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Noether's Theorem in Non-Local Field Theories

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Abstract: Explicit expressions are constructed for a locally conserved vector current associated with a continuous internal symmetry and for energy-momentum and angular-momentum density tensors associated with the Poincaré group in field theories with higher-order derivatives and in non-local field theories. We consider an example of non-local charged scalar field equations with broken C (charge conjugation) and CPT (charge conjugation, parity, and time reversal) symmetries. For this case, we find simple analytical expressions for the conserved currents.

Keywords: non-local field theories; Noether's theorem; internal symmetry; energy-momentum; angular-momentum; Poincaré group; charged scalar field; broken symmetries; CPT violation

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1. Introduction

According to Noether's theorem [1], the invariance of the Lagrangian function of a physical system with respect to continuous transformations leads to conservation laws and the corresponding existence of conserved charges. In its standard form, Noether's theorem refers to local field theories with derivatives, of no higher than second order, in the field equations.

Quantum field theories with higher derivatives are used for intermediate regularization procedures (see, e.g., [2]). The low-energy regime of quantum chromodynamics (QCD) is known to be successfully described by chiral perturbation theory based on a power-series expansion in derivatives [3,4]. Infinite higher-order derivatives, in the form of an infinite set of ordinary differential equations, appear in the treatment of the delay-time problem of the electromagnetic radiation-reaction [5,6], which can be considered as an example of non-local theory. Field theories with higher-order derivatives or non-local theories are also discussed in the context of general relativity [7,8]. In solid state physics, non-local field theories are successfully used to describe non-local interactions of atoms on scales up to the lattice parameter [9]. A conserved current associated with pairs of non-Noether or non-local symmetries is constructed in [10].

The charge conjugation, parity, and time reversal (CPT) theorem states that the CPT symmetry violation can be related to non-local interactions. Low-energy nuclear and atomic experiments provide strict constraints on the scale of a possible violation of CPT symmetry. A simple classification of the effects of the violation of the C, P, and T symmetries and their combinations is presented by Okun [11]. A class of inflationary models is based on a non-local field theory [12].

In this paper, the question of whether one can generalize Noether's theorem to non-local field theory is discussed.

As an initial step, we consider a Lagrangian that contains, along with a field $\Psi = (\phi, \phi^*)$, its higher derivatives $\partial_{\mu_1} \ldots \partial_{\mu_n} \Psi$, up to order $n \geq 1$:

$$\mathcal{L} = \mathcal{L}(\Psi, \partial_{\mu_1}\Psi, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_n}\Psi). \tag{1}$$

The Lagrangian given in (1) is still local as it is a function of the field and its finite-order derivatives are evaluated at a single point in space-time. To obtain a non-local field theory, one must include in (1) a dependence on an infinite number of field derivatives, i.e., by considering the limit $n \to \infty$.

In the remainder of this paper, we use a system of units such that $\hbar = c = 1$. Indices $\mu,\ \nu, ...,$ denoted by Greek letters from the middle of the alphabet, run from 0 to 3. Indices $\alpha,\ \beta, ...$ denote the spatial components of tensors and run from 1 to 3. We use a time-like metric $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, and indices enumerating members of internal-symmetry multiplets are suppressed.

2. Symmetries and the Conserved Currents

Any observable quantity can be expressed in terms of fields and their certain combinations. In general, the fields that appear in the Lagrangian belong to a representation space of the internal symmetry group. Linear transformations of the fields related to the internal symmetry group do not affect physical quantities, which is the case considered in the present paper. Thus, for infinitesimal transformations related to the internal symmetries, one can write the transformation matrix as follows:

$$U(\omega) = 1 - i\omega^a T^a, \tag{2}$$

where the ω^a are a set of infinitesimal real parameters and the T^a are generators of group transformations. If the matrix U is unitary, then the T^a are Hermitian matrices. For the $U(1)$ symmetry group, $T^a = 1$, and for $SU(2)$, the T^a are the Pauli matrices.

Along with the internal symmetry of a physical system, in the general case, one must consider the existence of external symmetries that are related to the invariance of physical quantities with respect to translations and the Lorentz transformations. Invariance under space-time translations leads to energy-momentum conservation, whereas the Lorentz invariance gives rise to the conservation of angular momentum. For an infinitesimal element of the Lorentz group, coordinate transformations can be realized by means of the matrix $a^\mu_\nu = \delta^\mu_\nu + \varepsilon^{\mu\cdot}_{\cdot\nu}$, where $\varepsilon^{\mu\nu}$ is an infinitesimal antisymmetric tensor. This implies that the infinitesimal Lorentz transformation matrix in the representation space of the field can be written in the most general form as follows:

$$S(a) = 1 - \frac{i}{2}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}, \tag{3}$$

where $\Sigma^{\mu\nu}$ is a matrix defined by the transformation properties of the field. Thus, the complete transformations of the coordinates and the field corresponding to internal and external symmetries can be expressed in matrix notation as follows:

$$x' = ax + b, \tag{4}$$
$$\Psi'(x') = U(\omega)S(a)\Psi(x), \tag{5}$$

where the notation used in Equation (4) dictates a particular order of the transformations, namely, translation is performed after the Lorentz transformation. In the opposite case, one must use $x' = a(x + b)$. In the particular case of a scalar field, we obtain a simple expression $\phi'(x') = \phi(x)$, with ϕ being a scalar with respect to the internal symmetries group and the Lorentz transformations.

The field $\Psi(x)$ in general belongs to a nontrivial representation of the internal symmetry group and a Poincaré group representation.

For the infinitesimal parameters ω^a, $\varepsilon^{\mu\nu}$, and b^μ, the variation of the field takes the form

$$\begin{aligned}\delta\Psi(x) &= \Psi'(x) - \Psi(x) \\ &= S(a)U(\omega)\Psi(a^{-1}x - b) - \Psi(x) \\ &= \left(-i\omega^a T^a\right)\Psi(x) - b^\mu\partial_\mu\Psi(x) - \frac{i}{2}\varepsilon^{\mu\nu}\left(\Lambda_{\mu\nu} + \Sigma_{\mu\nu}\right)\Psi(x).\end{aligned} \tag{6}$$

The intrinsic symmetry generates variation $\delta\Psi = (\delta\phi,\ \delta\phi^*)$ with $\delta\phi = -i\omega^a t^a\phi$ and $\delta\phi^* = i\omega^a\tilde{t}^a\phi^*$. We thus use $T^a = (t^a, -\tilde{t}^a)$. The spin generators act as $\delta\phi = -\frac{i}{2}\varepsilon_{\mu\nu}\sigma^{\mu\nu}\phi$ and $\delta\phi^* = \frac{i}{2}\varepsilon_{\mu\nu}\sigma^{\mu\nu*}\phi^*$. The rotation operators are defined by $\Lambda_{\mu\nu} = (\mathcal{R}_{\mu\nu}, -\mathcal{R}_{\mu\nu})$ where $\mathcal{R}_{\mu\nu} = x_\mu i\partial_\nu - x_\nu i\partial_\mu$ and $\Sigma^{\mu\nu} = (\sigma^{\mu\nu}, -\sigma^{\mu\nu*})$. The order of the transformations is as follows: the matrix $U(\omega)$ is applied first, followed by the Lorentz transformations and then translation. However, the order of the matrices $S(a)$ and $U(\omega)$ is interchangeable as the transformations of internal symmetries commute with those of external symmetries. Thus, the first term on the right-hand side of Equation (6) corresponds to transformations of internal symmetries, the second corresponds to translations, and the third corresponds to the Lorentz transformations.

Returning to Equation (1) for the infinitesimal parameters ω^a, $\varepsilon^{\mu\nu}$, and b^μ, one can now write the variation of the Lagrangian, $\delta\mathcal{L}(x) = \mathcal{L}'(x) - \mathcal{L}(x)$, as

$$-b^\sigma\partial_\sigma\mathcal{L} - \varepsilon^{\sigma\nu}x_\nu\partial_\sigma\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Psi}\delta\Psi + \sum_{n\geq 1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\ldots\partial_{\mu_n}\delta\Psi. \tag{7}$$

To derive the expression for the conserved current, one must use the generalized higher-order Euler–Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\Psi} + \sum_{n\geq 1}(-)^n\partial_{\mu_n}\ldots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_n}\Psi)} = 0. \tag{8}$$

By replacing the first term on the right-hand side of Equation (7) with the corresponding expression from the Euler–Lagrange Equation (8), one can rewrite the right-hand side of Equation (7) as follows:

$$\text{r.h.s.} = -\sum_{n\geq 1}(-)^n\partial_{\mu_n}\ldots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_n}\Psi)}\delta\Psi + \sum_{n\geq 1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\ldots\partial_{\mu_n}\delta\Psi. \tag{9}$$

The purpose is to represent the expression (9), using the form of the divergence of some quantity. The n-th order term under the first summation symbol in Equation (9) can be rewritten in the form

$$\begin{aligned}&-(-)^n\partial_{\mu_n}\ldots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_n}\Psi)}\delta\Psi \\ =&-(-)^n\partial_{\mu_n}\left(\partial_{\mu_{n-1}}\ldots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_n}\Psi)}\delta\Psi\right) \\ &-(-)^{n+1}\partial_{\mu_{n-1}}\ldots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_n}\Psi)}\partial_{\mu_n}\delta\Psi.\end{aligned} \tag{10}$$

The first term has the form of a divergence, whereas in the second term, the derivative ∂_{μ_n} is shifted to the right and acts on $\delta\Psi$. By rewriting the second term of (10) in the same way,

$$\begin{aligned} &-(-)^{n+1}\partial_{\mu_{n-1}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_n}\delta\Psi \\ =&-(-)^{n+1}\partial_{\mu_{n-1}}\left(\partial_{\mu_{n-2}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_n}\delta\Psi\right) \\ &-(-)^{n+2}\partial_{\mu_{n-2}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_{n-1}}\partial_{\mu_n}\delta\Psi, \end{aligned} \tag{11}$$

we again obtain a divergence and one more derivative of $\delta\Psi$ in the second term. This implies that through such recursion, one can shift the derivative to the right until it lies immediately before $\delta\Psi$. With each such procedure, the second term in the rewritten part of the expression changes sign and the first term has the form of a divergence.

Finally, the last term in the recursion can be obtained by shifting over n derivatives; this term will have an additional sign $(-1)^n$, and consequently, it will have a sign opposite to that of the second term of Equation (9) and will therefore vanish.

Thus, the result of this procedure for the right-hand side of Equation (7) has the form

$$\begin{aligned} \text{r.h.s.} &= \sum_{k=1}^{n}\delta^{\sigma}_{\mu_k}(-)^{k+1}\partial_\sigma\left(\partial_{\mu_{k-1}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\delta\Psi\right) \\ &= \partial_\sigma\sum_{k=1}^{n}(-)^{k+1}\left(\partial_{\mu_{k-1}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_{k-1}}\partial_\sigma\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\delta\Psi\right). \end{aligned} \tag{12}$$

Finally, Equation (7) can be written fully in the form of a divergence as follows:

$$\begin{aligned} \partial_\sigma[\sum_{n\geq1}\sum_{k=1}^{n}(-)^{k+1}\left(\partial_{\mu_{k-1}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_{k-1}}\partial_\sigma\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\delta\Psi\right) \\ +b^\sigma\mathcal{L}+\varepsilon^{\sigma\nu}x_\nu\mathcal{L}]=0, \end{aligned} \tag{13}$$

where $\delta\Psi$ in the parentheses is given by Equation (6). The terms that are linear in the parameter ω^a determine the set of conserved currents $\mathfrak{J}^{a\sigma}$ related to the internal symmetry group. The terms that are proportional to the vector $-b^\sigma$ determine the conserved second-rank tensor that can be identified with the energy-momentum tensor $\mathfrak{T}^{\sigma}_{\mu}$. Finally, the terms that are proportional to the tensor $\varepsilon^{\mu\nu}$ determine the conserved third-rank tensor $\mathfrak{M}^{\sigma}_{\mu\nu}$. The spatial components of this tensor correspond to the total angular momentum density of the system. In the case of $n=1$, we obtain the standard results. The set of conserved currents takes the form

$$\begin{aligned} \mathfrak{J}^{a\sigma} = \sum_{n\geq1}\sum_{k=1}^{n}(-)^{k+1}\left(\partial_{\mu_{k-1}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_{k-1}}\partial_\sigma\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\Psi)}\right) \\ \times\partial_{\mu_{k+1}}\dots\partial_{\mu_n}(-iT^a)\Psi(x), \end{aligned} \tag{14}$$

$$\begin{aligned} \mathfrak{T}^{\sigma}_{\mu} = \sum_{n\geq1}\sum_{k=1}^{n}(-)^{k+1}\left(\partial_{\mu_{k-1}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_{k-1}}\partial_\sigma\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\Psi)}\right) \\ \times\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\partial_\mu\Psi(x)-\delta^{\sigma}_{\mu}\mathcal{L}, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathfrak{M}^{\sigma}_{\mu\nu} = \sum_{n\geq1}\sum_{k=1}^{n}(-)^{k+1}\left(\partial_{\mu_{k-1}}\dots\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_{k-1}}\partial_\sigma\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\Psi)}\right) \\ \times\partial_{\mu_{k+1}}\dots\partial_{\mu_n}(-i)\left(\Lambda_{\mu\nu}+\Sigma_{\mu\nu}\right)\Psi(x)-\left(x_\mu\delta^{\sigma}_{\nu}-x_\nu\delta^{\sigma}_{\mu}\right)\mathcal{L}. \end{aligned} \tag{16}$$

Using Noether's theorem one can find the conserved currents with accuracy within an arbitrary factor. In Equations (14)–(16), the factors are chosen in such a way that the conserved quantity $\mathfrak{T}_0^0$ coincides with the energy density defined by the Legendre transform of the Lagrangian. The quantity $\mathfrak{M}^0_{\alpha\beta}$ then coincides with the angular momentum density of the system for the spatial indices α and β. Equation (15) is in agreement with [7].

We remark that a complete rotor, whose divergence is identically zero, can always be added to the conserved Noether current to achieve another conserved current, e.g., $\mathfrak{J}^{a\sigma} \& \to \& \mathfrak{J}^{a\sigma} + \epsilon^{\sigma\tau\rho\nu}\partial_\tau A^a_{\rho\nu}$, where $\epsilon^{\sigma\tau\rho\nu}$ is the totally antisymmetric Levi-Civita tensor and $A^a_{\rho\nu}$ is an arbitrary tensor. In general, the conserved Noether current is not a gauge invariant, even if the Lagrangian is. The construction of symmetric and gauge invariant energy–momentum tensors in electrodynamics and general relativity is discussed in [7,13].

In non-local field theory, we expand non-local operators of the Lagrangian in an infinite power series over the differential operators. The conserved currents are then given by Equations (14)–(16), with the summation over n extended to $+\infty$. This method is applied below to construct the conserved currents in a non-local charged scalar field theory.

3. Non-Local Charged Scalar Field

We consider an example of a non-local charged scalar field described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\phi^* \left(i\partial_t - \sqrt{-\Delta + m^2} \right) \phi + \text{c.c.} \tag{17}$$

The particles follow a relativistic dispersion law $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$. Because of the absence of negative-frequency solutions, the particles do not have antiparticles, which leads to a violation of the Lorentz invariance. After quantization of the system (17), the field commutator $[\phi(x), \phi(y)]$ does not disappear at space-like intervals $(x-y)^2 < 0$, so signals can propagate at speeds above the speed of light, ultimately violating causality. The condition $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$ is also required for the Lorentz invariance of the T product of the field operators. The system described by Lagrangian (17) is interesting from methodological and historical points of view, since the corresponding evolution equation was considered in the past as a possible relativistic generalization of the Schrödinger equation.

Let us check whether CPT invariance holds in the non-local field theory defined by (17). First we consider the charge-conjugation operation, C. In the momentum space given by $p^\alpha = -i\nabla = -(p^\alpha)^*$, with $\alpha = 1,2,3$, we replace the particle's momenta in (17) with the generalized momenta, $p_\mu \to p_\mu - eA_\mu$. Equating the functional derivative $\delta\mathcal{L}/\delta\phi^*$ to zero, we obtain the evolution equation in an external electromagnetic field:

$$(i\partial_t - eA_0)\phi = \sqrt{(\mathbf{p} - e\mathbf{A})^2 + m^2}\, \phi. \tag{18}$$

For the complex conjugate scalar field, one has

$$(i\partial_t + eA_0)\phi^* = -\sqrt{(\mathbf{p} - e\mathbf{A})^2 + m^2}\, \phi^*. \tag{19}$$

Together with the sign reversal of the charge e in Equation (19), a negative sign appears at the root. Obviously, the charge-conjugation symmetry is broken. Violation of the C symmetry means that the properties of a particle and its corresponding antiparticle are different or, as in our case, the corresponding antiparticles do not exist.

One can easily check that the Lagrangian given in (17) is invariant under the parity transformation, P: $\phi(t, \mathbf{x}) \to \phi(t, -\mathbf{x})$. By the same analysis, one can check that the time-reversal symmetry, T: $\phi(t, \mathbf{x}) \to \phi^*(-t, \mathbf{x})$, is conserved as well. Thus, the Lagrangian of (17) is symmetric under P and T transformations; whereas, the C symmetry is broken. The combined CPT symmetry is therefore broken, which is consistent with the fact that the theory is non-local.

The Lagrangian expressed in (17) is explicitly invariant under global phase rotations of ϕ, which may imply the existence of a conserved vector current. The Lagrangian given in (17) is also explicitly invariant under space-time translations and three-dimensional rotations. We thus expect the existence of conserved energy-momentum and angular momentum tensors. The dispersion law takes a relativistic form; therefore, the field theory of (17) is apparently invariant under boost transformations. This symmetry is, however, implicit, and we do not discuss its consequences here. We thus restrict ourselves to the case of $\epsilon^{0\alpha} = 0$, $\epsilon^{\alpha\beta} \neq 0$.

We will work in terms of a power series over the derivatives. Expanding $\mathcal{L}$, one can rewrite it as follows:

$$\mathcal{L} = \frac{1}{2}\left(\phi^* i\partial_t\phi - \sum_{l=0}^{\infty} f_l(m)\phi^*\Delta^l\phi\right) + \text{c.c.}, \tag{20}$$

where

$$f_l(m) = (-1)^l \frac{\Gamma(\frac{3}{2})}{l!\,\Gamma(\frac{3}{2}-l)\,m^{2l-1}}, \tag{21}$$

such that

$$\sum_{l=0}^{\infty} f_l(m)x^l = \sqrt{m^2 - x}. \tag{22}$$

3.1. Time-Like Components

One can easily find the zeroth component of the conserved currents as the Lagrangian expressed in (20) contains only the first derivative with respect to time and there are no mixed derivatives. This implies that the series in Equations (14)–(16) are truncated at the first term of the sum. Thus, the charge density $\mathfrak{J}^0$, the energy density $\mathfrak{T}^0_\mu$ and the angular momentum density $\mathfrak{M}^0_{\alpha\beta}$ take the following simple forms:

$$\begin{aligned} \mathfrak{J}^0 &= \phi^*\phi, &(23)\\ \mathfrak{T}^0_\mu &= \frac{1}{2}\phi^* i\overleftrightarrow{\partial}_\mu\phi, &(24)\\ \mathfrak{M}^0_{\alpha\beta} &= \frac{1}{2}\phi^*\mathcal{R}_{\alpha\beta}\phi + \frac{1}{2}(\mathcal{R}_{\alpha\beta}\phi)^*\phi, &(25) \end{aligned}$$

where $\overleftrightarrow{\partial}_\mu = \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$ and $\mathcal{R}_{\alpha\beta}$ is defined following Equation (6).

We turn to momentum space, substituting into Equations (23)–(25), plane waves for outgoing and incoming particles with momenta p' and p. The four-momentum operator in coordinate space is given by $p^\mu = (E, \mathbf{p}) = (i\partial_t, -i\nabla)$. In terms of the transition matrix elements, the conserved currents (23)–(25) take the forms

$$\begin{aligned} \mathfrak{J}^0(p', p) &= 1, &(26)\\ \mathfrak{T}^0_\mu(p', p) &= \frac{1}{2}(p' + p)_\mu, &(27)\\ \mathfrak{M}^0_{\alpha\beta}(p', p) &= \frac{1}{2}\left(\hat{x}_\alpha(p' + p)_\beta - \hat{x}_\beta(p' + p)_\alpha\right), &(28) \end{aligned}$$

where $\hat{x}_\mu = -i\partial/\partial p^\mu = i\partial/\partial p'^\mu$.

To find the spatial components of the conserved currents, one must specify the action of the derivatives in expressions (14)–(16). The rules that are useful for deriving the expressions for the conserved currents are given in Appendix A.

3.2. Vector Current

Following the rules listed in Appendix A, we find the spatial components of the vector current as follows:

$$
\begin{aligned}
\mathfrak{J}^{\alpha} =& \frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=1,3,5,\ldots}^{2l-1} (\partial_{\alpha_{k-1}}\ldots\partial_{\alpha_1}\phi^*)\delta^{\alpha_1\alpha_2}\ldots\delta^{\alpha\alpha_{k+1}}\ldots\delta^{\alpha_{2l-1}\alpha_{2l}}\partial_{\alpha_{k+1}}\ldots\partial_{\alpha_{2l}}\phi \\
&-\frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=2,4,6,\ldots}^{2l} (\partial_{\alpha_{k-1}}\ldots\partial_{\alpha_1}\phi^*)\delta^{\alpha_1\alpha_2}\ldots\delta^{\alpha_{k-1}\alpha}\ldots\delta^{\alpha_{2l-1}\alpha_{2l}}\partial_{\alpha_{k+1}}\ldots\partial_{\alpha_{2l}}\phi \\
&+\text{c.c.},
\end{aligned}
\tag{29}
$$

where $f_l(m)$ is given by Equation (21) and $\delta\phi = -i\phi$ and $\delta\phi^* = i\phi^*$ for the $U(1)$ symmetry group.

Let us write Equation (29) in the lowest-order approximation. Equation (21) yields $f_1(m) = -1/(2m)$. The space-like component of the vector current $\mathfrak{J}^{\sigma}$ reduces to the standard expression

$$
\mathfrak{J}^{\alpha} = \frac{1}{2m}\phi^* i\overset{\leftrightarrow}{\partial^{\alpha}}\phi + \ldots. \tag{30}
$$

By performing contractions of the indices and with the aid of Equation (A5) from Appendix A, we obtain

$$
\begin{aligned}
\mathfrak{J}^{\alpha} =& -\frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=1,3,5,\ldots}^{2l-1} (\triangle^{(k-1)/2}\phi^*)\,\triangle^{l-(k+1)/2}\,\partial^{\alpha}\phi \\
&+\frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=2,4,6,\ldots}^{2l} (\triangle^{(k-2)/2}\partial^{\alpha}\phi^*)\,\triangle^{l-k/2}\,\phi + \text{c.c.}
\end{aligned}
\tag{31}
$$

The sum of the first two terms is real, so adding the complex conjugate expression doubles the result. After some simple algebra and with the use of Equation (A7), we obtain

$$
\mathfrak{J}^{\alpha} = \phi^* i\mathcal{D}^{\alpha}\phi, \tag{32}
$$

where

$$
\mathcal{D}^{\alpha} = \frac{\overset{\leftrightarrow}{\partial}{}^{\alpha}}{\sqrt{m^2 - (\overset{\leftarrow}{\triangle})} + \sqrt{m^2 - (\overset{\rightarrow}{\triangle})}}. \tag{33}
$$

The detailed derivation of Equation (32) is given in Appendix B. In terms of the four-dimensional operator $i\mathcal{D}^{\sigma} \equiv (1, i\mathcal{D}^{\alpha})$, the four-dimensional vector current can be written as

$$
\mathfrak{J}^{\sigma} = \phi^* i\mathcal{D}^{\sigma}\phi. \tag{34}
$$

It is useful to rewrite the vector current in the momentum space. By substituting the plane waves $\phi^*(x) \sim e^{ip'x}$ and $\phi(x) \sim e^{-ipx}$ with momenta p' and p into Equation (34) and omitting the exponential factors from the final expression, we obtain

$$
\mathfrak{J}^{\sigma}(p', p) = \left(1, \frac{\mathbf{p}' + \mathbf{p}}{E(\mathbf{p}') + E(\mathbf{p})}\right). \tag{35}
$$

On the mass shell, the vector current is conserved:

$$
\partial_{\sigma}\mathfrak{J}^{\sigma} = 0. \tag{36}
$$

The variational derivative of the action functional $S = \int d^4x\mathcal{L}$ with respect to the vector field $A_\sigma(x)$,

$$\mathfrak{J}^\sigma(x) = -\frac{\delta S}{\delta A_\sigma(x)}, \tag{37}$$

introduced into $\mathcal{L}$ with the use of minimal substitution, is associated for $A_\sigma(x) = 0$ with a vector current. Current (37) is defined off the mass shell and it coincides with the Noether current (34) on the mass shell, as shown in Appendix C. Multiplying (35) with $(p' - p)_\sigma$ yields the result

$$(p' - p)_\sigma \mathfrak{J}^\sigma(p', p) = G^{-1}(p') - G^{-1}(p), \tag{38}$$

where $G(p) = (p^0 - \sqrt{m^2 + \mathbf{p}^2})^{-1}$ is the particle propagator. This equation can be recognized as the Ward identity.

The field $\phi_s(x)$, which behaves like a true scalar under Lorentz transformations, may be defined by the equation $\phi(x) = \left(i\partial_t + \sqrt{-\Delta + m^2}\right)^{1/2} \phi_s(x)$. In terms of $\phi_s(x)$, the Lagrangian (17) takes the explicitly covariant form $\mathcal{L} = \frac{1}{2}\phi_s^*(-\Box - m^2)\phi_s + \text{c.c.}$ The non-local operator $\left(i\partial_t + \sqrt{-\Delta + m^2}\right)^{1/2}$ eliminates from $\phi_s(x)$ the negative-frequency solutions. Since the proper Lorentz transformations do not mix plane waves with the positive and negative frequencies, the classical non-local field theory (17) appears to be Lorentz covariant. The interaction preserving the covariance can be introduced, e.g., by adding to $\mathcal{L}$ the term $\lambda|\phi_s|^4$.

Equations (14)–(16) are straightforward generalizations of the Noether currents of a local field theory. Noether's theorem applied to $\mathcal{L}(\phi_s(x))$ leads, however, to conserved currents that differ from those of Equations (14)–(16). A family of the conserved currents apparently exists when non-local field transformations are permitted. The conserved vector current of $\mathcal{L}(\phi_s(x))$ takes the form $(p' + p)^\sigma/\sqrt{2E(\mathbf{p}')2E(\mathbf{p})}$. Among the conserved currents, the expression (34) is highlighted by the coincidence with (37).

3.3. Energy-Momentum Tensor

An analysis that is fundamentally identical to that presented in the previous section leads to the conserved energy-momentum tensor. Considering that $\delta^\alpha_\mu\mathcal{L} = 0$ for the fields that satisfy the equations of motion, one can rewrite Equation (15) with the Lagrangian given in (17) in the form

$$\begin{aligned}
\mathfrak{T}^\alpha_\mu = &-\frac{1}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=1,3,5,\dots}^{2l-1} (\partial_{\alpha_{k-1}}\dots\partial_{\alpha_1}\phi^*)\delta^{\alpha_1\alpha_2}\dots\delta^{\alpha\alpha_{k+1}}\dots\delta^{\alpha_{2l-1}\alpha_{2l}}\partial_{\alpha_{k+1}}\dots\partial_{\alpha_{2l}}\partial_\mu\phi \\
&+\frac{1}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=2,4,6,\dots}^{2l} (\partial_{\alpha_{k-1}}\dots\partial_{\alpha_1}\phi^*)\delta^{\alpha_1\alpha_2}\dots\delta^{\alpha_{k-1}\alpha}\dots\delta^{\alpha_{2l-1}\alpha_{2l}}\partial_{\alpha_{k+1}}\dots\partial_{\alpha_{2l}}\partial_\mu\phi \\
&+\text{c.c.}.
\end{aligned} \tag{39}$$

The lowest-order $l = 1$ term of the expansion yields

$$\mathfrak{T}^\alpha_\mu = \frac{1}{4m}\phi^* i\overleftrightarrow{\partial}^\alpha i\overleftrightarrow{\partial}_\mu\phi + \dots. \tag{40}$$

By performing contractions of the indices in Equation (39), we obtain

$$\begin{aligned}
\mathfrak{T}^\alpha_\mu = &-\frac{1}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=1,3,5,\dots}^{2l-1} (\triangle^{(k-1)/2}\phi^*)\,\triangle^{l-(k+1)/2}\,\partial^\alpha\partial_\mu\phi \\
&+\frac{1}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=2,4,6,\dots}^{2l} (\triangle^{(k-2)/2}\partial^\alpha\phi^*)\,\triangle^{l-k/2}\,\partial_\mu\phi + \text{c.c.}
\end{aligned} \tag{41}$$

Using Equation (A7), the summation in Equation (41) can be performed in the same way as for the conserved current. The energy-momentum tensor finally takes the form

$$\mathfrak{T}^{\sigma}_{\mu} = \frac{1}{2}\phi^* i\mathcal{D}^{\sigma} i \overleftrightarrow{\partial}_{\mu}\phi. \tag{42}$$

A detailed derivation of this expression for $\sigma = 1, 2, 3$ is given in Appendix B. Equation (42) defines four conserved quantities, one for each component of the translation parameter b^{σ}. In momentum space,

$$\mathfrak{T}^{\sigma}_{\mu}(p', p) = \mathfrak{J}^{\sigma}(p', p)\frac{1}{2}(p' + p)_{\mu}, \tag{43}$$

where $\mathfrak{J}^{\sigma}(p', p)$ is given by Equation (35). Using Equation (38), we obtain the conservation condition for the energy-momentum tensor on the mass shell:

$$\partial_{\sigma}\mathfrak{T}^{\sigma}_{\mu} = 0. \tag{44}$$

3.4. Angular Momentum Tensor

The conservation of angular momentum arises from the invariance of the system with respect to rotation. Taking $\Sigma_{\mu\nu} = 0$ for the charged scalar field and substituting $\delta^{\alpha}_{\mu}\mathcal{L} = 0$ into Equation (16), one can write the expression for the angular momentum density in the following form:

$$\begin{aligned}
\mathfrak{M}^{\gamma}_{\alpha\beta} &= \frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=1,3,5,\ldots}^{2l-1} (\partial_{\alpha_{k-1}}\ldots\partial_{\alpha_1}\phi^*)\delta^{\alpha_1\alpha_2}\ldots\delta^{\gamma\alpha_{k+1}}\ldots\delta^{\alpha_{2l-1}\alpha_{2l}}\partial_{\alpha_{k+1}}\ldots\partial_{\alpha_{2l}}\mathcal{R}_{\alpha\beta}\phi \\
&- \frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=2,4,6,\ldots}^{2l} (\partial_{\alpha_{k-1}}\ldots\partial_{\alpha_1}\phi^*)\delta^{\alpha_1\alpha_2}\ldots\delta^{\alpha_{k-1}\gamma}\ldots\delta^{\alpha_{2l-1}\alpha_{2l}}\partial_{\alpha_{k+1}}\ldots\partial_{\alpha_{2l}}\mathcal{R}_{\alpha\beta}\phi \\
&+ \text{c.c.}
\end{aligned} \tag{45}$$

The first terms of the series expansion are

$$\mathfrak{M}^{\gamma}_{\alpha\beta} = \frac{1}{4m}\phi^* i\overleftrightarrow{\partial}^{\gamma}\mathcal{R}_{\alpha\beta}\phi + \frac{1}{4m}(\mathcal{R}_{\alpha\beta}\phi)^* i\overleftrightarrow{\partial}^{\gamma}\phi + \ldots. \tag{46}$$

By performing contraction of the indices in Equation (45), we obtain

$$\begin{aligned}
\mathfrak{M}^{\gamma}_{\alpha\beta} =& \frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=1,3,5,\ldots}^{2l-1} (\triangle^{(k-1)/2}\phi^*)\,\triangle^{l-(k+1)/2}\,\partial^{\gamma}\mathcal{R}_{\alpha\beta}\phi \\
&- \frac{i}{2}\sum_{l=1}^{\infty} f_l(m) \sum_{k=2,4,6,\ldots}^{2l} (\triangle^{(k-2)/2}\partial^{\gamma}\phi^*)\,\triangle^{l-k/2}\,\mathcal{R}_{\alpha\beta}\phi + \text{c.c.}
\end{aligned} \tag{47}$$

The arguments presented in Appendix B enable the summation of the series in Equation (47), yielding

$$\mathfrak{M}^{\sigma}_{\alpha\beta} = \frac{1}{2}\phi^* i\mathcal{D}^{\sigma}\mathcal{R}_{\alpha\beta}\phi + \frac{1}{2}(\mathcal{R}_{\alpha\beta}\phi)^* i\mathcal{D}^{\sigma}\phi. \tag{48}$$

For $\sigma = 0$, we recover Equation (25). $\mathcal{R}_{\alpha\beta}$ is not diagonal in the momentum representation, so the momentum-space representation of $\mathfrak{M}^{\gamma}_{\alpha\beta}$ offers no significant advantages. Using the equations of motion, one can verify that

$$\partial_{\sigma}\mathfrak{M}^{\sigma}_{\alpha\beta} = 0. \tag{49}$$

The conserved currents defined by Equation (48) correspond to the space-like components of the parameter $\varepsilon^{\alpha\beta}$, which describe a rotation; thus, the conserved charges are the components of the angular momentum tensor.

4. Conclusions

In non-local field theory with an internal symmetry and symmetries of the Poincaré group there exist conserved vector current, energy-momentum, and angular momentum tensors. Expressions (14)–(16) solve explicitly the problem of finding the corresponding Noether currents in terms of infinite series of the field's derivatives.

Equations (14)–(16) were used for the construction of the conserved currents in a non-local theory of a charged scalar field with explicit symmetries of phase rotations, translations, and spatial rotations. Using combinatorial arguments, we summed the infinite series over derivatives of fields and obtained the simple analytical expressions (34), (42), and (48) for the corresponding Noether currents.

Among the possible applications of the considered formalism, transformations related to the conformal symmetry group are of particular interest.

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Abbreviations

The following abbreviations are used in this manuscript:

CPT symmetry	Charge, parity, and time reversal symmetry
QCD	quantum chromodynamics
T product	time-ordered product

Appendix A. Field Derivatives

In this section, we consider algebraic rules for the manipulation of the field's derivatives in a Minkowski space. The proofs are valid, however, in the general case of $\mathbb{R}^{m,n}$. The fields Ψ and their derivatives are not assumed to be smooth; therefore, the sequence of the differentiation operations matters. As a result,

$$\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\Psi)}$$

is not necessarily symmetric under the permutation of indices. The conserved currents (14)–(16) are then calculated using the following formulas:

$$\frac{\partial}{\partial(\partial_\mu\Psi)}\partial_\tau\Psi = \delta^\mu_\tau, \tag{A1}$$

$$\frac{\partial}{\partial(\partial_\mu\partial_\nu\Psi)}\partial_\tau\partial_\sigma\Psi = \delta^\mu_\tau\delta^\nu_\sigma, \tag{A2}$$

$$\vdots$$

$$\frac{\partial}{\partial(\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\Psi)}\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n}\Psi = \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\ldots\delta^{\mu_n}_{\nu_n}. \tag{A3}$$

In particular,

$$\frac{\partial}{\partial(\partial_\mu\partial_\nu\Psi)}\Box\Psi = g^{\mu\nu} \tag{A4}$$

and

$$(\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi)\frac{\partial}{\partial(\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi)}\Box^l\Psi = \Box^l\Psi. \tag{A5}$$

After the replacements $\Box \to \Delta$ and $g^{\mu\nu} \to \delta^{\alpha\beta}$, these formulas also hold in Euclidean space. The Euclidean versions of Equations (A1)–(A5) were used to derive Equations (29), (39), and (45).

Appendix B. Series Summation

The factor 1/2 in Equation (31) vanishes with the addition of the complex conjugate part. The result can be written in the form

$$\begin{aligned}
\mathfrak{J}^\alpha &= i\sum_{l=1}^{\infty} f_l(m) \sum_{k=1,3,5,\dots}^{2l-1} (\triangle^{(k-1)/2}\phi^*)\,\triangle^{l-(k+1)/2}\,\partial^\alpha\phi \\
&- i\sum_{l=1}^{\infty} f_l(m) \sum_{k=2,4,6,\dots}^{2l} (\triangle^{(k-2)/2}\partial^\alpha\phi^*)\,\triangle^{l-k/2}\,\phi \\
&= i\sum_{l=1}^{\infty} f_l(m)\phi^*\left((\overrightarrow{\triangle})^{l-1} + (\overleftarrow{\triangle})(\overrightarrow{\triangle})^{l-2} + \dots + (\overleftarrow{\triangle})^{l-1}\right)\partial^\alpha\phi \\
&- i\sum_{l=1}^{\infty} f_l(m)\partial^\alpha\phi^*\left((\overrightarrow{\triangle})^{l-1} + (\overleftarrow{\triangle})(\overrightarrow{\triangle})^{l-2} + \dots + (\overleftarrow{\triangle})^{l-1}\right)\phi. \\
&= i\sum_{l=1}^{\infty} f_l(m)\phi^*\left((\overrightarrow{\triangle})^{l-1} + (\overleftarrow{\triangle})(\overrightarrow{\triangle})^{l-2} + \dots + (\overleftarrow{\triangle})^{l-1}\right)\overleftrightarrow{\partial}^\alpha\phi.
\end{aligned} \tag{A6}$$

The arrows over $\triangle$ indicate the direction in which the differentiation acts. The series can be summed up using the factorization formula

$$x^l - y^l = (x-y)\sum_{k=1}^{l} x^{k-1}y^{l-k}. \tag{A7}$$

This formula allows the simplification of the expression in brackets of Equation (A6):

$$\begin{aligned}
\mathfrak{J}^\alpha &= i\sum_{l=1}^{\infty} f_l(m)\phi^*\frac{1}{(\overrightarrow{\triangle}) - (\overleftarrow{\triangle})}[(\overrightarrow{\triangle})^l - (\overleftarrow{\triangle})^l]\overleftrightarrow{\partial}^\alpha\phi \\
&= i\sum_{l=0}^{\infty} f_l(m)\phi^*\frac{1}{(\overrightarrow{\triangle}) - (\overleftarrow{\triangle})}[(\overrightarrow{\triangle})^l - (\overleftarrow{\triangle})^l]\overleftrightarrow{\partial}^\alpha\phi \\
&= i\phi^*\frac{1}{(\overrightarrow{\triangle}) - (\overleftarrow{\triangle})}[\sqrt{m^2 - (\overrightarrow{\triangle})} - \sqrt{m^2 - (\overleftarrow{\triangle})}]\overleftrightarrow{\partial}^\alpha\phi \\
&= \phi^*\frac{i\overleftrightarrow{\partial}^\alpha}{\sqrt{m^2 - (\overrightarrow{\triangle})} + \sqrt{m^2 - (\overleftarrow{\triangle})}}\phi.
\end{aligned} \tag{A8}$$

In the transition to the second line, we use the fact that $\triangle^0 = 1$. The third line is obtained with the aid of Equation (22). We thus arrive at Equation (32).

On the way we proved a useful formula

$$\sum_{l=1}^{\infty} f_l(m)\sum_{k=1}^{l} x^{k-1}y^{l-k} = -\frac{1}{\sqrt{m^2 - x} + \sqrt{m^2 - y}}. \tag{A9}$$

The sum over k in Equation (41) can be written explicitly as follows:

$$\begin{aligned}\mathfrak{T}^{\alpha}_{\mu} &= \frac{1}{2}\sum_{l=1}^{\infty} f_l(m)\phi^* \left((\overrightarrow{\triangle})^{l-1} + (\overleftarrow{\triangle})(\overrightarrow{\triangle})^{l-2} + \ldots + (\overleftarrow{\triangle})^{l-1}\right)\partial^{\alpha}\partial_{\mu}\phi \\ &- \frac{1}{2}\sum_{l=1}^{\infty} f_l(m)\partial^{\alpha}\phi^* \left((\overrightarrow{\triangle})^{l-1} + (\overleftarrow{\triangle})(\overrightarrow{\triangle})^{l-2} + \ldots + (\overleftarrow{\triangle})^{l-1}\right)\partial_{\mu}\phi + \text{c.c.} \end{aligned} \tag{A10}$$

Here, the indices are those of tensors in Minkowski space (e.g., $\partial^{\alpha} = -\partial_{\alpha}$). Using the formula given in (A9), the energy–momentum tensor can be found to be

$$\mathfrak{T}^{\alpha}_{\mu} = \frac{1}{2}\phi^* \frac{i\overleftrightarrow{\partial}^{\alpha} i\overleftrightarrow{\partial}_{\mu}}{\sqrt{m^2 - (\overrightarrow{\triangle})} + \sqrt{m^2 - (\overleftarrow{\triangle})}}\phi. \tag{A11}$$

Equation (47) leads to

$$\begin{aligned}\mathfrak{M}^{\gamma}_{\alpha\beta} &= -\frac{i}{2}\sum_{l=1}^{\infty} f_l(m)\phi^* \left((\overrightarrow{\triangle})^{l-1} + (\overleftarrow{\triangle})(\overrightarrow{\triangle})^{l-2} + \ldots + (\overleftarrow{\triangle})^{l-1}\right)\partial^{\gamma}\mathcal{R}_{\alpha\beta}\phi \\ &+ \frac{i}{2}\sum_{l=1}^{\infty} f_l(m)\partial^{\gamma}\phi^* \left((\overrightarrow{\triangle})^{l-1} + (\overleftarrow{\triangle})(\overrightarrow{\triangle})^{l-2} + \ldots + (\overleftarrow{\triangle})^{l-1}\right)\mathcal{R}_{\alpha\beta}\phi + \text{c.c.} \end{aligned} \tag{A12}$$

By writing the complex conjugate part of the expression explicitly, one can simplify the above equation using the formula as expressed in (A9):

$$\begin{aligned}\mathfrak{M}^{\gamma}_{\alpha\beta} &= \frac{1}{2}\phi^* \frac{i\overleftrightarrow{\partial}^{\gamma}}{\sqrt{m^2 - (\overrightarrow{\triangle})} + \sqrt{m^2 - (\overleftarrow{\triangle})}}\mathcal{R}_{\alpha\beta}\phi \\ &+ \frac{1}{2}(\mathcal{R}_{\alpha\beta}\phi)^* \frac{i\overleftrightarrow{\partial}^{\gamma}}{\sqrt{m^2 - (\overrightarrow{\triangle})} + \sqrt{m^2 - (\overleftarrow{\triangle})}}\phi. \end{aligned} \tag{A13}$$

Finally, substituting the expression given in (33) into the above equation and combining the result with Equation (25), we obtain (48).

Appendix C. Vector Current from the Minimal Substitution

The minimal substitution provides a gauge invariance of theory. After the minimal substitution the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}\phi^* \left(i\partial_t - A^0 - \sum_{l=0}^{\infty}(-)^l f_l(m)(\mathbf{p} - \mathbf{A})^{2l}\right)\phi + \text{c.c.}, \tag{A14}$$

Based on Equation (37), we can immediately write $\mathfrak{J}^0 = \phi^*\phi$.

The variation of S under variation of $\mathbf{A}$ can be found using arguments similar to those of Appendix B:

$$\begin{aligned}
\delta S &= -\int d^4x \sum_{l=0}^{\infty}\sum_{k=1}^{l}(-)^l f_l(m)\phi^*(\mathbf{p})^{2k-2}(-p^\alpha\delta A^\alpha - \delta A^\alpha p^\alpha)(\mathbf{p})^{2l-2k}\phi \\
&= \int d^4x \sum_{l=0}^{\infty}\sum_{k=1}^{l} f_l(m)\phi^*(\overrightarrow{\Delta})^{k-1}(i\overrightarrow{\partial}_\alpha\delta A^\alpha + \delta A^\alpha i\overrightarrow{\partial}_\alpha)(\overrightarrow{\Delta})^{l-k}\phi \\
&= \int d^4x\delta A^\alpha \left(\sum_{l=0}^{\infty}\sum_{k=1}^{l} f_l(m)\phi^*(\overleftarrow{\Delta})^{k-1} i\overleftrightarrow{\partial}_\alpha(\overrightarrow{\Delta})^{l-k}\phi\right) \\
&= -\int d^4x\delta A^\alpha\left(\phi^* i\mathcal{D}_\alpha\phi\right),
\end{aligned} \tag{A15}$$

where we integrated by parts to remove derivatives from $\delta\mathbf{A}$. The bottom line is obtained using Equations (33) and (A9).

The current (37) coincides therefore with the Noether current (34).

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