x) = * \wedge f_a(y) \ne * \wedge \ f_a(y) \ne \max v_a \wedge f_a(y) \ne \min v_a) \vee (f_a(y) = * \wedge f_a(x) \ne * \wedge f_a(x) \ne \min v_a)\},$$
$$K(x,y) = \{a \in A \big| (f_a(x) > f_a(y)) \vee (f_a(x) = * \wedge f_a(y) = \min v_a)\}.$$

Obviously, $0 \le S_1,\ S_2,\ S_3 \le 1$*, and* $S_1 + S_2 + S_3 = 1$*. In order to calculate the possible advantage of the objects in the weak dominance, the concept of the joint dominance rate is proposed.*

Definition 6 (Joint dominance ratio)**.** [58] *Assume* IOIS $= (U, AT, VT, f)$*; for* $\forall A \subseteq AT$, $\forall X \in U$*, the joint dominance ratio of attribute values of x and y can be defined below:*

$$FP(x,y) = \begin{cases} 1, & x = y; \\ \frac{|P_A^{<}(x)| + |P_A^{>}(y)|}{2|A|}, & x \ne y. \end{cases}$$

where $P_A^{<}(x) = \{a \in A \big| f_a(x) \ne * \wedge (f_a(x) < avg(v_a))\}$, $P_A^{>}(y) = \{a \in A \big| f_a(y) \ne * \wedge (f_a(y) \ge avg(v_a))\}$*. It can be obtained readily that* $0 \le FP(x,y) \le 1$.

Additionally, it can be seen that the joint dominance ratio of the attribute values of objects is compared by means of the average value rather than unknown value$*$. Therefore, the subjectivity is not too strong.

Definition 7 (Set-pair dominance relation)**.** [58] *Assume that* IOIS $= (U, AT, V, f)$*; the set-pair dominance relation can be defined as:*

$$R_A^{\lambda\le} = \{(x,y) \big| S_A(x,y) = S_1 + S_2 p + S_3 q \wedge (S_1 + FP(x,y)) \ge \lambda \wedge S_3 = 0\} \cup I,$$

where $0.5 \le \lambda \le 1$*, and* $I = \{(x,x) \big| x \in U\}$*. The size of threshold* λ *is related to the size of knowledge granulation obtained. The larger* λ *is, the stronger the classification ability is because it obtains finer knowledge granulation it obtains. In addition, it can be set according to specific needs. From the set-pair dominance relation, the set-pair dominance class can be obtained.*

$$[x]_A^{\lambda\le} = \left\{y \in U \big| (x,y) \in R_A^{\lambda\le}\right\},$$

3. GM-SPD-RS Models

In IOIS, the set-pair dominance relation overcomes the shortcomings of similar dominance relation and restricted dominance relation, which are overly loose or overly strict, respectively. It can classify the same kind of objects better and more logically in real life. The GMRS can mine more useful information, delete useless information and adapt to more practical problems. Therefore, the combination of set-pair dominance relation and generalized multi-granulation has certain research significance. Moreover, the GM-SPD-VPRS and GM-SPD-GRS are proposed in the following.

Definition 8 (GMRS)**.** [39] *Assume* IS $= (U, AT, VT, f)$*, and denote* $\varphi \in (0.5, 1]$ *as the information level parameter; the approximations of X with regard to* $\sum_{i=1}^{m} A_i(x)$ *are defined below:*

$$\underline{GM}_{\sum_{i=1}^{m} A_i}(X) = \{x \in U | \frac{\sum_{i=1}^{m} S_X^{A_i}(x)}{m} \ge \varphi\};$$
$$\overline{GM}_{\sum_{i=1}^{m} A_i}(X) = \{x \in U | \frac{\sum_{i=1}^{m} (1 - S_{\sim X}^{A_i}(x))}{m} > 1 - \varphi\}.$$

If $[x]_{A_i} \subseteq X$*, then* $S_X^{A_i}(x) = 1$*; otherwise* $S_X^{A_i}(x) = 0$*. For* $X \subseteq U$ *and* $x \in U$*, the number of classes satisfying* $[x]_{A_i} \subseteq X$ *can be expressed as* $\sum\limits_{i=1}^{m} S_X^{A_i}(x)$*. Similarly, the number of classes satisfying* $[x]_{A_i} \cap X \neq \phi$ *can be expressed as* $\sum\limits_{i=1}^{m} (1 - S_{\sim X}^{A_i}(x))$*. Therefore, optimistic MRS and pessimistic MRS can be described by supporting characteristic functions.*

Proposition 1. *Based on* $S_X^{A_i}(x)$*, the approximation sets of optimistic MRS and pessimistic MRS can be expressed respectively, which are described below:*

$$\underline{OM}_{\sum\limits_{i=1}^{m} A_i}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} S_X^{A_i}(x)}{m} > 0\};$$

$$\overline{OM}_{\sum\limits_{i=1}^{m} A_i}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} (1 - S_{\sim X}^{A_i}(x))}{m} \geq 1\};$$

$$\underline{PM}_{\sum\limits_{i=1}^{m} A_i}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} S_X^{A_i}(x)}{m} \geq 1\};$$

$$\overline{PM}_{\sum\limits_{i=1}^{m} A_i}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} (1 - S_{\sim X}^{A_i}(x))}{m} > 0\}.$$

Proof. This can be proved easily from Definition 8. □

Definition 9 (GM-SPD-VPRS). *Let* IOIS $= (U, AT, VT, f)$*,* $X \subseteq U$*,* A_i *be a subset of any attribute in AT, where* $i = 1, 2, \ldots, m (m \leq 2^{|AT|})$*,* $\lambda \in [0.5, 1]$*,* $\varphi \in (0.5, 1]$*. Assume* $\beta \in [0, 0.5]$ *is the adjustable error classification level. The approximation sets of X with regard to* $\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}$ *are defined below:*

$$\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda \leq}}(x)}{m} \geq \varphi\};$$

$$\overline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} (1 - S_{\sim X}^{A_{i,\beta}^{\lambda \leq}}(x))}{m} > 1 - \varphi\}.$$

where $S_X^{A_{i,\beta}^{\lambda \leq}}(x) = \begin{cases} 1, c([x]_{A_i}^{\lambda \leq}, X) \leq \beta \\ 0, otherwise \end{cases}$ $(0 \leq i \leq m)$*. X is called definable if* $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}}(X) = \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}}(X)$*; otherwise, X is rough if* $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}}(X) \neq \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}}(X)$*.*

It should be pointed out that with the increase of β, the lower approximation set becomes larger, and the upper approximation becomes smaller. In addition, when $\beta = 0$, GM-SPD-VPRS model degenerates into GM-SPD-RS. Namely, the GM-SPD-VPRS model is an extension of GMRS model, set-pair dominance relation RS model and VPRS model, and it correspondingly meets some basic properties.

Theorem 1. *Let* IOIS $= (U, AT, VT, f)$*, the approximation sets of X with regard to* $\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda \leq}$ *at information level* φ *have the following properties:*

(1) $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \sim \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(\sim X)$;
(2) $\overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \sim \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(\sim X)$;
(3) $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(\phi) = \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(\phi) = \phi$;
(4) $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(U) = \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(U) = U$;
(5) $X \subseteq Y \Rightarrow \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$;
(6) $X \subseteq Y \Rightarrow \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$;
(7) $X \cap Y = \phi \Rightarrow \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y) = \phi$;
(8) $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cup Y) \supseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cup \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$;
(9) $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$;
(10) $\overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cup Y) \supseteq \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cup \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$;
(11) $\overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cap Y) \subseteq \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cap \overline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$.

Proof. (1)~(4) These can directly be proved by Definition 9.

(5) For any $x \in \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X)$, there is $(\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x))/m \geq \varphi$. Since $X \subseteq Y$, we have $S_X^{A_{i,\beta}^{\lambda\leq}}(x) \leq S_Y^{A_{i,\beta}^{\lambda\leq}}(x)$, therefore, $(\sum_{i=1}^{m} S_Y^{A_{i,\beta}^{\lambda\leq}}(x))/m \geq (\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x))/m \geq \varphi$, and $x \in \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$ is obtained.

(6) The proof is similar to (5).

(7) According to supporting characteristic function $S_X^{A_{i,\beta}^{\lambda\leq}}(x) = 1$, we can obtain $c([x]_{A_i}^{\lambda\leq}, X) \leq \beta$, as $X \cap Y = \phi$; then, there is $c([x]_{A_i}^{\lambda\leq}, X) > \beta$ and $S_X^{A_{i,\beta}^{\lambda\leq}}(x) = 0$. Thus, for any $x \in \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X)$, there is $(\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x))/m \geq \varphi$ and $(\sum_{i=1}^{m} S_Y^{A_{i,\beta}^{\lambda\leq}}(x))/m < \varphi$; we can get $x \notin \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$, that is, $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y) = \phi$.

(8) (9) For $\forall X, Y \subseteq U$, there is $X, Y \subseteq X \cup Y$. In addition, according to (5), we have $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cup Y)$ and $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y) \subseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cup Y)$. Namely, $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cup Y) \supseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cup \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$. Similarly, for $\forall X, Y \subseteq U$, there is $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$; according to (5), $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X)$ and $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$ can be obtained. Namely, $\underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(Y)$.

(10) (11) The proofs are similar to (8) and (9). □

Based on Definition 9, the rough regions of X with regard to $\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}$ at information level φ are as follows:

$$POS_{\beta}^{\lambda,\varphi}(X) = \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cap \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X);$$
$$BND_{\beta}^{\lambda,\varphi}(X) = \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) - \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X);$$
$$NEG_{\beta}^{\lambda,\varphi}(X) = \sim (\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cup \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X)).$$

Definition 10 (GM-SPD-GRS). *Let* IOIS $= (U, AT, VT, f)$, $X \subseteq U$, A_i *be a subset of any attribute in AT, where* $i = 1, 2, \ldots, m (m \leq 2^{|AT|})$, $\lambda \in [0.5, 1]$, $\varphi \in (0.5, 1]$. *Assume* $k \in N$, *where N is the natural number set. The approximation sets of X with regard to* $\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}$ *are defined as below:*

$$\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x)}{m} \geq \varphi\};$$
$$\overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} (1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x))}{m} > 1 - \varphi\}.$$

where $S_X^{A_{i,k}^{\lambda\leq}}(x) = \begin{cases} 1, & |[x]_{A_i}^{\lambda\leq}| - |[x]_{A_i}^{\lambda\leq} \cap X| \leq k \wedge [x]_{A_i}^{\lambda\leq} \cap X \neq \phi \\ 0, & otherwise \end{cases}$ $(0 \leq i \leq m)$. *X is definable if* $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X)$*; otherwise, X is rough if* $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \neq \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X)$.

It should be pointed out that the K in the GM-SPD-GRS model is different from the k in the classical GRS. k in the classical GRS is used to measure the relation between the equivalence classes and the target set, while K in the GM-SPD-GRS model is used to measure the relationship between the dominance class and the target set.

According to the Definition 10, we can find that with the increase of k, the lower approximation set becomes larger and the upper approximation becomes smaller. In addition, when $k = 0$, GM-SPD-GRS model degenerates into GM-SPD-RS. Namely, the GM-SPD-GRS is an extension of GMRS model, set-pair dominance relation RS model and GRS model, and it correspondingly meets some basic properties. The details are expressed as follows.

Theorem 2. *Let* IOIS $= (U, AT, VT, f)$ *be an IOIS, the approximation sets of X with regard to* $\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}$ *at information level* φ *satisfy the following properties:*

(1) $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \sim \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(\sim X)$;

(2) $\overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \sim \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(\sim X)$;

(3) $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(\phi) = \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(\phi) = \phi$;

(4) $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(U) = \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(U) = U$;

(5) $X \subseteq Y \Rightarrow \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$;

(6) $X \subseteq Y \Rightarrow \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$;

(7) $X \cap Y = \phi \Rightarrow \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y) = \phi$;

(8) $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cup Y) \supseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cup \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$;
(9) $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$;
(10) $\overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cup Y) \supseteq \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cup \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$;
(11) $\overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cap Y) \subseteq \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cap \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$.

Proof. (1)~(4) These can be directly demonstrated by the Definition 10.

(5) For any $x \in \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X)$, there is $(\sum\limits_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x))/m \geq \varphi$. Since $X \subseteq Y$, we have $S_X^{A_{i,k}^{\lambda\leq}}(x) \leq S_Y^{A_{i,k}^{\lambda\leq}}(x)$, therefore, $(\sum\limits_{i=1}^{m} S_Y^{A_{i,k}^{\lambda\leq}}(x))/m \geq (\sum\limits_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x))/m \geq \varphi$, and $x \in \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$ is obtained.

(6) It is similar to (5).

(7) According to supporting characteristic function $S_X^{A_{i,k}^{\lambda\leq}}(x) = 1$, we can get $(|[x]_{A_i}^{\lambda\leq}| - |[x]_{A_i}^{\lambda\leq} \cap X| \leq k) \wedge ([x]_{A_i}^{\lambda\leq} \cap X) \neq \phi$. As $X \cap Y = \phi$, then, there is $S_Y^{A_{i,k}^{\lambda\leq}}(x) = 0$. Thus, for any $x \in \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X)$, there is $(\sum\limits_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x))/m \geq \varphi$ and $(\sum\limits_{i=1}^{m} S_Y^{A_{i,k}^{\lambda\leq}}(x))/m < \varphi$. We can get $x \notin \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$, that is, $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y) = \phi$.

(8)(9) For $\forall X, Y \subseteq U$, there is $X, Y \subseteq X \cup Y$. In addition, according to (5), we have $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cup Y)$ and $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y) \subseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cup Y)$. Namely, $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cup Y) \supseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cup \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$. Similarly, for $\forall X, Y \subseteq U$, there is $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$; according to (5), $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X)$ and $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$ can be obtained. Namely, $\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X \cap Y) \subseteq \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cap \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(Y)$.

(10) (11) The proofs are similar to (8) and (9). □

Based on the Definition 10, the rough regions with regard to $\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}$ at information level φ are as follows:

$$POS_k^{\lambda,\varphi}(X) = \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cap \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X);$$
$$NEG_k^{\lambda,\varphi}(X) = \sim (\underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cup \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X));$$
$$UBND_k^{\lambda,\varphi}(X) = \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) - \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X);$$
$$LBND_k^{\lambda,\varphi}(X) = \underline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) - \overline{GM}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X).$$

4. DQGM-SPD-RS Models

The GM-SPD-VPRS and GM-SPD-GRS models provide two new methods for decision-making under IOIS. In addition, these two models expand the RS model in the light of relative quantization and absolute quantization, respectively, which have their own quantization advantages. Aiming at the decision-making in IOIS, we introduce the double-quantitative idea into GM-SPD-VPRS model and GM-SPD-GRS model. In [23], Li et al. fused absolute and relative quantitative approximation operators, and two novel scenarios were generated. On the other hand, [27,33–36] introduced logical operators into different RS models, enriching RS theory. Therefore, we can get six DQGM-SPD-RS models by fusing and combining them with logical disjunction and logical conjunction operators:

(1) Lower approximation sets quantify relative quantitative information, and upper approximation sets quantify absolute quantitative information.

(2) Lower approximation sets quantify absolute quantitative information, and upper approximation sets quantify relative quantitative information.

(3) Lower and upper approximation sets quantify information by employing a logical disjunction operator.

(4) Lower approximation sets and upper approximation sets quantify information by employing a logical conjunction operator.

(5) Lower approximation sets quantify information by employing a logical disjunction operator, and upper approximation sets quantify information by employing a logical conjunction operator.

(6) Lower approximation sets quantify information by employing a logical conjunction operator, and upper approximation sets quantify information by employing a logical disjunction operator.

As observed above, the RS from the sixth model make the lower approximation smaller and the upper approximation larger. From the perspective of view of RS, knowledge becomes rougher. Therefore, this paper discusses only the first five models.

4.1. Five GM-SPD-RS Models in IOIS

Definition 11 (GM-SPD-RS$^{\mathrm{I}}$)**.** *Let* IOIS $= (U, AT, VT, f)$*, the approximation sets of the* DQGM-SPD-RS$^{\mathrm{I}}$ *can be defined as follows:*

$$\underline{DQGM}^{\mathrm{I}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)}{m} \geq \varphi\};$$

$$\overline{DQGM}^{\mathrm{I}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum\limits_{i=1}^{m} (1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x))}{m} > 1 - \varphi\}.$$

Proposition 2. *Let* IOIS $= (U, AT, VT, f)$ *be an IOIS; considering the extremes of optimism and pessimism, double-quantitative optimistic multi-granulation set-pair dominance relation rough sets (DQOM-SPD-RS^{I}) and double-quantitative pessimistic multi-granulation set-pair dominance rough sets (DQPM-SPD-RS^{I}) under the first model are obtained:*

$$\underline{DQOM}^{\mathrm{I}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \{x \in U | \bigvee_{i=1}^{m} c([x]_{A_i}^{\lambda\leq}, X) \leq \beta\};$$

$$\overline{DQOM}^{\mathrm{I}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \bigwedge_{i=1}^{m} |[x]_{A_i}^{\lambda\leq} \cap X > k\};$$

$$\underline{DQPM}^{\mathrm{I}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \{x \in U | \bigwedge_{i=1}^{m} c([x]_{A_i}^{\lambda\leq}, X) \leq \beta\};$$

$$\overline{DQPM}^{\mathrm{I}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \bigvee_{i=1}^{m} |[x]_{A_i}^{\lambda\leq} \cap X > k\}.$$

Proof. This can be demonstrated according to Definition 11. □

According to the DQGM-SPD-RS$^{\mathrm{I}}$, DQOM-SPD-RS$^{\mathrm{I}}$, and DQPM-SPD-RS$^{\mathrm{I}}$ obtained above, the following properties hold.

Theorem 3. *Let* IOIS $= (U, AT, VT, f)$ *be an IOIS,*

(1) $\underline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \underline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X)$;

(2) $\overline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \overline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X)$;

(3) $\underline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \supseteq \underline{R}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}$, $\overline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \overline{R}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}$;

(4) $\underline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \underline{R}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}$, $\overline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \supseteq \overline{R}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}$;

(5) $\underline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \bigcup_{i=1}^{m} \underline{R}_{A_{i,\beta}^{\lambda\leq}}$, $\overline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \bigcup_{i=1}^{m} \overline{R}_{A_{i,k}^{\lambda\leq}}$;

(6) $\underline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \bigcap_{i=1}^{m} \underline{R}_{A_{i,\beta}^{\lambda\leq}}$, $\overline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \bigcap_{i=1}^{m} \overline{R}_{A_{i,k}^{\lambda\leq}}$.

Proof. This can be demonstrated based on Definition 11 and Proposition 2. □

According to Definition 11, the rough regions can be obtained:

$$POS^{\mathrm{I}}(X) = \underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cap \overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X);$$
$$NEG^{\mathrm{I}}(X) = \sim (\underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \cup \overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X));$$
$$UBN^{\mathrm{I}}(X) = \overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) - \underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X);$$
$$LBN^{\mathrm{I}}(X) = \underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) - \overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X).$$

According to Definition 11 and relevant rough regions, the corresponding decision rules 4.1 of DQOM-SPD-RS$^{\mathrm{I}}$ model can be achieved as follows:

Positive region decision rule (P^{I}):

If $(\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x))/m \geq \varphi$ and $(\sum_{i=1}^{m} (1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)))/m > 1 - \varphi$, then $x \in POS^{\mathrm{I}}(X)$;

Negative region decision rule (N^{I}):

If $(\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x))/m < \varphi$ and $(\sum_{i=1}^{m} (1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)))/m \leq 1 - \varphi$, then $x \in NEG^{\mathrm{I}}(X)$;

Upper boundary region decision rule (UB^{I}):

If $(\sum_{i=1}^{m} (1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)))/m > 1 - \varphi$ and $(\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x))/m < \varphi$, then $x \in UBN^{\mathrm{I}}(X)$;

Lower boundary region decision rule (LB^{I}):

If $(\sum_{i=1}^{m} (1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)))/m \leq 1 - \varphi$ and $(\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x))/m \geq \varphi$, then $x \in LBN^{\mathrm{I}}(X)$.

Definition 12 (GM-SPD-RS$^{\mathrm{II}}$)**.** *Let* IOIS $= (U, AT, VT, f)$*, the approximation sets of the* DQGM-SPD-RS$^{\mathrm{II}}$ *are defined:*

$$\underline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x)}{m} \geq \varphi\};$$
$$\overline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum_{i=1}^{m} (1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))}{m} > 1 - \varphi\}.$$

Proposition 3. *Let* IOIS $= (U, AT, VT, f)$ *be an IOIS; considering the extremes, double-quantitative optimistic multi-granulation set-pair dominance relation rough sets (DQOM-SPD-RS*$^{\text{II}}$*) and double-quantitative pessimistic multi-granulation set-pair dominance rough sets (DQPM-SPD-RS*$^{\text{II}}$*) under the second model are obtained:*

$$\underline{DQOM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \bigvee_{i=1}^{m} |[x]_{A_i}^{\lambda\leq}| - |[x]_{A_i}^{\lambda\leq} \cap X| \leq k \wedge [x]_{A_i}^{\lambda\leq} \cap X \neq \phi\};$$
$$\overline{DQOM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \{x \in U | \bigwedge_{i=1}^{m} c([x]_{A_i}^{\lambda\leq}, X) < 1 - \beta\};$$
$$\underline{DQPM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \{x \in U | \bigwedge_{i=1}^{m} |[x]_{A_i}^{\lambda\leq}| - |[x]_{A_i}^{\lambda\leq} \cap X| \leq k \wedge [x]_{A_i}^{\lambda\leq} \cap X \neq \phi\};$$
$$\overline{DQPM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \{x \in U | \bigvee_{i=1}^{m} c([x]_{A_i}^{\lambda\leq}, X) < 1 - \beta\}.$$

Proof. This can be demonstrated directly according to Definition 12. □

According to the DQGM-SPD-RS$^{\text{II}}$, DQOM-SPD-RS$^{\text{II}}$, and DQPM-SPD-RS$^{\text{II}}$ obtained above, it has the following properties.

Theorem 4. *Let* IOIS $= (U, AT, VT, f)$ *be an IOIS,*

(1) $\underline{DQPM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \underline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \underline{DQOM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X)$;

(2) $\overline{DQOM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \overline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \overline{DQPM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X)$;

(3) $\underline{DQOM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \supseteq \underline{R}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}, \overline{DQOM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \overline{R}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}$;

(4) $\underline{DQPM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \subseteq \underline{R}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}, \overline{DQPM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) \supseteq \overline{R}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}$;

(5) $\underline{DQOM}^{\text{I}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \bigcup\limits_{i=1}^{m} \underline{R}_{A_{i,k}^{\lambda\leq}}, \overline{DQOM}^{\text{I}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \bigcup\limits_{i=1}^{m} \overline{R}_{A_{i,\beta}^{\lambda\leq}}$;

(6) $\underline{DQPM}^{\text{I}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) = \bigcap\limits_{i=1}^{m} \underline{R}_{A_{i,k}^{\lambda\leq}}, \overline{DQPM}^{\text{I}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X) = \bigcap\limits_{i=1}^{m} \overline{R}_{A_{i,\beta}^{\lambda\leq}}$.

Proof. This can be demonstrated based on Definition 12 and Proposition 3. □

According to Definition 12, the rough regions can be achieved below:

$$POS^{\text{II}}(X) = \underline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cap \overline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X);$$
$$NEG^{\text{II}}(X) = \sim (\underline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) \cup \overline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X));$$
$$UBN^{\text{II}}(X) = \overline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) - \underline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i\beta}^{\leq}}(X);$$
$$LBN^{\text{II}}(X) = \underline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,k}^{\lambda\leq}}(X) - \overline{DQGM}^{\text{II}}_{\sum\limits_{i=1}^{m} A_{i,\beta}^{\lambda\leq}}(X).$$

The corresponding decision rules 4.2 of the DQOM-SPD-RS$^{\text{II}}$ model can be achieved as follows:
Positive region decision rule (P^{II}):
If $(\sum\limits_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x))/m \geq \varphi$ and $(\sum\limits_{i=1}^{m} (1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x)))/m > 1 - \varphi$, then $x \in POS^{\text{II}}(X)$;

Negative region decision rule (N^{II}):
If $(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x))/m < \varphi$ and $(\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x)))/m \leq 1 - \varphi$, then $x \in NEG^{\mathrm{II}}(X)$;
Upper boundary region decision rule (UB^{II}):
If $(\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x)))/m > 1 - \varphi$ and $(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x))/m < \varphi$, then $x \in UBN^{\mathrm{II}}(X)$;
Lower boundary region decision rule (LB^{II}):
If $(\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x)))/m \leq 1 - \varphi$ and $(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x))/m \geq \varphi$, then $x \in LBN^{\mathrm{II}}(X)$.

Definition 13 (GM-SPD-RS$^{\mathrm{III}}$)**.** *Let* IOIS $= (U, AT, VT, f)$, *the approximation sets of the* DQGM-SPD-RS$^{\mathrm{III}}$ *are defined:*

$$\underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x)}{m} \geq \varphi \ \vee \ \frac{\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)}{m} \geq \varphi\};$$
$$\overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x))}{m} > 1 - \varphi \ \vee \ \frac{\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))}{m} > 1 - \varphi\}.$$

Based on Definition 13, Proposition 4 can be obtained.

Proposition 4. *The DQGM-SPD-RS$^{\mathrm{III}}$ obtained by the logical disjunction operator can also be expressed below:*

$$\underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \max(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x), \ \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi\};$$
$$\overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \max(\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)), \ \sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1 - \varphi)\}.$$

Proof. According to the lower approximation of Definition 13, for $x \in U$, there is $\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x)/m \geq \varphi \vee \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)/m \geq \varphi$. Namely, we can obtain $(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x)) \geq m\varphi \vee (\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi$; therefore, $\max(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x), \ \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi$ is obtained. Similarly, according to the upper approximation of Definition 15, $(\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)))/m > 1 - \varphi \vee (\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x)))/m > 1 - \varphi$. Namely, we can obtain $(\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x))) > m(1 - \varphi) \vee (\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1 - \varphi)$; then, $\max(\sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)), \sum_{i=1}^{m}(1 - S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1 - \varphi)$ is obtained.

According to Definition 13, the rough regions can be obtained:

$$POS^{\mathrm{III}}(X) = \underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) \cap \overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X);$$
$$NEG^{\mathrm{III}}(X) = \sim (\underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) \cup \overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X));$$
$$UBN^{\mathrm{III}}(X) = \overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) - \underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X);$$
$$LBN^{\mathrm{III}}(X) = \underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) - \overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X).$$

The corresponding decision rules 4.3 of the DQOM-SPD-RS$^{\mathrm{III}}$ model can be achieved as follows:

Positive region decision rule (P^{III}):

If $\max(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi$ and $\max(\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),$ $\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1-\varphi)$, then $x \in POS^{\mathrm{III}}(X)$;

Negative region decision rule (N^{III}):

If $\max(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) < m\varphi$ and $\max(\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),$ $\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) \leq m(1-\varphi)$, then $x \in NEG^{\mathrm{III}}(X)$;

Upper boundary region decision rule (UB^{III}):

If $\max(\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),\ \sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1-\varphi)$ and $\max(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) < m\varphi$, then $x \in UBN^{\mathrm{III}}(X)$;

Lower boundary region decision rule (LB^{III}):

If $\max(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi$ and $\max(\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),\ \sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) \leq m(1-\varphi)$, then $x \in LBN^{\mathrm{III}}(X)$. □

Definition 14 (GM-SPD-RS$^{\mathrm{IV}}$)**.** *Let* IOIS $= (U, AT, VT, f)$*, the approximation sets of the* DQGM-SPD-RS$^{\mathrm{IV}}$ *can be defined as follows:*

$$\underline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x)}{m} \geq \varphi \ \wedge \ \frac{\sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)}{m} \geq \varphi\};$$
$$\overline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \frac{\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x))}{m} > 1-\varphi \ \wedge \ \frac{\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))}{m} > 1-\varphi\}.$$

Based on Definition 14, Proposition 5 can be obtained.

Proposition 5. *The DQGM-SPD-RS^{IV} obtained by the logical conjunction operator can also be expressed below:*

$$\underline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \min(\sum_{i=1}^{m} S_X^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_X^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi\};$$
$$\overline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{m} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x \in U | \min(\sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),\ \sum_{i=1}^{m}(1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1-\varphi)\}.$$

Proof. It's easy to prove.

Based on Definition 14, the rough regions can be obtained:

$$POS^{\mathrm{IV}}(X) = \underline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) \cap \overline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X);$$
$$NEG^{\mathrm{IV}}(X) = \sim (\underline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) \cup \overline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X));$$
$$UBN^{\mathrm{IV}}(X) = \overline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) - \underline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X);$$
$$LBN^{\mathrm{IV}}(X) = \underline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) - \overline{DQGM}^{\mathrm{IV}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X).$$

The decision rules 4.4 of the DQOM-SPD-RS$^{\mathrm{IV}}$ can be achieved as follows:

Positive region decision rule (P^{IV}):

If $\min(\sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,k}}(x), \sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,\beta}}(x)) \geq m\varphi$ and $\min(\sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,k}}(x)), \sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,\beta}}(x))) > m(1-\varphi)$, then $x \in POS^{\mathrm{IV}}(X)$;

Negative region decision rule (N^{IV}):

If $\min(\sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,k}}(x), \sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,\beta}}(x)) < m\varphi$ and $\min(\sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,k}}(x)), \sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,\beta}}(x))) \leq m(1-\varphi)$, then $x \in NEG^{\mathrm{IV}}(X)$;

Upper boundary region decision rule (UB^{IV}):

If $\min(\sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,k}}(x)), \sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,\beta}}(x))) > m(1-\varphi)$ and $\min(\sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,k}}(x), \sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,\beta}}(x)) < m\varphi$, then $x \in UBN^{\mathrm{IV}}(X)$;

Lower boundary region decision rule (LB^{IV}):

If $\min(\sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,k}}(x), \sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,\beta}}(x)) \geq m\varphi$ and $\min(\sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,k}}(x)), \sum_{i=1}^{m}(1 - S_{\sim X}^{A^{\lambda\leq}_{i,\beta}}(x))) \leq m(1-\varphi)$, then $x \in LBN^{\mathrm{IV}}(X)$. □

Definition 15 (GM-SPD-RS$^{\mathrm{V}}$)**.** *Let* IOIS $= (U, AT, VT, f)$, *the approximation sets of the* DQGM-SPD-RS$^{\mathrm{V}}$ *can be defined as follows:*

$$\underline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) = \{x \in U | \frac{\sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,k}}(x)}{m} \geq \varphi \vee \frac{\sum_{i=1}^{m} S_X^{A^{\lambda\leq}_{i,\beta}}(x)}{m} \geq \varphi\};$$
$$\overline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) = \{x \in U | \frac{\sum_{i=1}^{m} (1 - S_{\sim X}^{A^{\lambda\leq}_{i,k}}(x))}{m} > 1 - \varphi \wedge \frac{\sum_{i=1}^{m} (1 - S_{\sim X}^{A^{\lambda\leq}_{i,\beta}}(x))}{m} > 1 - \varphi\}.$$

Based on Definition 15, Proposition 6 can be obtained.

Proposition 6. *The DQGM-SPD-RS^{V} achieved by the logical disjunction operator and logical conjunction operator can also be expressed below:*

$$POS^{\mathrm{V}}(X) = \underline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) \cap \overline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X);$$
$$NEG^{\mathrm{V}}(X) = \sim (\underline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) \cup \overline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X));$$
$$UBN^{\mathrm{V}}(X) = \overline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) - \underline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X);$$
$$LBN^{\mathrm{V}}(X) = \underline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X) - \overline{DQGM}^{\mathrm{V}}_{\sum\limits_{i=1}^{m} A^{\lambda\leq}_{i,k,\beta}}(X).$$

The decision rule 4.5 of the DQOM-SPD-RS$^{\mathrm{V}}$ can be achieved as follows:

Positive region decision rule (P^{V}):

If $\max(\sum_{i=1}^{m} S_{X}^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_{X}^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi$ and $\min(\sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),\ \sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1-\varphi)$, then $x \in POS^{\mathrm{V}}(X)$;

Negative region decision rule (N^{V}):

If $\max(\sum_{i=1}^{m} S_{X}^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_{X}^{A_{i,\beta}^{\lambda\leq}}(x)) < m\varphi$ and $\min(\sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),\ \sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) \leq m(1-\varphi)$, then $x \in NEG^{\mathrm{V}}(X)$;

Upper boundary region decision rule (UB^{V}):

If $\min(\sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),\ \sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) > m(1-\varphi)$ and $\max(\sum_{i=1}^{m} S_{X}^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_{X}^{A_{i,\beta}^{\lambda\leq}}(x)) < m\varphi$, then $x \in UBN^{\mathrm{V}}(X)$;

Lower boundary region decision rule (LB^{V}):

If $\max(\sum_{i=1}^{m} S_{X}^{A_{i,k}^{\lambda\leq}}(x),\ \sum_{i=1}^{m} S_{X}^{A_{i,\beta}^{\lambda\leq}}(x)) \geq m\varphi$ and $\min(\sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,k}^{\lambda\leq}}(x)),\ \sum_{i=1}^{m} (1-S_{\sim X}^{A_{i,\beta}^{\lambda\leq}}(x))) \leq m(1-\varphi)$, then $x \in LBN^{\mathrm{V}}(X)$.

4.2. Rough Regions under the DQGM-SPD-RS^{I} Model

According to the decision rules given in the five models above, Algorithm 1 is given, taking the DQGM-SPD-RS$^{\mathrm{I}}$ model as an example in this section. Through Algorithm 1, the rough regions under the DQGM-SPD-RSI model can be calculated.

Algorithm 1. Rough regions under the DQGM-SPD-RS$^{\mathrm{I}}$ model

Input: IOIS $= (U, AT, VT, f)$, $X \subseteq U$, $\lambda \in [0.5, 1]$, information level parameter $\varphi \in (0.5, 1]$,
adjustable error classification level parameter $\beta \in [0, 0.5]$ and grade parameter $k \in N$
Output: $POS^{\mathrm{I}}(X)$, $NEG^{\mathrm{I}}(X)$, $UBN^{\mathrm{I}}(X)$, $LBN^{\mathrm{I}}(X)$

```
for each x ∈ X, A_i ⊆ AT do
    Compute [x]_{A_i}^{λ≤}, |[x]_{A_i}^{λ≤}|, and |[x]_{A_i}^{λ≤} ∩ X|
end for
Initialize POS^I(X) ← φ, NEG^I(X) ← φ, UBN^I(X) ← φ, LBN^I(X) ← φ
for each x ∈ X, A_i ⊆ AT do
    if c([x]_{A_i}^{λ≤}, X) ≤ β, then S_X^{A_{i,β}^{λ≤}}(x) = 1
        else S_X^{A_{i,β}^{λ≤}}(x) = 0
    end if
    if |[x]_{A_i}^{λ≤}| − |[x]_{A_i}^{λ≤} ∩ X| ≤ k ∧ [x]_{A_i}^{λ≤} ∩ X ≠ φ,
then S_X^{A_{i,k}^{λ≤}}(x) = 1
        else S_X^{A_{i,k}^{λ≤}}(x) = 0
    end if
end for
for each x ∈ X, A_i ⊆ AT do
POS^I(X) ← POS^I(X) ∪ {x ∈ X| (Σ_{i=1}^{m} S_X^{A_{i,β}^{λ≤}}(x))/m ≥ φ, (Σ_{i=1}^{m} (1 − S_{~X}^{A_{i,k}^{λ≤}}(x)))/m > 1 − φ}
NEG^I(X) ← NEG^I(X) ∪ {x ∈ X| (Σ_{i=1}^{m} S_X^{A_{i,β}^{λ≤}}(x))/m < φ, (Σ_{i=1}^{m} (1 − S_{~X}^{A_{i,k}^{λ≤}}(x)))/m ≤ 1 − φ }
UBN^I(X) ← UBN^I(X) ∪ {x ∈ X| (Σ_{i=1}^{m} (1 − S_{~X}^{A_{i,k}^{λ≤}}(x)))/m > 1 − φ, (Σ_{i=1}^{m} S_X^{A_{i,β}^{λ≤}}(x))/m < φ
LBN^I(X) ← LBN^I(X) ∪ {x ∈ X| (Σ_{i=1}^{m} (1 − S_{~X}^{A_{i,k}^{λ≤}}(x)))/m ≤ 1 − φ, (Σ_{i=1}^{m} S_X^{A_{i,β}^{λ≤}}(x))/m ≥ φ
end for
return POS^I(X), NEG^I(X), UBN^I(X), LBN^I(X)
```

It can be seen that the difference between the above algorithm and the other four models' algorithms is mainly in steps 14 to 17, which is the essence of each model. Therefore, the algorithm steps of DQGM-SPD-RS^{I} are given in this paper, and the algorithms of the other four models are similar.

5. Example Analysis

In IOIS, five DQGM-SPD-RS models have been defined in the previous section, and which provide a way for decision-making in IOIS. Inspired by the literature [37], we think that medical example is close to real life. It is interesting to demonstrate our models with medical example. It should be noted that the data in this paper are derived from the literature [37], we select the first 20 objects and randomly set up an IIS.

Let IODIS $= (U, AT \cup d, VT, f)$, $X \subseteq U$, $A_i \subseteq AT$, where $i = 1, 2, \ldots, m (m \leq 2^{|AT|})$. Suppose that $U = \{x_1, x_2, \ldots, x_{20}\}$ is 20 patients, $AT = \{a_1, a_2, a_3\}$ represent fever, headache, and cough, respectively, and $d = \{0, 1\}$ represent no cold and cold, respectively. Among them, $A_1 = \{a_1, a_2\}$, $A_2 = \{a_1, a_3\}$, $A_3 = \{a_2, a_3\}$, $VT = \{2, 1, 0, *\}$ represent different grades of condition attributes, where $*$ indicates unknown. Table 1 gives the specific conditions of 20 patients in detail.

Table 1. Medical data of patients.

U	a_1	a_2	a_3	d	U	a_1	a_2	a_3	d
x_1	0	0	0	0	x_{11}	1	2	$*$	1
x_2	1	1	1	0	x_{12}	2	0	0	0
x_3	$*$	2	1	1	x_{13}	0	$*$	1	0
x_4	2	1	2	0	x_{14}	$*$	1	2	1
x_5	1	0	$*$	1	x_{15}	0	1	2	1
x_6	2	2	2	1	x_{16}	1	$*$	0	0
x_7	0	$*$	0	0	x_{17}	0	2	1	0
x_8	1	2	1	0	x_{18}	2	1	$*$	1
x_9	2	2	$*$	1	x_{19}	0	$*$	0	0
x_{10}	$*$	1	1	1	x_{20}	1	2	2	1

From Table 1, we can obtain $U/d = \{D_1, D_2\}$, where $D_1 = \{x_1, x_2, x_4, x_7, x_8, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}\}$ and $D_2 = \{x_3, x_5, x_6, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}\}$. D_1 and D_2 stand for no cold and cold, respectively. We denote D_2 as the object set, namely, $X = D_2$. In addition, we assume that $\lambda = 1$. According to Algorithm 1, Tables 2–4 show the classification and statistical results of dominance classes under different granular sets, respectively.

Table 2. The computation results of dominance classes in A_1.

x	$[x]_{A_1}^{\lambda\leq}$	$\lvert[x]_{A_1}^{\lambda\leq}\rvert$	$\lvert[x]_{A_1}^{\lambda\leq}\cap \mathbf{X}\rvert$	x	$[x]_{A_1}^{\lambda\leq}$	$\lvert[x]_{A_1}^{\lambda\leq}\rvert$	$\lvert[x]_{A_1}^{\lambda\leq}\cap \mathbf{X}\rvert$
x_1	$x_1, \ldots, x_{20}$	20	10	x_8, x_{11}, x_{20}	$x_6, x_8, x_9, x_{11}, x_{20}$	5	4
x_2	$x_2, x_3, x_4, x_6, x_8, x_9, x_{11}, x_{16}, x_{18}, x_{20}$	10	6	x_{10}, x_{14}	$x_2, x_3, x_4, x_6, x_8, x_9, x_{10}, x_{11}, x_{14}, x_{18}, x_{20}$	11	8
x_3	$x_3, x_6, x_8, x_9, x_{11}, x_{20}$	6	5	x_{12}	$x_3, x_4, x_6, x_9, x_{12}, x_{18}$	6	4
x_4, x_{18}	$x_3, x_4, x_6, x_9, x_{18}$	5	4	x_{15}	$x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}$	17	9
x_5	$x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{11}, x_{12}, x_{16}, x_{18}, x_{20}$	12	7	x_{16}	$x_6, x_8, x_9, x_{11}, x_{16}, x_{20}$	6	4
x_6, x_9	x_6, x_9	2	2	x_{17}	$x_3, x_6, x_8, x_9, x_{11}, x_{16}, x_{17}, x_{20}$	8	5
x_7, x_{13}, x_{19}	$x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{11}, x_{13}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}$	14	6				

Table 3. The computation results of dominance classes in A_2.

x	$[x]_{A_2}^{\lambda\leq}$	$\lvert[x]_{A_2}^{\lambda\leq}\rvert$	$\lvert[x]_{A_2}^{\lambda\leq}\cap X\rvert$	x	$[x]_{A_2}^{\lambda\leq}$	$\lvert[x]_{A_2}^{\lambda\leq}\rvert$	$\lvert[x]_{A_2}^{\lambda\leq}\cap X\rvert$
x_1, x_7, x_{19}	$x_1, \ldots, x_{20}$	20	10	x_{12}	$x_3, x_4, x_6, x_9, x_{10}, x_{12}, x_{14}, x_{18}$	8	6
x_2, x_8	$x_2, x_4, x_6, x_8, x_{20}$	5	2	x_{13}, x_{17}	$x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{17}, x_{18}, x_{20}$	15	10
x_3, x_{10}	$x_2, x_3, x_4, x_6, x_8, x_{10}, x_{14}, x_{20}$	8	5	x_{14}	x_4, x_6, x_{14}, x_{20}	4	3
x_4, x_6	x_4, x_6	2	1	x_{15}	$x_3, x_4, x_5, x_6, x_{10}, x_{11}, x_{14}, x_{15}, x_{20}$	9	8
x_5, x_{11}	$x_2, x_4, x_5, x_6, x_8, x_9, x_{11}, x_{18}, x_{20}$	9	6	x_{16}	$x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{14}, x_{16}, x_{18}, x_{20}$	14	9
x_9, x_{18}	x_4, x_6, x_9, x_{18}	4	3	x_{20}	x_4, x_6, x_{20}	3	2

Table 4. The computation results of dominance classes in A_3.

x	$[x]_{A_3}^{\lambda\leq}$	$\lvert[x]_{A_3}^{\lambda\leq}\rvert$	$\lvert[x]_{A_1}^{\lambda\leq}\cap X\rvert$	x	$[x]_{A_3}^{\lambda\leq}$	$\lvert[x]_{A_3}^{\lambda\leq}\rvert$	$\lvert[x]_{A_1}^{\lambda\leq}\cap X\rvert$
x_1, x_{12}	$x_1, \ldots, x_{20}$	20	10	x_6, x_{20}	$x_{6,20}$	2	2
x_2, x_{10}	$x_2, x_3, x_4, x_6, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{17}, x_{20}$	13	8	x_7, x_{16}, x_{19}	$x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{19}, x_{20}$	16	8
x_3, x_8, x_{17}	$x_3, x_6, x_8, x_{17}, x_{20}$	5	3	x_9, x_{11}	$x_3, x_6, x_8, x_9, x_{11}, x_{17}, x_{20}$	7	5
x_4, x_{14}, x_{15}	$x_4, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{20}$	7	6	x_{13}	$x_3, x_6, x_8, x_{13}, x_{17}, x_{20}$	6	3
x_5	$x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{17}, x_{18}, x_{20}$	15	10	x_{18}	$x_2, x_3, x_4, x_6, x_8, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{17}, x_{18}, x_{20}$	13	9

In the following, for clarity, we firstly compute the classification of patients under each granulation set. We denote grade $k = 2$ and assume $\beta = 0.3$ and $\varphi = 2/3$ in this paper.

$$
\begin{aligned}
&\underline{A_1}_{\beta}^{\lambda\leq}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{10}, x_{11}, x_{14}, x_{18}, x_{20}\};\\
&\overline{A_1}_{\beta}^{\lambda\leq}(X) = U;\\
&\underline{A_2}_{\beta}^{\lambda\leq}(X) = \{x_9, x_{12}, x_{14}, x_{15}, x_{18}\};\\
&\overline{A_2}_{\beta}^{\lambda\leq}(X) = U;\\
&\underline{A_3}_{\beta}^{\lambda\leq}(X) = \{x_4, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{20}\};\\
&\overline{A_3}_{\beta}^{\lambda\leq}(X) = U.\\
&\underline{A_1}_{k}^{\lambda\leq}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{16}, x_{18}, x_{20}\};\\
&\overline{A_1}_{k}^{\lambda\leq}(X) = U - \{x_6, x_9\};\\
&\underline{A_2}_{k}^{\lambda\leq}(X) = \{x_4, x_6, x_9, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};\\
&\overline{A_2}_{k}^{\lambda\leq}(X) = U - \{x_2, x_4, x_6, x_8, x_{20}\};\\
&\underline{A_3}_{k}^{\lambda\leq}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{14}, x_{15}, x_{17}, x_{20}\};\\
&\overline{A_3}_{k}^{\lambda\leq}(X) = U - \{x_6, x_{20}\}.
\end{aligned}
$$

According to Tables 2–4, we can see that patients are classified differently under each granulation. In addition, patients are classified differently under GM-SPD-VPRS model and GM-SPD-GRS model. The main contributions of Sections 3 and 4 are described below.

On the basis of Definition 9 and Definition 10, the approximation sets of GM-SPD-VPRS and GM-SPD-GRS can be calculated.

$$\underline{GM}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = \{x_4, x_6, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\underline{GM}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = U;$$
$$\underline{GM}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\overline{GM}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = U - \{x_6, x_{20}\}.$$

Furthermore, based on Definitions 11 to 15, the approximation sets of DQGM-SPD-RS$^{\mathrm{I}}$ to DQGM-SPD-RS$^{\mathrm{V}}$ can be calculated. In particular, according to Proposition 2 and Proposition 3, we can obtain the approximation sets of DQOM-SPD-RS$^{\mathrm{I}}$, DQPM-SPD-RS$^{\mathrm{I}}$, DQOM-SPD-RS$^{\mathrm{II}}$, and DQPM-SPD-RS$^{\mathrm{II}}$, which are special circumstances of DQGM-SPD-RS$^{\mathrm{I}}$ and DQGM-SPD-RS$^{\mathrm{II}}$. The details are as follows:

For DQGM-SPD-RS$^{\mathrm{I}}$ model and DQGM-SPD-RS$^{\mathrm{II}}$ model, we have:

$$\underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = \{x_4, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = U - \{x_6, x_{20}\};$$
$$\underline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\overline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = U - \{x_2, x_4, x_6, x_8, x_9, x_{20}\};$$
$$\underline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = \{x_9, x_{14}\};$$
$$\overline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = U - \{x_6\};$$
$$\underline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\overline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = U;$$
$$\underline{DQOM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{20}\};$$
$$\overline{DQOM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = U;$$
$$\underline{DQPM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = \{x_4, x_6, x_9, x_{20}\};$$
$$\overline{DQPM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = U.$$

From the above, we can obtain such relationships readily:

$$\underline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \underline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X);$$
$$\overline{DQOM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) \subseteq \overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) \subseteq \overline{DQPM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X);$$
$$\underline{DQPM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) \subseteq \underline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) \subseteq \underline{DQOM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X);$$
$$\overline{DQOM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \overline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) \subseteq \overline{DQPM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X).$$

It is not difficult to find that these formulas are consistent with Theorems 3 and 4. In addition, the approximation sets of two special circumstances are not as clear as that of DQOM-SPD-RS, as the conditions are too loose or too harsh, respectively, which indicates the effectiveness of the new model.

For the DQGM-SPD-RS$^{\mathrm{III}}$ model to the DQGM-SPD-RS$^{\mathrm{V}}$ model, we have:

$$\underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = U;$$

$$\underline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x_4, x_6, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\overline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = U - \{x_6, x_{20}\};$$
$$\underline{DQGM}^{\mathrm{V}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\};$$
$$\overline{DQGM}^{\mathrm{V}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = U - \{x_6, x_{20}\}.$$

The approximation sets of the above models are not the same, because the calculation methods of each model are different. According to Definitions 3, we can calculate the approximation accuracy of each model.

$$\alpha_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X) = \frac{|\underline{GM}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X)|}{|\overline{GM}_{\sum_{i=1}^{3} A_{i,\beta}^{\lambda\leq}}(X)|} = \frac{9}{20} = 0.45;$$
$$\alpha_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X) = \frac{|\underline{GM}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X)|}{|\overline{GM}_{\sum_{i=1}^{3} A_{i,k}^{\lambda\leq}}(X)|} = \frac{11}{18} = 0.61.$$
$$\alpha^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \frac{|\underline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|}{|\overline{DQGM}^{\mathrm{I}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|} = \frac{9}{18} = 0.50;$$
$$\alpha^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \frac{|\underline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|}{|\overline{DQGM}^{\mathrm{II}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|} = \frac{11}{20} = 0.55;$$
$$\alpha^{\mathrm{III}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \frac{|\underline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|}{|\overline{DQGM}^{\mathrm{III}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|} = \frac{11}{20} = 0.55;$$
$$\alpha^{\mathrm{IV}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \frac{|\underline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|}{|\overline{DQGM}^{\mathrm{IV}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|} = \frac{9}{18} = 0.50;$$
$$\alpha^{\mathrm{V}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X) = \frac{|\underline{DQGM}^{\mathrm{V}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|}{|\overline{DQGM}^{\mathrm{V}}_{\sum_{i=1}^{3} A_{i,k,\beta}^{\lambda\leq}}(X)|} = \frac{11}{18} = 0.61.$$

Comparing the approximation accuracy of GM-SPD-VPRS and five DQGM-SPD-RS models, it is easy to find that the latter is higher than that of the former no matter which model. Comparing the approximation accuracy of GM-SPD-GRS and five DQGM-SPD-RS models, it can be found readily that the approximation accuracy of DQGM-SPD-RS$^{\mathrm{V}}$ is equal to that of GM-SPD-GRS, and the approximate accuracy of the other four DQGM-SPD-RS is slightly lower than that of GM-SPD-GRS. This is due to GM-SPD-GRS reflect the absolute quantization information.

In conclusion, approximate classification capability of DQGM-SPD-RS is better than that of the GM-SPD-VPRS. The classification of DQGM-SPD-RS is more reasonable than that of GM-SPD-GRS, people can choose according to different needs.

In the following, according to decision rules 4.1 to 4.5, rough regions under five novel models can be obtained. In particular, according to Propositions 2 and 3, we can obtain four rough regions in special cases. Table 5 gives the rough regions under each model in detail.

Table 5. Rough regions of patients under different models.

Model	$pos(X)$	$neg(X)$	$Ubn(X)$	$Lbn(X)$
DQGM-SPD-RS$^{\mathrm{I}}$	$\{x_4, x_9, x_{11}, x_{14}, x_{15}, x_{18}\}$	ϕ	$\{x_1, x_2, x_3, x_5, x_7, x_8, x_{10}, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}\}$	$\{x_6, x_{20}\}$
DQOM-SPD-RS$^{\mathrm{I}}$	$\{x_3, x_{10}, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}\}$	$\{x_2\}$	$\{x_1, x_5, x_7, x_{13}, x_{16}, x_{17}, x_{19}\}$	$\{x_4, x_6, x_8, x_9, x_{20}\}$
DQPM-SPD-RS$^{\mathrm{I}}$	$\{x_9\}$	$\{x_6\}$	$U - \{x_6, x_9\}$	ϕ
DQGM-SPD-RS$^{\mathrm{II}}$	$\{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\}$	ϕ	$\{x_1, x_2, x_5, x_7, x_{10}, x_{13}, x_{16}, x_{17}, x_{19}\}$	ϕ
DQOM-SPD-RS$^{\mathrm{II}}$	$\{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{20}\}$	ϕ	$\{x_1, x_2, x_5, x_7, x_{10}, x_{13}, x_{19}\}$	ϕ
DQPM-SPD-RS$^{\mathrm{II}}$	$\{x_4, x_6, x_9, x_{20}\}$	ϕ	$U - \{x_4, x_6, x_9, x_{20}\}$	ϕ
DQGM-SPD-RS$^{\mathrm{III}}$	$\{x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}, x_{20}\}$	ϕ	$\{x_1, x_2, x_5, x_7, x_{10}, x_{13}, x_{16}, x_{17}, x_{19}\}$	ϕ
DQGM-SPD-RS$^{\mathrm{IV}}$	$\{x_4, x_9, x_{11}, x_{14}, x_{15}, x_{18}\}$	ϕ	$\{x_1, x_2, x_3, x_5, x_7, x_8, x_{10}, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}\}$	$\{x_6, x_{20}\}$
DQGM-SPD-RS$^{\mathrm{V}}$	$\{x_3, x_4, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{18}\}$	ϕ	$\{x_1, x_2, x_5, x_7, x_{10}, x_{13}, x_{16}, x_{17}, x_{19}\}$	$\{x_6, x_{20}\}$

From Table 5, it can be observed that the rough regions obtained under different models are not exactly the same, which is due to different computing methods. In the above models, except for four special cases, patients x_4, x_9, x_{11}, x_{14}, x_{15}, and x_{18} are diagnosed as having a cold, but patients x_4 are not considered to have a cold in Table 1 of the preliminary medical data. After analyzing the symptoms of patients x_4, it is found that there is a possibility of misdiagnosis in the preliminary medical data of patients, and the results of other patients diagnosed as having a cold are consistent with those in Table 1. The patients x_6 and x_{20} are diagnosed as having a cold in the second model and the third model, while in models DQGM-SPD-RS$^{\mathrm{I}}$, DQOM-SPD-RS$^{\mathrm{I}}$, DQGM-SPD-RS$^{\mathrm{IV}}$, and DQGM-SPD-RS$^{\mathrm{V}}$, they are divided into lower boundaries, requiring further diagnosis. This is due to the difference in decision-making rules, which also reflected the rationality of the models. Meanwhile, the results are consistent with those in literature [37] to prove the validity of this method. Through these models, some preliminary diagnostic analysis can be conducted on patients. The appropriate model should be selected according to actual application needs.

6. Conclusions

This paper studies the problem of uncertain decision-making under IOIS. Combing with the advantages of set-pair dominance relation and generalized multi-granulation, we firstly propose GM-SPD-VPRS and GM-SPD-GRS and discuss the related properties. Since GM-SPD-VPRS and GM-SPD-GRS reflect the information from the perspective of relative and absolute quantization, respectively. The two rough sets are combined by logical disjunction and logical conjunction operators, and five double-quantification models, from DQGM-SPD-RS$^{\mathrm{I}}$ to DQGM-SPD-RS$^{\mathrm{V}}$, are obtained. Meanwhile, the related properties are discussed, and the algorithm of DQGM-SPD-RS$^{\mathrm{I}}$ is constructed. Finally, the models are validated through a medical case, which proves the effectiveness of the model.

The works of this paper can reflect the relative and absolute quantization information in IOIS without data preprocessing, are extensions of the classical RS, the existing VPRS and GRS, as well as the extension of the MRS. In the future, we can further extend the approach of this paper in the framework of two universes, and the approach proposed in this paper combined with intuitionistic fuzzy set is also a research problem.

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