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New Results for Oscillatory Behavior of Fourth-Order Differential Equations

Rami Ahmad El-Nabulsi [1,*,†], Osama Moaaz [2,†] and Omar Bazighifan [3,†]

1 Athens Institute for Education and Research, Mathematics and Physics Divisions, 10671 Athens, Greece
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; o_moaaz@mans.edu.eg
3 Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen; o.bazighifan@gmail.com
* Correspondence: nabulsiahmadrami@yahoo.fr
† These authors contributed equally to this work.

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Abstract: Our aim in the present paper is to employ the Riccati transformation which differs from those reported in some literature and comparison principles with the second-order differential equations, to establish some new conditions for the oscillation of all solutions of fourth-order differential equations. Moreover, we establish some new criterion for oscillation by using an integral averages condition of Philos-type, also Hille and Nehari-type. Some examples are provided to illustrate the main results.

Keywords: oscillatory solutions; nonoscillatory solutions; fourth-order; delay differential equations; Riccati transformation; comparison theorem

1. Introduction

In this work, we study the fourth-order nonlinear differential equation

$$\left(r\left(\ell\right)\left(y'''\left(\ell\right)\right)^{\alpha}\right)' + q\left(\ell\right) f\left(y\left(\tau\left(\ell\right)\right)\right) = 0,\ \ell \geq \ell_0, \tag{1}$$

where α is a quotient of odd positive integers, $r \in C^1\left([\ell_0,\infty),\mathbb{R}\right)$, $r\left(\ell\right) > 0$, $r'\left(\ell\right) > 0$, $q,\tau \in C\left([\ell_0,\infty),\mathbb{R}\right)$, $q\left(\ell\right) \geq 0$, $\tau\left(\ell\right) \leq \ell$, $\lim_{\ell\to\infty}\tau\left(\ell\right) = \infty$, $f \in C\left(\mathbb{R},\mathbb{R}\right)$ such that

$$f\left(u\right)/u^{\alpha} \geq k > 0, \text{ for } u \neq 0 \tag{2}$$

and under the condition

$$\int_{\ell_0}^{\infty} \frac{1}{r^{1/\alpha}\left(u\right)} \mathrm{d}u = \infty. \tag{3}$$

By a solution of Equation (1) we mean a function $y \in C^3[\ell_y,\infty)$, $\ell_y \geq \ell_0$, which has the property $r\left(\ell\right)\left(y'''\left(\ell\right)\right)^{\alpha} \in C^1[\ell_y,\infty)$, and satisfies Equation (1) on $[\ell_y,\infty)$. We consider only those solutions y of Equation (1) which satisfy $\sup\{|y\left(\ell\right)| : \ell \geq \ell_0\} > 0$, for all $\ell > \ell_y$. A solution of Equation (1) is called oscillatory if it has arbitrarily large zeros on $[\ell_y,\infty)$; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

One of the main reasons for this lies in the fact that differential and functional differential equations arise in many applied problems in natural sciences and engineering, see [1].

Fourth-order differential equations are quite often encountered in mathematical models of various physical, biological, and chemical phenomena. Applications include, for instance, problems of elasticity, deformation of structures, or soil settlement; see [2]. In mechanical and engineering problems, questions related to the existence of oscillatory and nonoscillatory solutions play an important role.

During the last years, significant efforts have been devoted to investigate the oscillatory behavior of fourth-order differential equations. For treatments on this subject, we refer the reader to the texts [3–29]. In what follows, we review some results that have provided the background and the motivation, for the present work.

In our paper, by careful observation and employing some inequalities of a different type, we provide a new criterion for the oscillation of differential Equation (1). Here, we offer different criteria for oscillation which can cover a larger area of different models of fourth-order differential equations. We introduce a generalized Riccati substitution to obtain a new Philos-type criteria and Hille- and Nehari-type. In the last section, we apply the main results to two different examples.

In the following, we show some previous results in the literature which relate to this paper: Many researchers in [30–33] have studied the oscillatory behavior of equation

$$\left(r\left(\ell\right)y^{(n-1)}\left(\ell\right)\left|y^{(n-1)}\left(\ell\right)\right|^{\alpha-1}\right)'+f\left(\ell,y\left(\tau\left(\ell\right)\right)\right)=0,$$

under the condition in Equation (3), where α is a positive real number. In [29], the authors studied the oscillation of the equation

$$\left(r\left(\ell\right)\left(y'''\left(\ell\right)\right)^{\alpha}\right)'+q\left(\ell\right)y^{\alpha}\left(\tau\left(\ell\right)\right)=0,$$

under the condition

$$\int_{\ell_0}^{\infty}\frac{1}{r^{1/\alpha}\left(u\right)}\mathrm{d}u<\infty. \tag{4}$$

By comparison theory, Baculikova et al. [6] proved that the equation

$$\left(r\left(\ell\right)\left(x^{(n-1)}\left(\ell\right)\right)^{\alpha}\right)'+q\left(\ell\right)f\left(x\left(\tau\left(\ell\right)\right)\right)=0$$

is oscillatory if the delay equations

$$y'\left(\ell\right)+q(\ell)f\left(\frac{\delta\tau^{n-1}\left(\ell\right)}{(n-1)!r^{\frac{1}{\alpha}}\left(\tau\left(\ell\right)\right)}\right)f\left(y^{\frac{1}{\alpha}}\left(\tau\left(\ell\right)\right)\right)=0$$

is oscillatory. Moaaz et al. [21] established the oscillation criterion for solutions of the equation

$$\left(r\left(\ell\right)\left(y'''\left(\ell\right)\right)^{\alpha}\right)'+\int_a^b q\left(\ell,\zeta\right)f\left(y\left(g\left(\ell,\zeta\right)\right)\right)d\sigma\left(\zeta\right)=0,$$

under the condition in Equation (4).

Next, we begin with the following lemmas.

Lemma 1. *[5] Let β be a ratio of two odd numbers. Then*

$$P^{(\beta+1)/\beta}-\left(P-Q\right)^{(\beta+1)/\beta}\leq\frac{1}{\beta}Q^{1/\beta}\left[\left(1+\beta\right)P-Q\right],\ \ PQ\geq0,\ \beta\geq1$$

and

$$Ux-Vx^{(\beta+1)/\beta}\leq\frac{\beta^{\beta}}{\left(\beta+1\right)^{\beta+1}}\frac{U^{\beta+1}}{V^{\beta}},\ V>0.$$

Lemma 2. *[9] If the function u satisfies $u^{(j)}>0$ for all $j=0,1,...,n$, and $u^{(n+1)}<0$, then*

$$\frac{n!}{\ell^{n}}u\left(\ell\right)-\frac{(n-1)!}{\ell^{n-1}}\frac{d}{d\ell}u\left(\ell\right)\geq0.$$

Lemma 3. *[25] The equation*

$$\left(a\left(\ell\right)\left(x'\left(\ell\right)^{\alpha}\right)\right)' + q\left(\ell\right)x^{\alpha}\left(\ell\right) = 0, \tag{5}$$

where $a \in C[\ell_0, \infty)$, $a\left(\ell\right) > 0$ and $q\left(\ell\right) > 0$, is nonoscillatory if and only if there exist a $\ell \geq \ell_0$ and a function $v \in C^1[\ell, \infty)$ such that

$$v'\left(\ell\right) + \frac{\alpha}{r^{1/\alpha}\left(\ell\right)}v^{1+1/\alpha}\left(\ell\right) + q\left(\ell\right) \leq 0,$$

for $\ell \geq \ell$.

Lemma 4. *[28] Suppose that $h \in C^n\left([\ell_0, \infty), (0, \infty)\right)$, $h^{(n)}$ is of a fixed sign on $[\ell_0, \infty)$, $h^{(n)}$ not identically zero and there exists a $\ell_1 \geq \ell_0$ such that*

$$h^{(n-1)}\left(\ell\right)h^{(n)}\left(\ell\right) \leq 0,$$

for all $\ell \geq \ell_1$. If we have $\lim_{\ell\to\infty} h\left(\ell\right) \neq 0$, then there exists $\ell_\lambda \geq \ell_1$ such that

$$h\left(\ell\right) \geq \frac{\lambda}{(n-1)!}\ell^{n-1}\left|h^{(n-1)}\left(\ell\right)\right|,$$

for every $\lambda \in (0, 1)$ and $\ell \geq \ell_\lambda$.

2. Main Results

In this section, we shall establish some oscillation criteria for Equation (1). For convenience, we denote

$$\eta\left(\ell\right) := \int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}\left(s\right)}\mathrm{d}s,\ F_{+}\left(\ell\right) := \max\left\{0, F\left(\ell\right)\right\},$$

$$\psi\left(\ell\right) := \rho\left(\ell\right)\left(kq\left(\ell\right)\left(\frac{\tau^3\left(\ell\right)}{\ell^3}\right)^{\alpha} + \frac{\mu\lambda_1^{(1+\alpha)/\alpha}\ell^2 - 2\lambda_1\alpha}{2r^{\frac{1}{\alpha}}\left(\ell\right)\eta^{\alpha+1}(\ell)}\right),$$

$$\phi\left(\ell\right) := \frac{\rho'_{+}\left(\ell\right)}{\rho\left(\ell\right)} + \frac{(\alpha+1)\lambda_1^{1/\alpha}\mu\ell^2}{2r^{\frac{1}{\alpha}}\left(\ell\right)\eta(\ell)},\quad \phi^*\left(\ell\right) := \frac{\vartheta'_{+}\left(\ell\right)}{\vartheta\left(\ell\right)} + \frac{2\lambda_2}{\eta(\ell)},$$

and

$$\psi^*\left(\ell\right) := \vartheta\left(\ell\right)\left(\int_{\ell}^{\infty}\left(\frac{k}{r\left(v\right)}\int_{v}^{\infty}q\left(s\right)\frac{\tau^{\alpha}\left(s\right)}{s^{\alpha}}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}v + \frac{\lambda_2^2 - \lambda_2 r^{\frac{-1}{\alpha}}\left(\ell\right)}{\eta^2(\ell)}\right),$$

where λ_1, λ_2 are constants and $\rho, \vartheta \in C^1\left([\ell_0, \infty), (0, \infty)\right)$.

Also, we define the generalized Riccati substitutions

$$\omega\left(\ell\right) := \rho\left(\ell\right)\left(\frac{r\left(\ell\right)\left(y'''\right)^{\alpha}\left(\ell\right)}{y^{\alpha}\left(\ell\right)} + \frac{\lambda_1}{\eta^{\alpha}(\ell)}\right), \tag{6}$$

and

$$\zeta\left(\ell\right) := \vartheta\left(\ell\right)\left(\frac{y'\left(\ell\right)}{y\left(\ell\right)} + \frac{\lambda_2}{\eta(\ell)}\right). \tag{7}$$

After studying the asymptotic behavior of the positive solutions of Equation (1), there are only two cases:

$$\begin{array}{lll} \text{Case (1)}: & y^{(j)}\left(\ell\right) > 0 & \text{for} \quad j = 0, 1, 2, 3. \\ \text{Case (2)}: & x^{(j)}\left(\ell\right) > 0 & \text{for} \quad j = 0, 1, 3 \text{ and } y''\left(\ell\right) < 0. \end{array}$$

Moreover, from Equation (1) and the condition in Equation (2), we have that $\left(r\left(\ell\right)\left(y'''\left(\ell\right)\right)^{\alpha}\right)'$. In the following, we will first study each case separately.

Lemma 5. *Assume that y be an eventually positive solution of Equation (1) and $y^{(j)}(\ell) > 0$ for all $j = 1,2,3$. If we have the function $\omega \in C^1[\ell,\infty)$ defined as (6), where $\rho \in C^1([\ell_0,\infty),(0,\infty))$, then*

$$\omega'(\ell) \leq -\psi(\ell) + \phi(\ell)\,\omega(\ell) - \frac{\alpha\mu\ell^2}{2\left(r(\ell)\,\rho(\ell)\right)^{1/\alpha}}\omega^{\frac{\alpha+1}{\alpha}}(\ell), \tag{8}$$

for all $\ell > \ell_1$, where ℓ_1 large enough.

Proof. Let y is an eventually positive solution of (1) and $y^{(j)}(\ell) > 0$ for all $j = 1,2,3$. Thus, from Lemma 4, we get

$$y'(\ell) \geq \frac{\mu}{2}\ell^2 y'''(\ell), \tag{9}$$

for every $\mu \in (0,1)$ and for all large ℓ. From Equation (6), we see that $\omega(\ell) > 0$ for $\ell \geq \ell_1$, and

$$\begin{aligned}\omega'(\ell) &= \rho'(\ell)\left(\frac{r(\ell)\,(y''')^{\alpha}(\ell)}{y^{\alpha}(\ell)} + \frac{\lambda_1}{\eta^{\alpha}(\ell)}\right) + \rho(\ell)\frac{\left(r\,(y''')^{\alpha}\right)'(\ell)}{y^{\alpha}(\ell)}\\ &\quad -\alpha\rho(\ell)\frac{y^{\alpha-1}(\ell)\,y'(\ell)\,r(\ell)\,(y''')^{\alpha}(\ell)}{y^{2\alpha}(\ell)} + \frac{\alpha\lambda_1\rho(\ell)}{r^{\frac{1}{\alpha}}(\ell)\,\eta^{\alpha+1}(\ell)}.\end{aligned}$$

Using Equation (9) and Equation (6), we obtain

$$\begin{aligned}\omega'(\ell) &\leq \frac{\rho'_+(\ell)}{\rho(\ell)}\omega(\ell) + \rho(\ell)\frac{\left(r(\ell)\,(y'''(\ell))^{\alpha}\right)'}{y^{\alpha}(\ell)}\\ &\quad -\alpha\rho(\ell)\frac{\mu}{2}\ell^2\frac{r(\ell)\,(y'''(\ell))^{\alpha+1}}{y^{\alpha+1}(\ell)} + \frac{\alpha\lambda_1\rho(\ell)}{r^{\frac{1}{\alpha}}(\ell)\,\eta^{\alpha+1}(\ell)}\\ &\leq \frac{\rho'(\ell)}{\rho(\ell)}\omega(\ell) + \rho(\ell)\frac{\left(r(\ell)\,(y'''(\ell))^{\alpha}\right)'}{y^{\alpha}(\ell)}\\ &\quad -\alpha\rho(\ell)\frac{\mu}{2}\ell^2 r(\ell)\left(\frac{\omega(\ell)}{\rho(\ell)\,r(\ell)} - \frac{\lambda_1}{r(\ell)\,\eta^{\alpha}(\ell)}\right)^{\frac{\alpha+1}{\alpha}} + \frac{\alpha\lambda_1\rho(\ell)}{r^{\frac{1}{\alpha}}(\ell)\,\eta^{\alpha+1}(\ell)}.\end{aligned} \tag{10}$$

Using Lemma 1 with $P = \omega(\ell)/(\rho(\ell)\,r(\ell))$, $Q = \lambda_1/(r(\ell)\,\eta^{\alpha}(\ell))$ and $\beta = \alpha$, we get

$$\begin{aligned}\left(\frac{\omega(\ell)}{\rho(\ell)\,r(\ell)} - \frac{\lambda_1}{r(\ell)\,\eta^{\alpha}(\ell)}\right)^{\frac{\alpha+1}{\alpha}} &\geq \left(\frac{\omega(\ell)}{\rho(\ell)\,r(\ell)}\right)^{\frac{\alpha+1}{\alpha}}\\ &\quad -\frac{\lambda_1^{1/\alpha}}{\alpha r^{\frac{1}{\alpha}}(\ell)\,\eta(\ell)}\left((\alpha+1)\frac{\omega(\ell)}{\rho(\ell)\,r(\ell)} - \frac{\lambda_1}{r(\ell)\,\eta^{\alpha}(\ell)}\right).\end{aligned} \tag{11}$$

From Lemma 2, we have that $y(\ell) \geq \frac{\ell}{3}y'(\ell)$ and hence,

$$\frac{y(\tau(\ell))}{y(\ell)} \geq \frac{\tau^3(\ell)}{\ell^3}. \tag{12}$$

From Equations (1), (10), and (11), we obtain

$$\begin{aligned}\omega'(\ell) \leq{}& \frac{\rho'_+(\ell)}{\rho(\ell)}\omega(\ell) - k\rho(\ell)\,q(\ell)\left(\frac{\tau^3(\ell)}{\ell^3}\right)^{\alpha} - \alpha\rho(\ell)\frac{\mu}{2}\ell^2 r(\ell)\left(\frac{\omega(\ell)}{\rho(\ell)\,r(\ell)}\right)^{\frac{\alpha+1}{\alpha}}\\ &-\alpha\rho(\ell)\frac{\mu}{2}\ell^2 r(\ell)\left(\frac{-\lambda_1^{1/\alpha}}{\alpha r^{\frac{1}{\alpha}}(\ell)\,\eta(\ell)}\left((\alpha+1)\frac{\omega(\ell)}{\rho(\ell)\,r(\ell)} - \frac{\lambda_1}{r(\ell)\,\eta^{\alpha}(\ell)}\right)\right) + \frac{\alpha\lambda_1\rho(\ell)}{r^{\frac{1}{\alpha}}(\ell)\,\eta^{\alpha+1}(\ell)}.\end{aligned}$$

This implies that

$$\begin{aligned} \omega'(\ell) &\leq \left(\frac{\rho'_{+}(\ell)}{\rho(\ell)}+\frac{(\alpha+1)\lambda_1^{1/\alpha}\mu\ell^2}{2r^{\frac{1}{\alpha}}(\ell)\,\eta(\ell)}\right)\omega(\ell)-\frac{\alpha\mu\ell^2}{2r^{1/\alpha}(\ell)\,\rho^{1/\alpha}(\ell)}\omega^{\frac{\alpha+1}{\alpha}}(\ell) \\ &\quad -\rho(\ell)\left(kq(\ell)\left(\frac{\tau^3(\ell)}{\ell^3}\right)^{\alpha}+\frac{\mu\lambda_1^{(1+\alpha)/\alpha}\ell^2-2\lambda_1\alpha}{2r^{\frac{1}{\alpha}}(\ell)\,\eta^{\alpha+1}(\ell)}\right). \end{aligned}$$

Thus,

$$\omega'(\ell)\leq-\psi(\ell)+\phi(\ell)\,\omega(\ell)-\frac{\alpha\mu\ell^2}{2\left(r(\ell)\,\rho(\ell)\right)^{1/\alpha}}\omega^{\frac{\alpha+1}{\alpha}}(\ell).$$

The proof is complete. □

Lemma 6. *Assume that y be an eventually positive solution of Equation (1), $y^{(j)}(\ell)>0$ for $j=1,3$ and $y''(\ell)<0$. If we have the function $\zeta\in C^1[\ell,\infty)$ defined as Equation (7), where $\vartheta\in C^1([\ell_0,\infty),(0,\infty))$, then*

$$\zeta'(\ell)\leq-\psi^{*}(\ell)+\phi^{*}(\ell)\,\zeta(\ell)-\frac{1}{\vartheta(\ell)}\zeta^2(\ell), \tag{13}$$

for all $\ell>\ell_1$, where ℓ_1 large enough.

Proof. Let y is an eventually positive solution of Equation (1), $y^{(j)}>0$ for $j=1,3$ and $y''(\ell)<0$. From Lemma 2, we get that $y(\ell)\geq\ell y'(\ell)$. By integrating this inequality from $\tau(\ell)$ to ℓ, we get

$$y(\tau(\ell))\geq\frac{\tau(\ell)}{\ell}y(\ell).$$

Hence, from Equation (2), we have

$$f(y(\tau(\ell)))\geq k\frac{\tau^{\alpha}(\ell)}{\ell^{\alpha}}y^{\alpha}(\ell). \tag{14}$$

Integrating Equation (1) from ℓ to u and using $y'(\ell)>0$, we obtain

$$\begin{aligned} r(u)\left(y'''(u)\right)^{\alpha}-r(\ell)\left(y'''(\ell)\right)^{\alpha} &= -\int_{\ell}^{u}q(s)\,f(y(\tau(s)))\,ds \\ &\leq -ky^{\alpha}(\ell)\int_{\ell}^{u}q(s)\frac{\tau^{\alpha}(s)}{s^{\alpha}}\mathrm{d}s. \end{aligned}$$

Letting $u\to\infty$, we see that

$$r(\ell)\left(y'''(\ell)\right)^{\alpha}\geq ky^{\alpha}(\ell)\int_{\ell}^{\infty}q(s)\frac{\tau^{\alpha}(s)}{s^{\alpha}}\mathrm{d}s$$

and so

$$y'''(\ell)\geq y(\ell)\left(\frac{k}{r(\ell)}\int_{\ell}^{\infty}q(s)\frac{\tau^{\alpha}(s)}{s^{\alpha}}\mathrm{d}s\right)^{1/\alpha}.$$

Integrating again from ℓ to ∞, we get

$$y''(\ell)\leq-y(\ell)\int_{\ell}^{\infty}\left(\frac{k}{r(v)}\int_{v}^{\infty}q(s)\frac{\tau^{\alpha}(s)}{s^{\alpha}}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}v. \tag{15}$$

From the definition of $\zeta(\ell)$, we see that $\zeta(\ell) > 0$ for $\ell \geq \ell_1$. By differentiating, we find

$$\zeta'(\ell) = \frac{\vartheta'(\ell)}{\vartheta(\ell)}\zeta(\ell) + \vartheta(\ell)\frac{y''(\ell)}{y(\ell)} - \vartheta(\ell)\left(\frac{\zeta(\ell)}{\vartheta(\ell)} - \frac{\lambda_2}{\eta(\ell)}\right)^2 + \frac{\vartheta(\ell)\lambda_2}{r^{1/\alpha}(\ell)\eta^2(\ell)}. \tag{16}$$

Using Lemma 1 with $P = \zeta(\ell)/\vartheta(\ell)$, $Q = \lambda_2/\eta(\ell)$ and $\beta = 1$, we get

$$\left(\frac{\zeta(\ell)}{\vartheta(\ell)} - \frac{\lambda_2}{\eta(\ell)}\right)^2 \geq \left(\frac{\zeta(\ell)}{\vartheta(\ell)}\right)^2 - \frac{\lambda_2}{\eta(\ell)}\left(\frac{2\zeta(\ell)}{\vartheta(\ell)} - \frac{\lambda_2}{\eta(\ell)}\right). \tag{17}$$

From Equations (1), (16) and (17), we obtain

$$\begin{aligned}\zeta'(\ell) \leq\ & \frac{\vartheta'(\ell)}{\vartheta(\ell)}\zeta(\ell) - \vartheta(\ell)\int_\ell^\infty\left(\frac{k}{r(v)}\int_v^\infty q(s)\frac{\tau^\alpha(s)}{s^\alpha}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}v \\ & -\vartheta(\ell)\left(\left(\frac{\zeta(\ell)}{\vartheta(\ell)}\right)^2 - \frac{\lambda_2}{\eta(\ell)}\left(\frac{2\zeta(\ell)}{\vartheta(\ell)} - \frac{\lambda_2}{\eta(\ell)}\right)\right) + \frac{\lambda_2\vartheta(\ell)}{r^{\frac{1}{\alpha}}(\ell)\eta^2(\ell)}.\end{aligned}$$

This implies that

$$\begin{aligned}\zeta'(\ell) \leq\ & \left(\frac{\vartheta'_+(\ell)}{\vartheta(\ell)} + \frac{2\lambda_2}{\eta(\ell)}\right)\zeta(\ell) - \frac{1}{\vartheta(\ell)}\zeta^2(\ell) \\ & -\vartheta(\ell)\left(\int_\ell^\infty\left(\frac{k}{r(v)}\int_v^\infty q(s)\frac{\tau^\alpha(s)}{s^\alpha}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}v + \frac{\lambda_2^2 - \lambda_2 r^{\frac{-1}{\alpha}}(\ell)}{\eta^2(\ell)}\right).\end{aligned}$$

Thus,

$$\zeta'(\ell) \leq -\psi^*(\ell) + \phi^*(\ell)\zeta(\ell) - \frac{1}{\vartheta(\ell)}\zeta^2(\ell).$$

The proof is complete. □

Lemma 7. *Assume that y will eventually be a positive solution of Equation (1). If there exists a positive function $\rho \in C([\ell_0, \infty))$ such that*

$$\int_{\ell_0}^\infty\left(\psi(s) - \left(\frac{2}{\mu s^2}\right)^\alpha\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)\mathrm{d}s = \infty, \tag{18}$$

for some $\mu \in (0,1)$, then y does not fulfill Case (1).

Proof. Assume that y is eventually positive solution of Equation (1). From Lemma 5, we get that Equation (8) holds. Using Lemma 1 with

$$U = \phi(\ell),\ V = \alpha\mu\ell^2/\left(2(r(\ell)\rho(\ell))^{1/\alpha}\right) \text{ and } x = \omega,$$

we get

$$\omega'(\ell) \leq -\psi(\ell) + \left(\frac{2}{\mu\ell^2}\right)^\alpha\frac{r(\ell)\rho(\ell)(\phi(\ell))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}. \tag{19}$$

Integrating from ℓ_1 to ℓ, we get

$$\int_{\ell_1}^\ell\left(\psi(s) - \left(\frac{2}{\mu s^2}\right)^\alpha\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)\mathrm{d}s \leq \omega(\ell_1),$$

for every $\mu \in (0,\ 1)$, which contradicts Equation (18). The proof is complete. □

Lemma 8. *Assume that y will eventually be a positive solution of Equation (1), $y^{(j)}(\ell) > 0$ for $j = 1,3$ and $y''(\ell) < 0$. If there exists a positive function $\vartheta \in C([\ell_0,\infty))$ such that*

$$\int_{\ell_0}^{\infty}\left(\psi^*(s) - \frac{1}{4}\vartheta(s)(\phi^*(s))^2\right)\mathrm{d}s = \infty, \tag{20}$$

then y does not fulfill Case (2).

Proof. Assume that y is eventually a positive solution of Equation (1). From Lemma 6, we get that Equation (13) holds. Using Lemma 1 with

$$U = \phi^*(\ell), \; V = 1/\vartheta(\ell), \alpha = 1 \text{ and } x = \zeta,$$

we get

$$\omega'(\ell) \le -\psi^*(\ell) + \frac{1}{4}\vartheta(\ell)(\phi^*(\ell))^2. \tag{21}$$

Integrating from ℓ_1 to ℓ, we get

$$\int_{\ell_1}^{\ell}\left(\psi^*(s) - \frac{1}{4}\vartheta(s)(\phi^*(s))^2\right)\mathrm{d}s \le \omega(\ell_1),$$

which contradicts Equation (20). The proof is complete. □

Theorem 1. *Assume that there exist positive functions $\rho, \vartheta \in C([\ell_0,\infty))$ such that Equations (18) and (20) hold, for some $\mu \in (0,1)$. Then every solution of Equation (1) is oscillatory.*

In the next theorem, we establish new oscillation results for Equation (1) by using the integral averaging technique.

Definition 1. *Let*

$$D = \{(\ell,s) \in \mathbb{R}^2 : \ell \ge s \ge \ell_0\} \text{ and } D_0 = \{(\ell,s) \in \mathbb{R}^2 : \ell > s \ge \ell_0\}.$$

A kernel function $H_i \in C(D,\mathbb{R})$ is said to belong to the function class $\Im$, written by $H \in \Im$, if, for $i = 1,2$,

(i) $H_i(\ell,s) = 0$ *for* $\ell \ge \ell_0$, $H_i(\ell,s) > 0$, $(\ell,s) \in D_0$;

(ii) $H_i(\ell,s)$ *has a continuous and nonpositive partial derivative* $\partial H_i/\partial s$ *on* D_0 *and there exist functions* $\rho, \vartheta \in C^1([\ell_0,\infty),(0,\infty))$ *and* $h_i \in C(D_0,\mathbb{R})$ *such that*

$$\frac{\partial}{\partial s}H(\ell,s) + \phi(s)H(\ell,s) = h_1(\ell,s)H_1^{\alpha/(\alpha+1)}(\ell,s) \tag{22}$$

and

$$\frac{\partial}{\partial s}H_2(\ell,s) + \phi^*(s)H_2(\ell,s) = h_2(\ell,s)\sqrt{H_2(\ell,s)}. \tag{23}$$

Lemma 9. *Assume that y be an eventually positive solution of Equation (1) and $y^{(j)}(\ell) > 0$ for $j = 1,2,3$. If there exist functions $\rho \in C([\ell_0,\infty),(0,\infty))$ and $H_1 \in \Im$ such that Equation (22) holds and*

$$\limsup_{\ell\to\infty}\frac{1}{H_1(\ell,\ell_1)}\int_{\ell_1}^{\ell}\left(H_1(\ell,s)\psi(s) - \frac{2^{\alpha}r(s)\rho(s)h_1^{\alpha+1}(\ell,s)H_1^{\alpha}(\ell,s)}{(\alpha+1)^{\alpha+1}(\mu s^2)^{\alpha}}\right)\mathrm{d}s = \infty, \tag{24}$$

then y does not fulfill Case (1).

Proof. Assume that y is eventually positive solution of Equation (1). From Lemma 5, we get that Equation (8) holds. Multiplying Equation (8) by $H(\ell,s)$ and integrating the resulting inequality from ℓ_1 to ℓ; we find that

$$\begin{aligned}\int_{\ell_1}^{\ell} H(\ell,s)\,\psi(s)\,\mathrm{d}s &\leq \omega(\ell_1)\,H(\ell,\ell_1)+\int_{\ell_1}^{\ell}\left(\frac{\partial}{\partial s}H(\ell,s)+\phi(s)\,H(\ell,s)\right)\omega(s)\,\mathrm{d}s\\ &\quad-\int_{\ell_1}^{\ell}\frac{\alpha\mu s^2}{2\,(r(s)\,\rho(s))^{1/\alpha}}H(\ell,s)\,\omega^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s.\end{aligned}$$

From Equation (22), we get

$$\begin{aligned}\int_{\ell_1}^{\ell} H(\ell,s)\,\psi(s)\,\mathrm{d}s &\leq \omega(\ell_1)\,H(\ell,\ell_1)+\int_{\ell_1}^{\ell} h_1(\ell,s)\,H_1^{\alpha/(\alpha+1)}(\ell,s)\,\omega(s)\,\mathrm{d}s\\ &\quad-\int_{\ell_1}^{\ell}\frac{\alpha\mu s^2}{2\,(r(s)\,\rho(s))^{1/\alpha}}H(\ell,s)\,\omega^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s.\end{aligned}\tag{25}$$

Using Lemma 1 with $C=\frac{\alpha\mu s^2}{2(r(s)\rho(s))^{1/\alpha}}H(\ell,s)$, $D=h_1(\ell,s)\,H_1^{\alpha/(\alpha+1)}(\ell,s)$ and $x=\omega(s)$, we get

$$h_1(\ell,s)\,H_1^{\alpha/(\alpha+1)}(\ell,s)\,\omega(s)-\frac{\alpha\mu s^2}{2\,(r(s)\,\rho(s))^{1/\alpha}}H(\ell,s)\,\omega^{\frac{\alpha+1}{\alpha}}(s)\leq\frac{2^{\alpha}r(s)\,\rho(s)\,h_1^{\alpha+1}(\ell,s)\,H_1^{\alpha}(\ell,s)}{(\alpha+1)^{\alpha+1}\,(\mu s^2)^{\alpha}},$$

which, with Equation (25) gives

$$\frac{1}{H(\ell,\ell_1)}\int_{\ell_1}^{\ell}\left(H(\ell,s)\,\psi(s)-\frac{2^{\alpha}r(s)\,\rho(s)\,h_1^{\alpha+1}(\ell,s)\,H_1^{\alpha}(\ell,s)}{(\alpha+1)^{\alpha+1}\,(\mu s^2)^{\alpha}}\right)\mathrm{d}s\leq\omega(\ell_1),$$

which contradicts Equation (24). The proof is complete. □

Lemma 10. *Assume that y be an eventually positive solution of Equation (1), $y^{(j)}(\ell)>0$ for $j=1,3$ and $y''(\ell)<0$. If there exist functions $\vartheta\in C([\ell_0,\infty),(0,\infty))$ and $H_2\in\Im$ such that Equation (23) holds and*

$$\limsup_{\ell\to\infty}\frac{1}{H_2(\ell,\ell_1)}\int_{\ell_1}^{\ell}\left(H_2(\ell,s)\,\psi^*(s)-\frac{\vartheta(s)\,h_2^2(\ell,s)}{4}\right)\mathrm{d}s=\infty,\tag{26}$$

then y does not fulfill Case (2).

Proof. Assume that y is eventually positive solution of Equation (1). From Lemma 6, we get that Equation (13) holds. Multiplying Equation (13) by $H_2(\ell,s)$ and integrating the resulting from ℓ_1 to ℓ, we obtain

$$\begin{aligned}\int_{\ell_1}^{\ell} H_2(\ell,s)\,\psi^*(s)\,\mathrm{d}s &\leq \zeta(\ell_1)\,H_2(\ell,\ell_1)\\ &\quad+\int_{\ell_1}^{\ell}\left(\frac{\partial}{\partial s}H_2(\ell,s)+\phi^*(s)\,H_2(\ell,s)\right)\zeta(s)\,\mathrm{d}s\\ &\quad-\int_{\ell_1}^{\ell}\frac{1}{\vartheta(s)}H_2(\ell,s)\,\zeta^2(s)\,\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\ell_1}^{\ell} H_2(\ell,s)\,\psi^*(s)\,\mathrm{d}s &\le \zeta(\ell_1)\,H_2(\ell,\ell_1)+\int_{\ell_1}^{\ell} h_2(\ell,s)\sqrt{H_2(\ell,s)}\zeta(s)\,\mathrm{d}s \\ &\quad -\int_{\ell_1}^{\ell}\frac{1}{\vartheta(s)}H_2(\ell,s)\,\zeta^2(s)\,\mathrm{d}s \\ &\le \zeta(\ell_1)\,H_2(\ell,\ell_1)+\int_{\ell_1}^{\ell}\frac{\vartheta(s)\,h_2^2(\ell,s)}{4}\mathrm{d}s,\end{aligned}$$

and so

$$\frac{1}{H_2(\ell,\ell_1)}\int_{\ell_1}^{\ell}\left(H_2(\ell,s)\,\psi^*(s)-\frac{\vartheta(s)\,h_2^2(\ell,s)}{4}\right)\mathrm{d}s\le\zeta(\ell_1),$$

which contradicts Equation (26). The proof is complete. □

Theorem 2. *Assume that there exist positive functions $\rho,\vartheta\in C([\ell_0,\infty))$ and functions $H_1,H_2\in\Im$ such that Equation (24) and Equation (26) hold, for some $\mu\in(0,1)$. Then every solution of Equation (1) is oscillatory.*

In the next theorem, we establish new oscillation results for (1) by using the theory of comparison with the second order differential equation:

Theorem 3. *Let (3) hold. Assume that the equation*

$$\left[\frac{r(\ell)}{\ell^{2\alpha}}\left(y'(\ell)\right)^{\alpha}\right]'+\psi(\ell)\,y^{\alpha}(\ell)=0 \tag{27}$$

and

$$y''(\ell)+y(\ell)\int_{\ell}^{\infty}\left[\frac{k}{r(v)}\int_{v}^{\infty}q(s)\,\frac{\tau^{\alpha}(s)}{s^{\alpha}}ds\right]^{1/\alpha}dv=0, \tag{28}$$

are oscillatory, then every solution of (1) is oscillatory.

Proof. Suppose to the contrary that Equation (1) has a eventually positive solution x and by virtue of Lemma 3. If we set $\rho(\ell)=1,\ \lambda_1=0$ in Equation (8), then we get

$$\omega'(\ell)+\frac{\alpha\mu\ell^2}{2r(\ell)^{1/\alpha}}\omega^{\frac{\alpha+1}{\alpha}}+\psi(\ell)(\ell)\le 0.$$

Thus, we can see that Equation (27) is nonoscillatory. Which is a contradiction. If we now set $\vartheta(\ell)=1,\ \lambda_2=0$ in Equation (13), then we obtain

$$\zeta'(\ell)+\psi^*(\ell)+\zeta^2(\ell)\le 0.$$

Hence, Equation (28) is nonoscillatory, which is a contradiction.

Theorem 3 is proved. □

It is well known (see [26]) that if

$$\int_{\ell_0}^{\infty}\frac{1}{r(\ell)}d\ell=\infty,\ \text{and}\ \liminf_{\ell\to\infty}\left(\int_{\ell_0}^{\ell}\frac{1}{r(s)}ds\right)\int_{\ell}^{\infty}q(s)\,ds>\frac{1}{4},$$

then Equation (5) with $\alpha=1$ is oscillatory.

From the previous results that we have concluded and Theorem 3, we can easily obtain Hille and Nehari type oscillation criteria for Equation (1), in the next theorem:

Theorem 4. *Let Equation (3) hold. Assume that*

$$\int_{\ell_0}^{\infty}\frac{\ell^2}{r(\ell)}d\ell=\infty,\ \text{and}\ \liminf_{\ell\to\infty}\left(\int_{\ell_0}^{\ell}\frac{s^2}{r(s)}ds\right)\int_{\ell}^{\infty}\psi(s)\,ds>\frac{1}{2\lambda_1},$$

for some constant $\lambda_1\in(0,1)$ *and*

$$\liminf_{\ell\to\infty}\int_{\ell}^{\infty}\left(\int_{\ell}^{\infty}\left[\frac{k}{r(v)}\int_{v}^{\infty}q(s)\frac{\tau^{\alpha}(s)}{s^{\alpha}}ds\right]dv\right)ds>\frac{1}{4}.$$

Then every solution of Equation (1) is oscillatory.

3. Discussion and Application

Theorems 1 and 2 can be used in a wide range of applications for oscillation of Equation (1) depending on the appropriate choice of functions ρ and ϑ. To applying the conditions of theorems, we search on for suitable restitution for functions ρ and ϑ such that $\rho,\vartheta\in C^1([\ell_0,\infty),(0,\infty))$.

In the following, by using our results, we study the oscillation behavior of some differential equations with a fourth-order.

Example 1. *Consider a differential equation*

$$y^{(4)}(\ell)+\frac{q_0}{\ell^4}y(\ell)=0,\ \ell\geq 1, \tag{29}$$

where $q_0>0$. *Note that* $\alpha=1$, $r(\ell)=1$, $q(\ell)=q_0/\ell^4$ *and* $\tau(\ell)=\ell$. *Hence, we have*

$$\eta(\ell_0)=\infty,\ \psi(\ell)=\frac{q_0}{\ell},\ \phi(\ell)=\frac{3}{\ell},\ \phi^*(\ell)=\frac{1}{\ell}\ \text{and}\ \psi^*(\ell)=\frac{q_0}{6\ell}.$$

If we set $\rho(\ell)=\ell^3$, $\vartheta(\ell)=\ell$ *and* $k=1$, *then condition Equation (18) becomes*

$$\begin{aligned}\int_{\ell_0}^{\infty}\left(\psi(s)-\left(\frac{2}{\mu s^2}\right)^{\alpha}\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)ds &= \int_{\ell_0}^{\infty}\left(\frac{q_0}{s}-\frac{9}{2\mu s}\right)ds\\ &= \left(q_0-\frac{9}{2\mu}\right)\int_{\ell_0}^{\infty}\frac{1}{s}ds.\end{aligned}$$

Therefore, from Lemma 7, if $q_0>9/2\mu$, *then Equation (29) has no positive solution y satisfies* $y''(t)>0$. *Moreover, condition Equation (20) becomes*

$$\begin{aligned}\int_{\ell_0}^{\infty}\left(\psi^*(s)-\frac{1}{4}\vartheta(s)(\phi^*(s))^2\right)\mathrm{d}s &= \int_{\ell_0}^{\infty}\left(\frac{q_0}{6s}-\frac{1}{4s}\right)\mathrm{d}s\\ &= \infty,\ \text{if}\ \ q_0>\frac{3}{2},\end{aligned}$$

From Lemma 8, if $q_0>3/2$, *then Equation (29) has no positive solution y satisfies* $y''(t)<0$.

Remark 1. *In Example 1, by using Theorem 1, the new criterion for oscillation of Equation (29) is*

$$q_0>\max\left\{\frac{9}{2\mu},\frac{3}{2}\right\}.$$

Example 2. *Consider a differential equation*

$$\left(\ell^3\left(y'''(\ell)\right)^3\right)'+\frac{v}{\ell^7}y^3(\varepsilon\ell)=0,\ \ell\geq 1, \tag{30}$$

where $\upsilon > 0$ *and* $0 < \varepsilon < 1$ *is a constant. Note that* $\alpha = 3$, $r(\ell) = \ell^3$, $q(\ell) = \upsilon/\ell^7$ *and* $\tau(\ell) = \varepsilon\ell$. *Hence,*

$$\eta(\ell_0) = \infty,\ \psi(\ell) = \frac{\upsilon\varepsilon^9}{\ell},\ \phi(\ell) = \frac{6}{\ell},\ \phi^*(\ell) = \frac{1}{\ell} \text{ and } \psi^*(\ell) = \left(\frac{\upsilon\varepsilon^3}{48}\right)^{1/3}\frac{1}{\ell}.$$

If we set $\rho(\ell) = \ell^6$, $\vartheta(\ell) = \ell$ *and* $k = 1$, *then*

$$\begin{aligned}\int_{\ell_0}^{\infty}\left(\psi(s) - \left(\frac{2}{\mu s^2}\right)^{\alpha}\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)ds &= \int_{\ell_0}^{\infty}\left(\frac{\upsilon\varepsilon^9}{s} - \frac{81}{2\mu^3 s}\right)ds \\ &= \left(\upsilon\varepsilon^9 - \frac{81}{2\mu^3}\right)\int_{\ell_0}^{\infty}\frac{1}{s}ds\end{aligned}$$

and

$$\int_{\ell_0}^{\infty}\left(\psi^*(s) - \frac{1}{4}\vartheta(s)(\phi^*(s))^2\right)\mathrm{d}s = \int_{\ell_0}^{\infty}\left(\left(\frac{\upsilon\varepsilon^3}{48}\right)^{1/3}\frac{1}{s} - \frac{1}{4s}\right)\mathrm{d}s$$

for some constant $\mu, \varepsilon \in (0, 1)$. *Hence, by Theorem 1, every solution of Equation (30) is oscillatory if*

$$\upsilon > \max\left\{\frac{3}{4\varepsilon^3}, \frac{81}{2\varepsilon^9\mu^3}\right\}.$$

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