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On Hypergroups with a β -Class of Finite Height

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Received: 13 December 2019; Accepted: 10 January 2020; Published: 15 January 2020



Abstract: In every hypergroup, the equivalence classes modulo the fundamental relation β are the union of hyperproducts of element pairs. Making use of this property, we introduce the notion of height of a β -class and we analyze properties of hypergroups where the height of a β -class coincides with its cardinality. As a consequence, we obtain a new characterization of 1-hypergroups. Moreover, we define a hierarchy of classes of hypergroups where at least one β -class has height 1 or cardinality 1, and we enumerate the elements in each class when the size of the hypergroups is $n \leq 4$, apart from isomorphisms.

Keywords: hypergroup; semihypergroup; 1-hypergroup; fundamental relation; height

1. Introduction

The term algebraic hyperstructure designates a suitable generalization of a classical algebraic structure, like a group, a semigroup, or a ring. In classical algebraic structures, the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a set. In the last few decades, many scholars have been working in the field of algebraic hyperstructures, also called hypercompositional algebra. In fact, algebraic hyperstructures have found applications in many fields, including geometry, fuzzy/rough sets, automata, cryptography, artificial intelligence and probability [1], relational algebras [2], and sensor networks [3].

Certain equivalence relations, called fundamental relations, introduce natural correspondences between algebraic hyperstructures and classical algebraic structures. These equivalence relations have the property of being the smallest strongly regular equivalence relations such that the corresponding quotients are classical algebraic structures [4–11]. For example, if (H, \circ) is a hypergroup, then the fundamental relation β is transitive [12–14] and the quotient set H/β is a group. Moreover, if $\varphi: H \rightarrow H/\beta$ is the canonical projection, then the kernel $\omega_H = \varphi^{-1}(1_{H/\beta})$ is a subhypergroup, which is called the *heart* of (H, \circ) . The heart of a hypergroup (H, \circ) plays a very important role in hypergroup theory because it gives detailed information on the partition of H determined by the relation β , since $\beta(x) = \omega_H \circ x = x \circ \omega_H$ for all $x \in H$.

In this work, we focus on the fundamental relation β in hypergroups, and we introduce a new classification of hypergroups in terms of the minimum number of hyperproducts of two elements whose union is the β -class that contains these hyperproducts. Our main aim is to deepen the understanding of the properties of the fundamental relation β in hypergroups and to enumerate the non-isomorphic hypergroups fulfilling certain conditions on the cardinality of the heart. This task belongs to an established research field that deals with fundamental relations and enumerative problems in hypercompositional algebra [5,6,13–15]. The plan of this article is the following: After introducing some basic definitions and notations to be used throughout this article, in Section 3, we

define the notion of *height* $h(\beta(x))$ of an equivalence class $\beta(x)$. We give examples of hypergroups with infinite size where the height of all β -classes is finite. Denoting cardinality by $|\cdot|$, if (H, \circ) is a hypergroup with a β -class of finite size such that $|\beta(x)| = h(\beta(x))$, then $|a \circ y| = |y \circ a| = 1$, for all $a \in \omega_H$ and $y \in \beta(x)$. Moreover, when ω_H is finite, we prove that $|\omega_H| = h(\omega_H)$ if and only if (H, \circ) is a 1-hypergroup. In Section 4, we use the notion of height of a β -class to introduce new classes of hypergroups. We enumerate the elements in each class when the size of the hypergroups is not larger than 4, apart from isomorphisms. In particular, we prove that there are 4023 non-isomorphic hypergroups of size $n \leq 4$ with a β -class of size 1. Moreover, excluding the hypergroups (H, \circ) with $|H| = 4$ and $|H/\beta| = 1$, there exist 8154 non-isomorphic hypergroups of size $n \leq 4$ with $h(\omega_H) = 1$.

2. Basic Definitions and Results

Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H . A *hyperoperation* \circ on H is a map from $H \times H$ to $\mathcal{P}^*(H)$. For all $x, y \in H$, the set $x \circ y$ is called the *hyperproduct* of x and y . The hyperoperation \circ is naturally extended to subsets as follows: If $A, B \subseteq H$, then $A \circ B = \bigcup_{x \in A, y \in B} x \circ y$.

A *semihypergroup* is a non-empty set H endowed with an associative hyperproduct \circ , that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. A semihypergroup (H, \circ) is a *hypergroup* if for all $x \in H$ we have $x \circ H = H \circ x = H$; this property is called *reproducibility*. A non-empty subset K of a semihypergroup (H, \circ) is called a *subsemihypergroup* of (H, \circ) if it is closed with respect to \circ that is, $x \circ y \subseteq K$ for all $x, y \in K$. A non-empty subset K of a hypergroup (H, \circ) is called a *subhypergroup* if $x \circ K = K \circ x = K$, for all $x \in K$. If a subhypergroup is isomorphic to a group, then we say that it is a subgroup of (H, \circ) .

Given a semihypergroup (H, \circ) , the relation β^* of H is the transitive closure of the relation $\beta = \bigcup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation in H and, for every integer $n > 1$, β_n is defined as follows:

$$x\beta_n y \iff \exists (z_1, \dots, z_n) \in H^n : \{x, y\} \subseteq z_1 \circ z_2 \circ \dots \circ z_n.$$

The relations β and β^* are among the so-called *fundamental relations* [16]. Their relevance in semihypergroup and hypergroup theory stems from the following facts [17]: If (H, \circ) is a semihypergroup (resp., a hypergroup), the quotient set H/β^* equipped with the operation $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$ for all $x, y \in H$ and $z \in x \circ y$, is a semigroup (resp., a group). The canonical projection $\varphi : H \rightarrow H/\beta^*$ is a good homomorphism, that is, $\varphi(x \circ y) = \varphi(x) \otimes \varphi(y)$ for all $x, y \in H$. If (H, \circ) is a hypergroup, then H/β^* is a group and the kernel $\omega_H = \varphi^{-1}(1_{H/\beta^*})$ of φ is the *heart* of (H, \circ) . Moreover, if $|\omega_H| = 1$, then (H, \circ) is called *1-hypergroup*.

Let A be a non-empty subset of a semihypergroup (H, \circ) . We say that A is a *complete part* of (H, \circ) if, for every $n \in \mathbb{N} - \{0\}$ and $(x_1, x_2, \dots, x_n) \in H^n$,

$$(x_1 \circ \dots \circ x_n) \cap A \neq \emptyset \implies (x_1 \circ \dots \circ x_n) \subseteq A.$$

Clearly, the set H is a complete part, and the intersection $\mathcal{C}(X)$ of all the complete parts containing a non-empty set X is called the *complete closure* of X . If X is a complete part of (H, \circ) then $\mathcal{C}(X) = X$.

If (H, \circ) is a semihypergroup and $\varphi : H \rightarrow H/\beta^*$ is the canonical projection, then, for every non-empty set $A \subseteq H$, we have $\mathcal{C}(A) = \varphi^{-1}(\varphi(A))$. Moreover, if (H, \circ) is a hypergroup, then

$$\mathcal{C}(A) = \varphi^{-1}(\varphi(A)) = A \circ \omega_H = \omega_H \circ A.$$

A hypergroup (H, \circ) is said to be *complete* if $x \circ y = \mathcal{C}(x \circ y)$, for all $(x, y) \in H^2$. If (H, \circ) is a complete hypergroup, then

$$x \circ y = \mathcal{C}(a) = \beta^*(a),$$

for every $(x, y) \in H^2$ and $a \in x \circ y$.

A subhypergroup K of a hypergroup (H, \circ) is said to be *conjugable* if it satisfies the following property: for all $x \in H$, there exists $x' \in H$ such that $xx' \subseteq K$. The interested reader can find all relevant definitions, many properties, and applications of fundamental relations, even in more abstract contexts, also in [18–28].

For later reference, we collect in the following theorem some classic results of hypergroup theory from [12,17,26].

Theorem 1. *Let (H, \circ) be a hypergroup. Then,*

1. *The relation β is transitive, which is $\beta = \beta^*$;*
2. *$\beta(x) = x \circ \omega_H = \omega_H \circ x$, for all $x \in H$;*
3. *a subhypergroup K of (H, \circ) is conjugable if and only if it is a complete part of (H, \circ) ;*
4. *the heart of (H, \circ) is the smallest conjugable subhypergroup (or complete part) of (H, \circ) , that is, ω_H is the intersection of all conjugable subhypergroups (or complete part) of (H, \circ) .*

3. Locally Finite Hypergroups

Let (H, \circ) be a hypergroup and let \sim be the following equivalence relation on the set $H \times H$: $(x, y) \sim (z, w) \Leftrightarrow x \circ y = z \circ w$. Let \mathcal{T} be a transversal of the equivalence classes of the relation \sim . For every $x \in H$, there exists a non-empty set $A \subseteq \mathcal{T}$ such that $\beta(x) = \bigcup_{(a,b) \in A} a \circ b$. In fact, by reproducibility of (H, \circ) , if $y \in \beta(x)$, then there exist $z, w \in H$ such that $y \in z \circ w$. Clearly, we have $z \circ w \cap \beta(x) \neq \emptyset$ and $z \circ w \subseteq \beta(x)$ because $\beta(x)$ is a complete part of H . Moreover, there exists $(a, b) \in \mathcal{T}$ such that $(z, w) \sim (a, b)$ and $y \in z \circ w = a \circ b$. Hence, there exists a non-empty set $A \subseteq \mathcal{T}$ such that $\beta(x) \subseteq \bigcup_{(a,b) \in A} a \circ b$ and $a \circ b \cap \beta(x) \neq \emptyset$ for all $(a, b) \in A$. The other inclusion follows from the fact that $\beta(x)$ is a complete part of (H, \circ) .

In conclusion, each β -class is the union of hyperproducts of pairs of elements that can be chosen within a transversal of \sim . This fact suggests the following definitions.

Definition 1. *Let (H, \circ) be a hypergroup and let \mathcal{T} be a transversal of the equivalence classes of the relation \sim . For every $x \in H$, the class $\beta(x)$ is called *locally finite* if there exists a finite set $A \subseteq \mathcal{T}$ such that $\beta(x) = \bigcup_{(a,b) \in A} a \circ b$. If a class $\beta(x)$ is not locally finite, we say that it is *locally infinite*.*

Definition 2. *Let $\beta(x)$ be a β -class of a hypergroup (H, \circ) . If $\beta(x)$ is locally finite, then the minimum positive integer m such that there is a non-empty set $M \subseteq \mathcal{T}$ such that $|M| = m$ and $\beta(x) = \bigcup_{(a,b) \in M} a \circ b$ is called *height* of $\beta(x)$, and we write $h(\beta(x)) = m$. If $\beta(x)$ is locally infinite, we write $h(\beta(x)) = \infty$.*

Definition 3. *A hypergroup (H, \circ) is *locally finite* if all β -classes are locally finite. In particular, (H, \circ) is called *locally n -finite* if $h(\beta(x)) \leq n$ for every $x \in H$, and there is at least one element $y \in H$ such that $h(\beta(y)) = n$. Moreover, (H, \circ) is *strongly locally n -finite* if $h(\beta(x)) = n$ for every $x \in H$.*

Clearly, (H, \circ) is locally 1-finite if and only if (H, \circ) is strongly locally 1-finite. Examples of hypergroups locally 1-finite are the complete hypergroups. Indeed, if (H, \circ) is a complete hypergroup, then, for every $x \in H$, there exist $y, z \in H$ such that $x \in y \circ z$ and $\beta(x) = y \circ z$.

Example 1. *In the set $H = \{1, 2, 3, 4, 5, 6\}$, consider the hyperproducts defined by the following tables:*

\circ_1	1	2	3	4	5	6	\circ_2	1	2	3	4	5	6
1	1,2	1,3	2,3	4,5,6	4,6	5,6	1	1,2	1,3	2,3	4,5	4,6	5,6
2	2,3	1,2	1,3	5,6	4,5,6	4,6	2	2,3	1,2	1,3	5,6	4,5	4,6
3	1,3	2,3	1,2	4,6	5,6	4,5,6	3	1,3	2,3	1,2	4,6	5,6	4,5
4	4,5	4,6	5,6	1,2	1,3	2,3	4	4,5	4,6	5,6	1,2	1,3	2,3
5	5,6	4,5	4,6	2,3	1,2	1,3	5	5,6	4,5	4,6	2,3	1,2	1,3
6	4,6	5,6	4,5	1,3	2,3	1,2	6	4,6	5,6	4,5	1,3	2,3	1,2

Then, (H, \circ_1) and (H, \circ_2) are hypergroups such that $|H/\beta| = 2$. In particular, (H, \circ_1) is a locally 2-finite hypergroup since $h(\omega_H) = 2$ and $h(\beta(4)) = 1$, while (H, \circ_2) is a strongly locally 2-finite hypergroup because $h(\omega_H) = h(\beta(4)) = 2$.

Example 2. Let (H, \circ) be a hypergroup and $\text{Aut}(H)$ the automorphism group of H . If $f \in \text{Aut}(H)$, let $\langle f \rangle$ be the subgroup of $\text{Aut}(H)$, generated by f . In $H \times \langle f \rangle$, we define the following hyperproduct: For all $(a, f^m), (b, f^n) \in H \times \langle f \rangle$, let

$$(a, f^m) \star (b, f^n) = \{(c, f^{m+n}) \mid c \in a \circ f^m(b)\} = (a \circ f^m(b)) \times \{f^{m+n}\}.$$

Firstly, we show that $(H \times \langle f \rangle, \star)$ is a hypergroup. Then, we describe its β -classes and the heart. As a consequence, we obtain that (H, \circ) is a locally n -finite hypergroup (resp., strongly locally n -finite hypergroup, complete hypergroup, or 1-hypergroup) if and only if $(H \times \langle f \rangle, \star)$ is a locally n -finite hypergroup (resp., strongly locally n -finite hypergroup, complete hypergroup, or 1-hypergroup).

1. The hyperproduct \star on $H \times \langle f \rangle$ is associative. In fact, if $(a, f^m), (b, f^n), (c, f^r) \in H \times \langle f \rangle$, then we obtain:

$$\begin{aligned} ((a, f^m) \star (b, f^n)) \star (c, f^r) &= ((a \circ f^m(b)) \times \{f^{m+n}\}) \star (c, f^r) \\ &= \bigcup_{z \in a \circ f^m(b)} (z, f^{m+n}) \star (c, f^r) \\ &= \bigcup_{z \in a \circ f^m(b)} (z \circ f^{m+n}(c)) \times \{f^{m+n+r}\} \\ &= (a \circ f^m(b)) \circ f^{m+n}(c) \times \{f^{m+n+r}\} \\ &= a \circ (f^m(b) \circ f^{m+n}(c)) \times \{f^{m+n+r}\} \\ &= a \circ (f^m(b \circ f^n(c))) \times \{f^{m+n+r}\} \\ &= \bigcup_{w \in b \circ f^n(c)} (a \circ f^m(w)) \times \{f^{m+n+r}\} \\ &= \bigcup_{w \in b \circ f^n(c)} (a, f^m) \star (w, \{f^{n+r}\}) \\ &= (a, f^m) \star ((b \circ f^n(c)) \times \{f^{n+r}\}) = (a, f^m) \star ((b, f^n) \star (c, f^r)). \end{aligned}$$

Consequently, we have that

$$(a, f^m) \star (b, f^n) \star (c, f^r) = (a \circ f^m(b) \circ f^{m+n}(c)) \times \{f^{m+n+r}\}.$$

By induction, if $(a_1, f^{n_1}), (a_2, f^{n_2}), \dots, (a_r, f^{n_r})$ are elements in $H \times \langle f \rangle$, then the hyperproduct $(a_1, f^{n_1}) \star (a_2, f^{n_2}) \star \dots \star (a_r, f^{n_r})$ is the set

$$(a_1 \circ f^{n_1}(a_2) \circ f^{n_1+n_2}(a_3) \circ \dots \circ f^{n_1+n_2+\dots+n_{r-1}}(a_r)) \times \{f^{n_1+n_2+\dots+n_r}\}.$$

2. The hyperproduct \star is reproducible. Indeed, we have $(b, f^m) \star (H \times \langle f \rangle) \subseteq H \times \langle f \rangle$, for all elements $(b, f^m) \in H \times \langle f \rangle$. On the other hand, if $(a, f^n) \in H \times \langle f \rangle$, then, by reproducibility of (H, \circ) , there exists $x \in H$ such that $a \in b \circ x$. Now, if we consider $(f^{-m}(x), f^{n-m})$, then $(a, f^n) \in (b, f^m) \star (f^{-m}(x), f^{n-m})$ because $a \in b \circ x = b \circ f^m(f^{-m}(x))$ and $f^m f^{n-m} = f^n$. Hence, $H \times \langle f \rangle \subseteq (b, f^m) \star (H \times \langle f \rangle)$. Thus, we have

$$(b, f^m) \star (H \times \langle f \rangle) = H \times \langle f \rangle, \text{ for all } (b, f^m) \in H \times \langle f \rangle.$$

In the same way, one shows that $(H \times \langle f \rangle) \star (b, f^m) = H \times \langle f \rangle$.

3. From 1 and 2, $(H \times \langle f \rangle, \star)$ is a hypergroup. If β' and β are the fundamental relations in $(H \times \langle f \rangle, \star)$ and (H, \circ) respectively, we have

$$\beta'(x, f^n) = \beta(x) \times \{f^n\},$$

for all $(x, f^n) \in H \times \langle f \rangle$. Indeed, if $(y, f^n) \in \beta'((x, f^m))$, then there exist $r \in \mathbb{N} - \{0\}$ and $(a_1, f^{n_1}), (a_2, f^{n_2}), \dots, (a_r, f^{n_r}) \in H \times \langle f \rangle$ such that

$$\{(y, f^n), (x, f^m)\} \subseteq (a_1, f^{n_1}) \star (a_2, f^{n_2}) \star \dots \star (a_r, f^{n_r}).$$

By point 1, we have $\{y, x\} \subseteq a_1 \circ f^{n_1}(a_2) \circ f^{n_1+n_2}(a_3) \circ \dots \circ f^{n_1+n_2+\dots+n_{r-1}}(a_r)$ and $f^n = f^{n_1+n_2+\dots+n_r} = f^m$. Hence, $y \in \beta(x)$, $f^n = f^m$ and $(y, f^n) \in \beta(x) \times \{f^n\}$. Thus, $\beta'((x, f^m)) \subseteq \beta(x) \times \{f^n\}$. On the other hand, if $(y, f^n) \in \beta(x) \times \{f^n\}$, then $y\beta x$ and there exist $r \in \mathbb{N} - \{0\}$ and $a_1, a_2, \dots, a_r \in H$ such that $\{x, y\} \subseteq a_1 \circ a_2 \circ \dots \circ a_r$. Now, if in $H \times \langle f \rangle$ we consider $(a_1, f^0), (a_2, f^0), \dots, (a_{r-1}, f^0), (a_r, f^n)$, we obtain

$$\{(x, f^n), (y, f^n)\} \subseteq (a_1, f^0) \star (a_2, f^0) \star \dots \star (a_{r-1}, f^0) \star (a_r, f^n),$$

and we have $(y, f^n) \in \beta'((x, f^n))$. Hence, $\beta(x) \times \{f^n\} \subseteq \beta'((x, f^n))$.

4. The set $\omega_H \times \{f^0\}$ is a subhypergroup of $(H \times \langle f \rangle, \star)$. In fact, if $(a, f^0) \in \omega_H \times \{f^0\}$, we have

$$\begin{aligned} (a, f^0) \star (\omega_H \times \{f^0\}) &= \bigcup_{b \in \omega_H} (a, f^0) \star (b, f^0) \\ &= \bigcup_{b \in \omega_H} (a \circ f^0(b)) \times \{f^0\} \\ &= \bigcup_{b \in \omega_H} (a \circ b) \times \{f^0\} \\ &= (a \circ \omega_H) \times \{f^0\} = \omega_H \times \{f^0\}. \end{aligned}$$

Hence, $(a, f^0) \star (\omega_H \times \{f^0\}) = \omega_H \times \{f^0\}$, for all $(a, f^0) \in \omega_H \times \{f^0\}$. In the same way one proves that $(\omega_H \times \{f^0\}) \star (a, f^0) = \omega_H \times \{f^0\}$, so $\omega_H \times \{f^0\}$ is a subhypergroup.

5. The subhypergroup $\omega_H \times \{f^0\}$ is the heart $\omega_{H \times \langle f \rangle}$ of $(H \times \langle f \rangle, \star)$.

Indeed, for all $(a, f^n) \in H \times \langle f \rangle$, there exists $b \in H$ such that $a \circ b \subseteq \omega_H$. If we consider $(f^{-n}(b), f^{-n})$, then $(a, f^n) \star (f^{-n}(b), f^{-n}) = (a \circ f^n(f^{-n}(b))) \times \{f^0\} = (a \circ b) \times \{f^0\} \subseteq \omega_H \times \{f^0\}$, so $\omega_H \times \{f^0\}$ is conjugable. From point 4 of Theorem 1, we obtain that $\omega_{H \times \langle f \rangle} \subseteq \omega_H \times \{f^0\}$. Finally, since $\omega_{H \times \langle f \rangle}$ and $\omega_H \times \{f^0\}$ are β' -classes, we deduce that $\omega_{H \times \langle f \rangle} = \omega_H \times \{f^0\}$.

Proposition 1. Let (H, \circ) be a hypergroup. Then, $h(\beta(y)) \leq |\beta(x)|$, for all $x, y \in H$.

Proof. Let $x, y \in H$. By reproducibility of H , there exists $b \in H$ such that $y \in x \circ b$ and so we have

$$\beta(y) = \omega_H \circ y \subseteq \omega_H \circ x \circ b = \beta(x) \circ b = \bigcup_{a \in \beta(x)} a \circ b.$$

Clearly, for every $z \in \beta(y)$, there exists $a_z \in \beta(x)$ such that $z \in a_z \circ b$. Moreover, since $\beta(y)$ is a complete part of H , we have $a_z \circ b \subseteq \beta(y)$. Hence, there exists a non-empty subset A of $\beta(x)$ such that $\beta(y) = \bigcup_{a \in A} a \circ b$ and so we obtain $h(\beta(y)) \leq |A| \leq |\beta(x)|$. \square

By the previous proposition, we have the following results:

Corollary 1. Let (H, \circ) be a hypergroup. Then, $h(\beta(x)) \leq \min\{|\omega_H|, |\beta(x)|\}$, for every $x \in H$.

Corollary 2. Let (H, \circ) be a hypergroup. If there exists a β -class of size 1, then (H, \circ) is a strongly locally 1-finite hypergroup.

Proposition 2. Let (H, \circ) be a hypergroup. If $x \in H$ is such that $\beta(x)$ is finite and $|\beta(x)| = h(\beta(x))$, then $|a \circ y| = |y \circ a| = 1$ for all $a \in \omega_H$ and $y \in \beta(x)$.

Proof. Let $n = |\beta(x)|$. For all $a \in \omega_H$ and $y \in \beta(x)$, we have $a \circ y \subseteq \omega_H \circ y = \beta(y) = \beta(x)$. If $|a \circ y| = |\beta(x)|$, then $a \circ y = \beta(x)$ and so $n = |\beta(x)| = h(\beta(x)) = 1$. Hence, $|a \circ y| = 1$. Now, let $|a \circ y| \neq |\beta(x)|$ and by contradiction we suppose that $2 \leq |a \circ y| = k < n$. Let $\beta(x) - a \circ y = \{x_1, x_2, \dots, x_{n-k}\}$, by reproducibility of H , there exist $y_1, y_2, \dots, y_{n-k} \in H$ such that $x_i \in a \circ y_i$ for all $i \in \{1, 2, \dots, n-k\}$. Since $\beta(x)$ is a complete part of H and $a \circ y_i \cap \beta(x) \neq \emptyset$, we deduce that $a \circ y_i \subseteq \beta(x)$, for all $i \in \{1, 2, \dots, n-k\}$. Therefore, $\beta(x) = a \circ y \cup a \circ y_1 \cup \dots \cup a \circ y_{n-k}$ and so $n = h(\beta(x)) \leq n - k + 1 < n$, impossible. Thus, $|a \circ y| = 1$, for all $a \in \omega_H$ and $y \in \beta(x)$. In an analogous way, we have that $|y \circ a| = 1$, for every $y \in \beta(x)$. \square

An immediate consequence of the previous proposition is the following corollary:

Corollary 3. Let (H, \circ) be a hypergroup. If ω_H is finite and $|\omega_H| = h(\omega_H)$, then ω_H is a subgroup of (H, \circ) .

In the preceding corollary, the finiteness of ω_H is a critical hypothesis. Indeed, in the next example, we show a hypergroup where $|\omega_H| = h(\omega_H) = \infty$ and ω_H is not a group.

Example 3. Let $(\mathbb{Z}, +)$ be the group of integers. In the set $H = \mathbb{Z} \times \mathbb{Z}$, we define the following hyperproduct:

$$(a, b) \star (c, d) = \{(a, b + d), (c, b + d)\}.$$

Routine computations show that (H, \star) is a hypergroup and $(\mathbb{Z} \times \{0\}, \star)$ is a subhypergroup of (H, \star) . Hereafter, we firstly describe the core ω_H , then we compute $h(\omega_H)$.

To prove that $\omega_H = \mathbb{Z} \times \{0\}$, we will show that $\mathbb{Z} \times \{0\}$ is the smallest conjugable subhypergroup of (H, \star) . For every element $(x, y) \in H$, we can consider the element $(0, -y)$. We obtain $(x, y) \star (0, -y) = \{(x, 0), (0, 0)\} \subset \mathbb{Z} \times \{0\}$, hence $\mathbb{Z} \times \{0\}$ is a conjugable subhypergroup. Now, let K be a conjugable subhypergroup of (H, \star) and let $(a, 0) \in \mathbb{Z} \times \{0\}$. Since K is conjugable, there exists $(x, y) \in H$ such that $(a, 0) \star (x, y) \subseteq K$, and so $\{(a, y), (x, y)\} \subseteq K$. By reproducibility of K , there exists $(x', y') \in K$ such that $(a, y) \in (x, y) \star (x', y') = \{(x, y + y'), (x', y + y')\} \subseteq K$. Clearly, $y' = 0$ because $y = y + y'$ and $(x', 0) \in K$. Since $(x', 0), (a, y) \in K$, there exists $(z, w) \in K$ such that $(x', 0) \in (a, y) \star (z, w) = \{(a, y + w), (z, y + w)\} \subseteq K$. Consequently, we have $y + w = 0$ and $(a, 0) \in K$. Hence, $\mathbb{Z} \times \{0\} \subseteq K$ and $\omega_H = \mathbb{Z} \times \{0\}$ since $\mathbb{Z} \times \{0\}$ is conjugable. Obviously, $|\omega_H| = |\mathbb{Z}|$.

Finally, we prove that $h(\omega_H) = |\mathbb{Z}|$. By Proposition 1, we have $h(\omega_H) \leq |\omega_H| = |\mathbb{Z}|$. If $h(\omega_H) < |\mathbb{Z}|$, then there exist n hyperproducts $(a_i, b_i) \star (c_i, d_i)$ of elements in H such that

$$\mathbb{Z} \times \{0\} = \omega_H = \bigcup_{i=1}^n (a_i, b_i) \star (c_i, d_i) = \bigcup_{i=1}^n \{(a_i, b_i + d_i), (c_i, b_i + d_i)\}.$$

This result is impossible since $\bigcup_{i=1}^n \{a_i, c_i\} \neq \mathbb{Z}$.

Now, we give two examples of hypergroups (H, \circ) whose heart is a group and $|\omega_H| \geq 2$. In particular, we have that $|H/\beta| = 2$, $h(\omega_H) = 1$ and $h(\beta(a)) = |\omega_H|$, if $a \in H - \omega_H$.

Example 4. Consider the group $(\mathbb{Z}, +)$ and a set $A = \{a_i\}_{i \in \mathbb{Z}}$ such that $\mathbb{Z} \cap A = \emptyset$. In the set $H = \mathbb{Z} \cup A$, we define the following hyperproduct:

- $m \circ n = \{m + n\}$ if $m, n \in \mathbb{Z}$;

- $m \circ a_n = a_n \circ m = \{a_{m+n}\}$ if $m \in \mathbb{Z}$ and $a_n \in A$;
- $a \circ b = \mathbb{Z}$ if $a, b \in A$.

Routine computations show that (H, \circ) is a hypergroup. We have $H/\beta \cong \mathbb{Z}_2$, $\omega_H = \mathbb{Z}$ and $\beta(a) = A$ if $a \in A$. Clearly, we have $h(\omega_H) = 1$ and $h(\beta(a)) = |\beta(a)| = |\mathbb{Z}|$ because $m \circ a$ is a singleton, for all $m \in \mathbb{Z}$ and $a \in A$.

Example 5. Let (G, \cdot) be a group of size $n \geq 2$ and let $G = \{g_1, g_2, \dots, g_n\}$. Moreover, let σ be a n -cycle of the symmetric group defined over $X = \{1, 2, \dots, n\}$. If $A = \{a_1, a_2, \dots, a_n\}$ is a set disjoint with G , in $H = G \cup A$, we can define the following hyperproduct:

- $g_h \circ g_k = \{g_h g_k\}$,
- $g_h \circ a_k = a_k \circ g_h = \{a_{\sigma^h(k)}\}$,
- $a \circ b = G$ if $a, b \in A$.

Then, (H, \circ) is a hypergroup such that $H/\beta \cong \mathbb{Z}_2$, $\omega_H = G$ and $\beta(a) = A$ if $a \in A$. Moreover, we have $h(\omega_H) = 1$ and $h(\beta(a)) = |\beta(a)| = |G|$.

In the next theorem, we characterize the locally n -finite hypergroups such that $|\omega_H| = h(\omega_H)$.

Theorem 2. Let (H, \circ) be a hypergroup such that $|\omega_H|$ is finite. The following conditions are equivalent:

1. (H, \circ) is an 1-hypergroup that is $|\omega_H| = 1$;
2. $|\omega_H| = h(\omega_H)$.

Proof. The implication 1. \Rightarrow 2. is trivial, hence we prove that 2. \Rightarrow 1. Let n a positive integer such that $|\omega_H| = h(\omega_H) = n$. By Corollary 3, the heart ω_H is a subgroup of (H, \circ) . Let e be the identity of ω_H . If there exist $a, b \in H$ such that $a \circ b = \omega_H$, then we have $|\omega_H| = h(\omega_H) = 1$ and $\omega_H = \{e\}$. Now, by contradiction, we suppose that $a \circ b \neq \omega_H$, for all $a, b \in H$, and let x be an element of H . By reproducibility of (H, \circ) , there exists $x' \in H$ such that $e \in x \circ x'$. Clearly, we have $x \circ x' \subset \omega_H$, since ω_H is a complete part of (H, \circ) . Therefore, there exists an integer k , with $1 \leq k < n$, and $n - k$ elements of ω_H such that $\emptyset \neq \omega_H - x \circ x' = \{a_1, a_2, \dots, a_{n-k}\}$. Hence, we obtain

$$\omega_H = x \circ x' \cup \{a_1, a_2, \dots, a_{n-k}\} = x \circ x' \cup e \circ a_1 \cup e \circ a_2 \cup \dots \cup e \circ a_{n-k},$$

and so $k = 1$ because $n = h(\omega_H) \leq n - k + 1$. Therefore, we obtain $x \circ x' = \{e\}$ and so $\{e\}$ is a conjugable subhypergroup of (H, \circ) . Consequently, we have $\{e\} = \omega_H = e \circ e$, a contradiction. \square

The following result is an immediate consequence of Theorem 2.

Corollary 4. Let (H, \circ) be a finite hypergroup, then (H, \circ) is a group if and only if $|\beta(x)| = h(\beta(x))$, for all $x \in H$.

Proof. The implication \Rightarrow is obvious. On the other hand, if we suppose that $|\beta(x)| = h(\beta(x))$ for all $x \in H$, then we have $|\omega_H| = h(\omega_H)$ and so $|\omega_H| = 1$, by Theorem 2. From Corollary 1 and the hypothesis, we have $|\beta(x)| = h(\beta(x)) = 1$. Hence, (H, \circ) is a group. \square

4. Hypergroups with at Least One β -Class of Height Equal to 1

From Theorem 2, the 1-hypergroups are characterized by the fact that $h(\omega_H) = |\omega_H| = 1$. In this section, we use the notion of height of a β -class to introduce new classes of hypergroups. We enumerate the elements in each class when the size of hypergroups is $n \leq 4$, apart from isomorphisms. We give the following definition:

Definition 4. Let (H, \circ) be a hypergroup. We say that

- a. (H, \circ) is a $(1, \beta)$ -hypergroup if there exists $x \in H$ such that $|\beta(x)| = 1$;
- b. (H, \circ) is a locally 1-finite hypergroup if $h(\beta(x)) = 1$, for all $x \in H$;
- c. (H, \circ) is a 1-weak hypergroup if $h(\omega_H) = 1$;
- d. (H, \circ) is a weakly locally 1-finite hypergroup if there exists $x \in H$ such that $h(\beta(x)) = 1$;

In the following, we denote by $\mathfrak{U}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and \mathfrak{D} the classes of 1-hypergroups, $(1, \beta)$ -hypergroups, locally 1-finite hypergroups, 1-weak hypergroups, and weakly locally 1-finite hypergroups, respectively. By Definition 4 and Corollary 2, we have the inclusions $\mathfrak{U} \subseteq \mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}$. Actually, these inclusions are strict, as shown in the following example.

Example 6. In this example, we show four hypergroups $(A, \circ), (B, \circ), (C, \circ)$ and (D, \circ) such that

- 1. $(A, \circ) \in \mathfrak{A}$ and $(A, \circ) \notin \mathfrak{U}$,
- 2. $(B, \circ) \in \mathfrak{B}$ and $(B, \circ) \notin \mathfrak{A}$,
- 3. $(C, \circ) \in \mathfrak{C}$ and $(C, \circ) \notin \mathfrak{B}$,
- 4. $(D, \circ) \in \mathfrak{D}$ and $(D, \circ) \notin \mathfrak{C}$.

They are the following:

- 1. $A = \{1, 2, 3\}$ with the hyperproduct

\circ	1	2	3
1	1	2	3
2	2	1	3
3	3	3	1,2

- 2. $B = \{1, 2, 3, 4\}$ with the hyperproduct

\circ	1	2	3	4
1	1	1,2	3,4	3,4
2	1,2	1,2	3,4	3,4
3	3,4	3,4	1,2	1,2
4	3,4	3,4	1,2	1,2

- 3. $C = \{1, 2, 3, 4\}$ with the hyperproduct

\circ	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1,2	1,2
4	4	3	1,2	1,2

- 4. $D = \{1, 2, 3, 4, 5\}$ with the hyperproduct

\circ	1	2	3	4	5
1	1,2	2,3	1,3	4,5	4,5
2	2,3	1,3	1,2	4,5	4,5
3	1,3	1,2	2,3	4,5	4,5
4	4,5	4,5	4,5	1,2	2,3
5	4,5	4,5	4,5	2,3	1,2

4.1. $(1, \beta)$ -Hypergroups of Size $n \leq 4$

In [26], Corsini introduced the class of 1-hypergroups and listed the 1-hypergroups of size $n \leq 4$, apart from isomorphisms. In this subsection, our interest is to study the hypergroups in class \mathfrak{A} and, in particular, to determine their number, apart from isomorphisms. Since the class of 1-hypergroups is a subclass of \mathfrak{A} , we recall the result proved by Corsini in [26].

Theorem 3. *If (H, \circ) is a 1-hypergroup with $|H| \leq 4$, then (H, \circ) is a complete hypergroup. Moreover, (H, \circ) is isomorphic to either a group or one of the hypergroups described in the following three hyperproduct tables:*

\circ	1	2	3
1	1	2,3	2,3
2	2,3	1	1
3	2,3	1	1

\circ	1	2	3	4
1	1	2,3,4	2,3,4	2,3,4
2	2,3,4	1	1	1
3	2,3,4	1	1	1
4	2,3,4	1	1	1

\circ	1	2	3	4
1	1	2,3	2,3	4
2	2,3	4	4	1
3	2,3	4	4	1
4	4	1	1	2,3

Therefore, there exist eight 1-hypergroups of size $|H| \leq 4$.

Now, we study the hypergroups $(H, \circ) \in \mathfrak{A} - \mathfrak{U}$ of size $n \leq 4$. Clearly, $|\omega_H|$ and $|H/\beta|$ can take the values 2 or 3 and so we distinguish the following cases:

1. $|H| = 3, |\omega_H| = 2$ and $|H/\beta| = 2$;
 2. $|H| = 4, |\omega_H| = 2$ and $|H/\beta| = 3$;
 3. $|H| = 4, |\omega_H| = 3$ and $|H/\beta| = 2$.
1. In this case, we can suppose $H = \{a, b, x\}, \omega_H = \{a, b\}$ and $\beta(x) = \{x\}$. Clearly, since $H/\beta \cong \mathbb{Z}_2$, for reproducibility of H , we have the following partial hyperproduct table of (H, \circ) :

\circ	a	b	x
a			x
b			x
x	x	x	a, b

To complete this table, the undetermined entries must correspond to the hyperproduct table of the subhypergroup ω_H . Apart from isomorphisms, there are eight hypergroups of size 2. Their hyperproduct tables were determined in [29] and are reproduced here below:

$W_1: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a & b \\ b & b & a \\ \hline \end{array}$	$W_2: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a & b \\ b & b & a, b \\ \hline \end{array}$	$W_3: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a & a, b \\ b & b & a, b \\ \hline \end{array}$	$W_4: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a, b & a, b \\ b & a & b \\ \hline \end{array}$
$W_5: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a & a, b \\ b & a, b & b \\ \hline \end{array}$	$W_6: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a & a, b \\ b & a, b & a, b \\ \hline \end{array}$	$W_7: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a, b & a, b \\ b & a, b & a \\ \hline \end{array}$	$W_8: \begin{array}{ c cc } \hline \circ & a & b \\ \hline a & a, b & a, b \\ b & a, b & a, b \\ \hline \end{array}$

Hence, in this case, we have eight hypergroups.

2. Without loss of generality, we suppose that $H = \{a, b, x, y\}, \omega_H = \{a, b\}, \beta(x) = \{x\}$ and $\beta(y) = \{y\}$. Since $H/\beta \cong \mathbb{Z}_3$, we have the following partial hyperproduct table of (H, \circ) :

\circ	a	b	x	y
a			x	y
b			x	y
x	x	x	y	a, b
y	y	y	a, b	x

As in the previous case, the entries that are left empty must be determined so that ω_H is a hypergroup of order 2. Hence, also in this case, we have eight hypergroups.

- Let $H = \{a, b, c\}$ and $\beta(x) = \{x\}$. We have the following partial hyperproduct table:

\circ	a	b	c	x
a				x
b				x
c				x
x	x	x	x	a, b, c

In this case, it is straightforward to see that we get as many hypergroups as there are of size three, apart from isomorphisms. In [30], this number is found to be equal to 3999.

From Theorem 3 and the preceding arguments, we summarize the number of non-isomorphic $(1, \beta)$ -hypergroups with $|H| \leq 4$ in Table 1.

Table 1. The number of non-isomorphic $(1, \beta)$ -hypergroups with $|H| \leq 4$.

$ H = 1$	$ H = 2$	$ H = 3$	$ H = 4$
1	1	10	4011

Result 1. There are 4023 non-isomorphic $(1, \beta)$ -hypergroups of size $n \leq 4$.

4.2. Locally 1-Finite Hypergroups of Size $n \leq 4$

In this subsection, we focus on hypergroups $(H, \circ) \in \mathfrak{B} - \mathfrak{A}$ with $|H| \leq 4$. By Definition 4, we have that $|\beta(x)| \geq 2$, for all $x \in H$.

If $|H| = 2$, then $|H/\beta| = 1$ and (H, \circ) is isomorphic to one of the hypergroups (W_i, \circ) listed in the previous subsection, for $i = 2, 3, \dots, 8$.

If $|H| = 3$, then $|H/\beta| = 1$, otherwise at least one β -class has size 1 and $(H, \circ) \in \mathfrak{A}$. Hence, to determine the hypergroups in $(H, \circ) \in \mathfrak{B} - \mathfrak{A}$ of size 3, we must assume that there exist $a, b \in H$ such that $a \circ b = H$. With the help of computer-assisted computations, we found that in this case there are exactly 3972 hypergroups, apart from isomorphisms.

If $|H| = 4$, then there are two possible cases, namely $|H/\beta| = 1$ and $|H/\beta| = 2$. In the first case, the only information we can deduce about (H, \circ) is that there are at least two elements $a, b \in H$ such that $a \circ b = H$. The number of hypergroups having that property is huge, and at present we are not able to enumerate them because the computational task exceeds our available resources. A detailed analysis of this case is challenging and may be the subject of further research. In the other case, if $x \in H - \omega_H$, then we have $|\omega_H| = |\beta(x)| = 2$ with $h(\omega_H) = h(\beta(x)) = 1$. Moreover, ω_H is isomorphic to one of the hypergroups (W_i, \circ) for $i = 1, 2, \dots, 8$ listed beforehand, and there exist $a, b \in H$ such that $a \circ b = \beta(x)$. On the basis of the information gathered from the preceding arguments, we are able to perform an exhaustive search of all possible hyperproduct tables with the help of a computer algebra system. In Table 2, we report the number of the hypergroups such that $|\omega_H| = |\beta(x)| = 2$ and $h(\omega_H) = h(\beta(x)) = 1$, depending on the structure of ω_H , apart from isomorphisms.

Table 2. The number of non-isomorphic hypergroups in $\mathfrak{B} - \mathfrak{A}$ with $|H| = 4$, according to the structure of ω_H .

W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
3	25	17	17	31	26	12	20

Since the hypergroups corresponding to the cases in which the heart is one of the W_2, W_3, \dots, W_8 are quite a few, we list hereafter only those whose heart is isomorphic to W_1 . Apart from isomorphisms, we have the following hyperproduct tables:

\circ	a	b	x	y	\circ	a	b	x	y	\circ	a	b	x	y
a	a	b	x	y	a	a	b	x, y	x, y	a	a	b	x, y	x, y
b	b	a	y	x	b	b	a	x, y	x, y	b	b	a	x, y	x, y
x	x, y	x, y	a, b	a, b	x	x	y	a, b	a, b	x	x, y	x, y	a, b	a, b
y	x, y	x, y	a, b	a, b	y	y	x	a, b	a, b	y	x, y	x, y	a, b	a, b

On the basis of the previous arguments, the number of hypergroups (H, \circ) belonging to $\mathfrak{B} - \mathfrak{A}$ is summarized in Table 3, in relation to the size of H :

Table 3. The number of non-isomorphic hypergroups in $\mathfrak{B} - \mathfrak{A}$, depending on their size.

$ H = 2$	$ H = 3$	$ H = 4$ and $ H/\beta = 2$
7	3972	151

Finally, since there are 4023 hypergroups in class \mathfrak{A} , see Result 1, we obtain the following result:

Result 2. Excluding the hypergroups (H, \circ) such that $|H| = 4$ and $|H/\beta| = 1$, there are 8153 non-isomorphic locally 1-finite hypergroups of size $n \leq 4$.

4.3. 1-Weak Hypergroups of Size $n \leq 4$

In this subsection, we determine the hypergroups $(H, \circ) \in \mathfrak{C}$ of size $n \leq 4$, apart from isomorphisms. We observe that, if $(H, \circ) \in \mathfrak{C} - \mathfrak{B}$, then there is at least one β -class of height different from 1. Moreover, since $\mathfrak{A} \subset \mathfrak{B}$, we have $|\beta(x)| > 1$, for all $x \in H$. Hence, if $(H, \circ) \in \mathfrak{C} - \mathfrak{B}$, then $|H| \geq 4$ and $|H/\beta| \geq 2$.

Lemma 1. Let (H, \circ) be a hypergroup in $\mathfrak{C} - \mathfrak{B}$ such that $|H| = 4$, then ω_H is isomorphic to the group \mathbb{Z}_2 .

Proof. By hypotheses, we have $|H/\beta| = 2$ and $|\omega_H| = 2$. By Proposition 1, if $\beta(x)$ is the class different from ω_H , then $h(\beta(x)) = 2 = |\beta(x)|$. Moreover, for Proposition 2, we have $|a \circ y| = |y \circ a| = 1$, for all $a \in \omega_H$ and $y \in \beta(x)$. Now, if by contradiction we suppose that there exist $a, b \in \omega_H$ such that $|a \circ b| \neq 1$, then $a \circ b = \omega_H$ and $|b \circ x| = |a \circ (b \circ x)| = 1$ because $b \circ x \subset \beta(x)$, $a \circ (b \circ x) \subset \beta(x)$ and $h(\beta(x)) = 2$. This fact is impossible since $a \circ (b \circ x) = (a \circ b) \circ x = \omega_H \circ x = \beta(x)$ and $|\beta(x)| = 2$. Hence, $|a \circ b| = 1$, for all $a, b \in \omega_H$, and so $\omega_H \cong \mathbb{Z}_2$. \square

Theorem 4. Let (H, \circ) be a hypergroup such that the heart ω_H is isomorphic to a torsion group. If ε is the identity of ω_H , then $x \in \varepsilon \circ x \cap x \circ \varepsilon$, for all $x \in H - \omega_H$.

Proof. Let $x \in H - \omega_H$. By reproducibility of H , there exists $e \in H$ such that $x \in x \circ e$. Clearly $e \in \omega_H$; moreover, we have $x \in x \circ e \subseteq (x \circ e) \circ e = x \circ (e \circ e) = x \circ e^2$ and so $x \in x \circ e^2$. Obviously, by induction, we obtain $x \in x \circ e^n$, for all $n \in \mathbb{N} - \{0\}$. Finally, since ω_H is isomorphic to a torsion group, there exists $m \in \mathbb{N} - \{0\}$ such that $e^m = \varepsilon$, hence $x \in x \circ \varepsilon$. In the same way, we have $x \in \varepsilon \circ x$. \square

By reproducibility, Lemma 1, and Theorem 4, the hypergroups (H, \circ) in $\mathfrak{C} - \mathfrak{B}$ with $|H| = 4$ have the following partial hyperproduct table, apart from isomorphisms:

\circ	a	b	x	y
a	a	b	x	y
b	b	a	y	x
x	x	y		
y	y	x		

Since $h(\omega_H) = 1$, we have $\omega_H \in \{x \circ x, x \circ y, y \circ x, y \circ y\}$. Now, we prove that $\omega_H = x \circ x = x \circ y = y \circ x = y \circ y$. In fact, if we suppose that $\omega_H = x \circ y$, then we have:

$$\begin{aligned} x \circ x &= x \circ (y \circ b) = (x \circ y) \circ b = \omega_H \circ b = \omega_H; \\ y \circ x &= (b \circ x) \circ (y \circ b) = b \circ (x \circ y) \circ b = b \circ \omega_H \circ b = \omega_H; \\ y \circ y &= (b \circ x) \circ y = b \circ (x \circ y) = b \circ \omega_H = \omega_H. \end{aligned}$$

We obtain the same result also if we suppose that $\omega_H = x \circ x$ or $\omega_H = y \circ x$ or $\omega_H = y \circ y$. Hence, in class $\mathfrak{C} - \mathfrak{B}$, there is only one hypergroup of size 4, apart from isomorphisms. Its hyperproduct table is the following:

\circ	a	b	x	y
a	a	b	x	y
b	b	a	y	x
x	x	y	a, b	a, b
y	y	x	a, b	a, b

We note that this hypergroup is a special case of the hypergroup described in Example 5. The group (G, \cdot) is \mathbb{Z}_2 and the cycle σ is a transposition.

Result 3. Excluding the hypergroups (H, \circ) such that $|H| = 4$ and $|H/\beta| = 1$, in the class \mathfrak{C} there are 8154 hypergroups of size $n \leq 4$, apart from isomorphisms.

We complete this section by showing a result concerning the weakly locally 1-finite hypergroups.

Theorem 5. If $(H, \circ) \in \mathfrak{D} - \mathfrak{C}$, then $|H| \geq 5$.

Proof. By hypothesis, there is a class $\beta(x)$ different from ω_H such that $h(\beta(x)) = 1$ and $h(\omega_H) \geq 2$. Because of the inclusions $\mathfrak{U} \subset \mathfrak{A} \subset \mathfrak{C}$, we have $|\omega_H| \geq 2$ and $|\beta(x)| \geq 2$, otherwise $(H, \circ) \in \mathfrak{C}$. Now, if we suppose that $|\omega_H| = 2$, by Proposition 1, we obtain $2 \leq h(\omega_H) \leq |\omega_H| = 2$ and so $h(\omega_H) = |\omega_H| = 2$. Consequently, with the help of Theorem 2, we have the contradiction $(H, \circ) \in \mathfrak{U} \subset \mathfrak{C}$. Hence, $|\omega_H| \geq 3$, $|\beta(x)| \geq 2$ and $|H| \geq 5$. \square

Recall that the hypergroup (D, \circ) shown in Example 6 belongs to \mathfrak{D} but not to \mathfrak{C} due to the previous theorem that the hypergroup has the smallest cardinality, among all hypergroups sharing that property.

5. Conclusions

In hypergroup theory, the relation β is the smallest strongly regular equivalence relation whose corresponding quotient set is a group. If (H, \circ) is a hypergroup and $\varphi : H \rightarrow H/\beta$ is the canonical projection, then the kernel $\omega_H = \varphi^{-1}(1_{H/\beta})$ is the hearth of (H, \circ) . If the hearth is a singleton, then (H, \circ) is a 1-hypergroup. We remark that the hearth is a β -class and also a subhypergroup of (H, \circ) . In particular, if $\omega_H = \{e\}$, then we have $\omega_H = e \circ e$. More generally, every β -class is the union of hyperproducts of pairs of elements of H . In this work, we defined the height of a β -class as the minimum number of such hyperproducts. This concept yields a new characterization of 1-hypergroups, see Theorem 2, and allows us to introduce new hypergroup classes, depending on the relationship

between height and cardinality of the β -classes; see Definition 4. These classes include 1-hypergroups as particular cases. Apart from isomorphisms, we were able to enumerate the elements of those classes when $|H| \leq 4$, with only one exception. In fact, the problem of enumerating the non-isomorphic hypergroups where $|H| = 4$, $|H/\beta| = 1$ and $h(\omega_H) = 1$ remains open.

In conclusion, as a direction for further research, we point out that many hypergroups that arose in the analysis of the hypergroup classes introduced in the present work are join spaces or transposition hypergroups [15,31]. For example, the 10 hypergroups of size three in Table 1 are transposition hypergroups. Transposition hyperstructures are very important in hypercompositional algebra. Hence, it would be interesting to enumerate the join spaces or the transposition hypergroups belonging to the hypergroup classes introduced in Definition 4, at least for small cardinalities. Another question that is stimulated by the concept of height concerns the height of the β -classes of the coset hypergroups, i.e., the hypergroups that are quotient of a non-commutative group with respect to a non-normal subgroup [32]. We leave these observations and suggestions as a possible subject for new works.

Author Contributions: Conceptualization, formal analysis, writing—original draft: M.D.S., D.F. (Domenico Freni), and G.L.F.; software, writing—review and editing: D.F. (Dario Fasino). All authors have read and agreed to the published version of the manuscript.

Funding: This research has been carried out in the framework of the departmental research projects “Topological, Categorical and Dynamical Methods in Algebra” and “Innovative Combinatorial Optimization in Networks”, Department of Mathematics, Computer Science and Physics (PRID 2017), University of Udine, Italy. The work of Giovanni Lo Faro has been partly supported by INdAM-GNSAGA, and the work of Dario Fasino has been partly supported by INdAM-GNCS.

Conflicts of Interest: The authors declare no conflict of interest.

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