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# A Symmetry of the Einstein–Friedmann Equations for Spatially Flat, Perfect Fluid, Universes

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**Abstract:** We report a symmetry property of the Einstein–Friedmann equations for spatially flat Friedmann–Lemaître–Robertson–Walker universes filled with a perfect fluid with any constant equation of state. The symmetry transformations form a one-parameter Abelian group.

**Keywords:** Einstein–Friedmann equations; cosmology; classical general relativity

## 1. Introduction

In the recent literature, there have been many studies of the symmetries of the Einstein–Friedmann equations of spatially homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology. These studies are ultimately inspired by (although not always directly related to) string dualities [1–21] or by methods introduced in supersymmetric quantum mechanics [22–24]. Here, we discuss a simpler symmetry arising in the context of pure Einstein gravity. Adopting the notation of [25,26], we use metric signature  $-+++$  and units in which the speed of light and Newton's constant are unity.

The FLRW line element of spatially homogeneous and isotropic cosmology in comoving coordinates  $(t, x, y, z)$  is

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2), \quad (1)$$

where the dynamics are contained in the evolution of the cosmic scale factor  $a(t)$ . The Einstein–Friedmann equations for a spatially flat universe filled with a perfect fluid with energy density  $\rho(t)$  and isotropic pressure  $P(t)$  are

$$H^2 = \frac{8\pi}{3} \rho, \quad (2)$$

$$\dot{\rho} + 3H(P + \rho) = 0, \quad (3)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} (\rho + 3P). \quad (4)$$

These equations exhibit a special symmetry that maps a barotropic perfect fluid with equation of state  $P = P(\rho)$  into itself, with a rescaled energy density and pressure but with the same equation of state.

## 2. The Symmetry Transformation

Assume that the energy content of the universe is a single perfect fluid with constant equation of state  $P = w\rho$  with  $w = \text{constant}$ ; then, the Einstein–Friedmann Equations (2)–(4) assume the form

$$\dot{\rho} + 3(w+1)\frac{\dot{a}}{a}\rho = 0, \quad (5)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(3w+1)\rho. \quad (6)$$

The solution of these equations is well known and easy to derive (e.g., [25]):

$$a(t) = a_0 t^{\frac{2}{3(w+1)}}, \quad (7)$$

$$\rho(a) = \frac{\rho_0}{a^{3(w+1)}}. \quad (8)$$

The change of variables

$$a \longrightarrow \tilde{a} = a^s, \quad (9)$$

$$dt \longrightarrow d\tilde{t} = s a^{\frac{3(w+1)(s-1)}{2}} dt = s \tilde{a}^{\frac{3(w+1)(s-1)}{2s}} d\tilde{t}, \quad (10)$$

$$\rho \longrightarrow \tilde{\rho} = a^{-3(w+1)(s-1)}\rho, \quad (11)$$

with inverse

$$a = \tilde{a}^{1/s}, \quad (12)$$

$$dt = \frac{a^{-\frac{3(w+1)(s-1)}{s}}}{s} d\tilde{t}, \quad (13)$$

$$\rho = \tilde{\rho} \tilde{a}^{\frac{3(w+1)(s-1)}{s}} \quad (14)$$

leaves the Einstein–Friedmann Equations (2), (5) and (6) unchanged. In particular, the matter source, the barotropic perfect fluid, maintains the equation of state  $P = w\rho$  with the same equation of state parameter  $w$ . This is in contrast with other symmetries of the same equations which change an “ordinary” fluid satisfying the weak energy condition into a phantom fluid with different equation of state parameter and are ultimately inspired by string theory dualities [1–21], and with other solutions of the Einstein–Friedmann equations obtained using methods of supersymmetric quantum mechanics [22–24].

To check invariance of the equations, begin from the Friedmann Equation (2), which becomes, in terms of tilded quantities:

$$\frac{1}{\tilde{a}^{2/s}} \left[ \frac{d(\tilde{a}^{1/s})}{d\tilde{t}} \right]^2 \left( \frac{d\tilde{t}}{dt} \right)^2 = \frac{8\pi}{3} \tilde{\rho} \tilde{a}^{\frac{3(w+1)(s-1)}{s}} \quad (15)$$

or

$$\frac{1}{s^2} \frac{\tilde{a}^{\frac{2}{s}-2}}{\tilde{a}^{2/s}} \left( \frac{d\tilde{a}}{d\tilde{t}} \right)^2 s^2 \tilde{a}^{\frac{3(w+1)(s-1)}{s}} = \frac{8\pi}{3} \tilde{\rho} \tilde{a}^{\frac{3(w+1)(s-1)}{s}}, \quad (16)$$

which simplifies to

$$\left(\frac{1}{\tilde{a}} \frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{8\pi}{3} \tilde{\rho}. \quad (17)$$

Let us consider now the covariant conservation Equation (5), which now reads

$$\frac{d}{d\tilde{t}} \left( \tilde{\rho} \tilde{a}^{\frac{3(w+1)(s-1)}{s}} \right) \frac{d\tilde{t}}{dt} + 3(w+1)\tilde{\rho} \frac{\tilde{a}^{\frac{3(w+1)(s-1)}{s}}}{\tilde{a}^{1/s}} \frac{d\tilde{t}}{dt} \frac{d}{d\tilde{t}} \left( \tilde{a}^{1/s} \right) = 0. \quad (18)$$

Expanding, one has

$$\tilde{a}^{\frac{3(w+1)(s-1)}{s}} \frac{d\tilde{\rho}}{d\tilde{t}} + \frac{3(w+1)(s-1)}{s} \tilde{a}^{\frac{3(w+1)(s-1)}{s}-1} \frac{d\tilde{a}}{d\tilde{t}} \tilde{\rho} + \frac{3(w+1)}{s} \tilde{a}^{\frac{3(w+1)(s-1)-1}{s}} \tilde{a}^{\frac{1}{s}-1} \tilde{\rho} \frac{d\tilde{a}}{d\tilde{t}} = 0. \quad (19)$$

Grouping similar terms then yields

$$\tilde{a}^{\frac{3(w+1)(s-1)}{s}} \left[ \frac{d\tilde{\rho}}{d\tilde{t}} + 3(w+1) \frac{1}{\tilde{a}} \frac{d\tilde{a}}{d\tilde{t}} \tilde{\rho} \right] = 0. \quad (20)$$

Since  $\tilde{a} \neq 0$ , the conservation equation in the tilded world follows.

Coming to the acceleration Equation (6), one first notices that

$$\frac{da}{dt} = \frac{d}{d\tilde{t}} \left( \tilde{a}^{1/s} \right), \quad \frac{d\tilde{t}}{dt} = \tilde{a}^{\frac{(3w+1)(s-1)}{2s}} \frac{d\tilde{a}}{d\tilde{t}}, \quad (21)$$

$$\frac{d^2a}{dt^2} = \frac{d\tilde{t}}{dt} \frac{d}{d\tilde{t}} \left( \frac{da}{dt} \right) = \tilde{a}^{\frac{(3w+1)(s-1)}{s}} \left[ \frac{(3w+1)(s-1)}{2} \tilde{a} \left( \frac{1}{\tilde{a}} \frac{d\tilde{a}}{d\tilde{t}} \right)^2 + s \frac{d^2\tilde{a}}{d\tilde{t}^2} \right]. \quad (22)$$

Using the Friedmann equation for tilded quantities, this becomes

$$\frac{d^2a}{dt^2} = \tilde{a}^{\frac{(3w+1)(s-1)}{s}+1} \left[ \frac{(3w+1)(s-1)}{2} \frac{8\pi}{3} \tilde{\rho} + \frac{s}{\tilde{a}} \frac{d^2\tilde{a}}{d\tilde{t}^2} \right]. \quad (23)$$

The acceleration Equation (6) now reads

$$\frac{\tilde{a}^{\frac{(3w+1)(s-1)}{s}+1}}{\tilde{a}^{1/s}} \left[ (3w+1)(s-1) \frac{4\pi}{3} \tilde{\rho} + \frac{s}{\tilde{a}} \frac{d^2\tilde{a}}{d\tilde{t}^2} \right] = -\frac{4\pi}{3} (3w+1) \tilde{\rho} \tilde{a}^{\frac{(3w+1)(s-1)}{s}}. \quad (24)$$

By simplifying a factor  $\tilde{a}^{\frac{(3w+1)(s-1)}{s}}$  on both sides and collecting similar terms, one obtains

$$\frac{s}{\tilde{a}} \frac{d^2\tilde{a}}{d\tilde{t}^2} = -\frac{4\pi}{3} (3w+1) \tilde{\rho} [1 + (s-1)] \quad (25)$$

and the acceleration equation for tilded quantities follows.

### 3. A Group of Symmetry Transformations

The transformations (9)–(11) form a commutative group, as shown below. First, we show that the composition of two such transformations is a change of variables of the same form. Let

$$\hat{L}_s : (a, dt, \rho) \longrightarrow (\tilde{a}, d\tilde{t}, \tilde{\rho}) = \left( a^s, s a^{\frac{3(w+1)(s-1)}{2}} dt, a^{-3(w+1)(s-1)} \rho \right), \quad (26)$$

$$\hat{L}_p : (a, dt, \rho) \longrightarrow (\tilde{a}, d\tilde{t}, \tilde{\rho}) = \left( a^p, p a^{\frac{3(w+1)(p-1)}{2}} dt, a^{-3(w+1)(p-1)} \rho \right); \quad (27)$$

then, the composition of the two transformation gives

$$a \rightarrow \tilde{a} \rightarrow \tilde{\tilde{a}} = \tilde{a}^p = a^{ps} \equiv a^r, \quad (28)$$

$$dt \rightarrow d\tilde{t} \rightarrow d\tilde{\tilde{t}} = p\tilde{a}^{\frac{3(w+1)(p-1)}{2}} d\tilde{t} = pa^{\frac{3(w+1)(p-1)s}{2}} sa^{\frac{3(w+1)(s-1)}{2}} dt = psa^{\frac{3(w+1)(ps-1)}{2}} dt \equiv ra^{\frac{3(w+1)(r-1)}{2}} dt, \quad (29)$$

$$\rho \rightarrow \tilde{\rho} \rightarrow \tilde{\tilde{\rho}} = \tilde{a}^{-3(w+1)(p-1)} \tilde{\rho} = a^{-3(w+1)(p-1)s} a^{-3(w+1)(s-1)} \rho = a^{-3(w+1)(r-1)} \rho, \quad (30)$$

where  $r = ps$ . Therefore, the composition of two transformations gives the same kind of transformation and the order of these two operations does not matter:

$$\hat{L}_s \circ \hat{L}_p = \hat{L}_p \circ \hat{L}_s \equiv \hat{L}_{sp} \equiv \hat{L}_r. \quad (31)$$

There is a neutral element for the operation of composition of maps: the transformation  $\hat{L}_1$  with  $s = 1$  is the identity since

$$\hat{L}_1 : (a, dt, \rho) \longrightarrow (\tilde{a}, d\tilde{t}, \tilde{\rho}) = (a, dt, \rho). \quad (32)$$

Finally, each transformation  $\hat{L}_s$  with  $s \neq 0$  has a (left and right) inverse  $\hat{L}_{1/s}$  since

$$\hat{L}_s \circ \hat{L}_{1/s} = \hat{L}_{1/s} \circ \hat{L}_s = \hat{L}_{(s \cdot 1/s)} = \hat{I}_d. \quad (33)$$

Therefore, the transformations  $\hat{L}_s$  given by Equations (9)–(11) form a one-parameter Abelian group parametrized by the real number  $s \neq 0$ .

#### 4. Symmetry of the Solutions

Since the form of the Einstein–Friedmann Equations (2), (5) and (6) does not change under the symmetry operation (9)–(11), the solution corresponding to the same perfect fluid will still be given by Equations (7) and (8) but now with tilded scale factor:

$$\tilde{a}(t) = a_0 \tilde{t}^{\frac{2}{3(w+1)}}, \quad (34)$$

$$\tilde{\rho}(\tilde{a}) = \frac{\rho_0}{\tilde{a}^{3(w+1)}}. \quad (35)$$

Let us verify this property explicitly. We have, using Equation (7),

$$d\tilde{t} = sa^{\frac{3(w+1)(s-1)}{2}} dt = sa_0^{\frac{3(w+1)(s-1)}{2}} t^{s-1} dt \quad (36)$$

and, integrating,

$$\tilde{t} = \int d\tilde{t} = a_0^{\frac{3(w+1)(s-1)}{2}} t^s, \quad (37)$$

where we set to zero an additive integration constant assuming that  $\tilde{t}(0) = 0$  if  $s > 0$  (apart from dimensions, the multiplicative constant on the right-hand side of Equation (37) is not physically relevant because it can be absorbed into a rescaling of the time coordinate). Inverting this relation yields

$$t = \frac{\tilde{t}^{1/s}}{a_0^{\frac{3(w+1)(s-1)}{2}}}. \quad (38)$$

Therefore, it is

$$\tilde{a}(\tilde{t}) = a^s = a_0^s t^{\frac{2s}{3(w+1)}} = a_0^{\frac{2s^2-3(w+1)s+3(w+1)}{2s}} \tilde{t}^{\frac{2}{3(w+1)}} \equiv \tilde{a}_0 \tilde{t}^{\frac{2}{3(w+1)}}. \quad (39)$$

Similarly, one checks that

$$\tilde{\rho}(\tilde{a}) = a^{3(w+1)(s-1)} \rho = a^{3(w+1)(s-1)} \frac{\rho_0}{a^{3(w+1)}} = \frac{\rho_0}{\tilde{a}^{3(w+1)}}. \quad (40)$$

## 5. Conclusions

We have reported a symmetry property of the Einstein–Friedmann equations for a spatially flat FLRW universe filled with a single barotropic perfect fluid with constant equation of state  $P = w\rho$ . It is easy to check that this symmetry does not hold for spatially curved universes.

As for the physical meaning of this symmetry, let us note that the scale factor and the comoving time scale as

$$a \rightarrow a^s, \quad (41)$$

$$t \rightarrow t^s, \quad (42)$$

respectively. This scaling could be superficially interpreted by saying that the scaling of proper spatial distances and the proper time of observers comoving with the cosmic perfect fluid by the same power points to some scale-invariance property of the Einstein–Friedmann equations for spatially flat sections, with this property failing when a spatial scale associated with the curvature of the three-dimensional spatial sections is present. However, these equations are definitely *not* scale invariant, and this is the meaning of the completely different scaling of the energy density (11) (in the units used, in which energy and mass have the dimensions of a length, energy density should scale as the inverse square of a length  $\ell^{-2s}$ , but it does not). This fact simply reflects the lack of scale invariance of the Einstein equations even in vacuo or in the presence of conformally invariant matter (such as, e.g., a radiation fluid with  $w = 1/3$ ).

Next, one could be tempted to view the symmetry of the Einstein–Friedmann Equations (2), (5) and (6) as deriving from a conformal transformation of the spacetime metric  $ds^2 \rightarrow d\tilde{s}^2 = \Omega^2 ds^2$  for some conformal factor  $\Omega(x^\mu)$ , followed by a suitable redefinition of the comoving time coordinate, but this is not possible in general, as it is easy to check.

The symmetry map (9)–(11) for the special case of a radiation fluid with equation of state parameter  $w = 1/3$  was already noted in the context of an analogy between the cosmic radiation era and the freezing of bodies of water in environmental physics [27].

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