# Quantum Correction for Newton's Law of Motion

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**Abstract:** A description of the motion in noninertial reference frames by means of the inclusion of high time derivatives is studied. Incompleteness of the description of physical reality is a problem of any theory, both in quantum mechanics and classical physics. The "stability principle" is put forward. We also provide macroscopic examples of noninertial mechanics and verify the use of high-order derivatives as nonlocal hidden variables on the basis of the equivalence principle when acceleration is equal to the gravitational field. Acceleration in this case is a function of high derivatives with respect to time. The definition of dark metrics for matter and energy is presented to replace the standard notions of dark matter and dark energy. In the Conclusion section, problem symmetry is noted for noninertial mechanics.

## 1. Introduction

The problem of physics axiomatizing being one of Hilbert's problems entails a search for a unified axiom of both classical and quantum physics. In this paper, the incompleteness problem of the quantum-mechanical description of physical reality is replaced with the incompleteness problem of classical physics. The implementation of the search for a unified axiom of classical and quantum physics is also suggested through complementing classical physics. This is related to quantum physics being much richer in variables than classical physics, and complementing classical physics with hidden variables is more reasonable than doing it with quantum physics, which has been practiced for the last 100 years by numerous authors in the effort to sew classical and quantum physics together.

## 2. Why Newton's Law of Motion is a Second-Order Derivative Equation

Since 1935, the contradiction of classical and quantum mechanics, and the search for a satisfactory quantum axiom have been important problems, but the search for this quantum axiom has been unsuccessful. A more successful solution in providing the consistency of classical and quantum physics would be a unified axiom. According to Gödel's theorem, there are provisions in any theory that cannot be proven in the framework of this theory, and that no theory is complete. The axioms of any theory have not been proved, but guessed, so any system of axioms can be replaced by another. The main idea of Newton's laws in *Principia* postulates a description dynamics of mechanical systems with second-order differential equations. There are cases of describing reality using higher-order differential equations, but this is not Newtonian mechanics. It is difficult to find an inertial reference frame since there always exist random weak external fields and forces, but we can assume that an inertial reference frame theoretically exists. A possible example of non-Newtonian mechanics is quantum mechanics. A noninertial reference

frame is needed in order to add one of the most important properties of micro-objects of quantum mechanics—nonlocality. In this case, the role of nonlocal hidden variables is played by acceleration and its higher derivatives with respect to time. In a noninertial reference frame, the oscillations of two classical particles correlate since the acceleration and its high-order derivatives do not depend on their coordinates. The description of mechanical systems in noninertial mechanics is performed using high-order derivatives of differential equations. Let us assume that $q$ are coordinates of a noninertial reference frame. Then, averaged $q$ is denoted by $Q$:

$$Q = \langle q(t) \rangle = \int\limits_{-\tau}^{+\tau} \psi^* q \psi \, dt.$$

Here, $\tau$ is a time interval for averaging, and

$$\psi = \psi_0 e^{i \frac{\Delta p \Delta q + \Delta E \Delta t}{\hbar}} = \psi_0 e^{i \frac{\frac{\Delta p}{\Delta t} - F}{\frac{\hbar}{\Delta q \Delta t}}} = \psi_0 e^{i \frac{f_0}{f_Q}} \tag{1}$$

is wave function with inertial force $f_0$ dependent on high-order derivative coordinates on time; $f_Q$ corresponds with inertial forces and constant force.

Noninertial reference frames are a method for describing the influence of random fields on both the particle to be described and the observer. The transition from a noninertial reference frame to an inertial one causes a free particle to randomly oscillate, correlating with vibrations of other free particles. Transformations of noninertial reference frames differ from Galileo–Lorentz transformations by residual terms in the Taylor expansion. Then, free particles in inertial reference frames are described with uncertainty in coordinates and momentum, time, and energy, equal to the rest of the terms of the Taylor expansion. If the transformation of a noninertial reference frame to another, described by the Taylor expansion, contains a remainder term with index N, then we can say that this free particle conserves its time derivative of the N-th order. Such a free particle is described by N derivatives and preserves this state until interactions with other bodies (forces) perturb this state. If such a particle interacts with other bodies (a force acts on it), then the dynamics of such a particle is described by differential equations of the (N + 1)-th order. In other words, the influence of force adds one more derivative to the description of particle dynamics. Considering a particle in an inertial reference frame instead of noninertial ones, one either introduces inertial forces, that is, a change from higher-derivative description to the description without higher derivatives, but with inertia forces, or takes into account remainder terms of Taylor expansion.

Modern physics (both classical and quantum) is the physics of inertial reference frames. The case of a noninertial reference frame usually comes down to the introduction of inertia forces into the inertial reference frame. The use of inertia forces makes it possible to reduce the problems of the dynamics of a physical system in a noninertial reference frame to the tasks in an inertial reference frame by artificially introducing inertial forces or applying the d'Alembert principle. At the same time, an inertial reference frame does not exist in nature, since any reference frame is always influenced by infinitesimal disturbing fields or forces. In this study, we propose to consider only noninertial reference frames as real. Since the introduction of the d'Alembert principle to the present, the reality of inertia forces has been a debated issue. The question of the reality of inertia forces can be reduced to the question of the reality of inertial reference frames. How can one describe physical systems in noninertial reference frames without introducing inertia forces? The reference frame is inertial if Newton's laws hold. They postulate the description of physical systems by second-order differential equations. The rejection of higher-order derivatives of coordinates is associated with the problem of inertia forces in inertial reference frames. Thus, to answer the above question, we must consider a more general case of higher-order differential equations and expand classical

physics with a description using higher-order derivatives. The transition from an inertial reference frame to a noninertial one without introducing inertia forces means a transition from a description of physical systems of second-order differential equations to their description using higher-order differential equations. The rejection of the use of higher-order derivatives with respect to coordinates in classical Newtonian physics does not mean that they do not exist; they exist in certain cases, but this is not Newtonian physics.

In the most general case, a transformation from the noninertial reference frame to another can be expressed as:

$$Q = q(\tau, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)})$$
$$T = t.$$

Conversion of coordinates of a point particle between two noninertial reference frames, provided that $\tau$ is a time interval for averaging, is expressed as

$$Q = q(t) + \dot{q}(t)\tau + \Delta q(t),$$
$$\Delta q(t) = \sum_{k=2}^{n}(-1)^k \frac{1}{k!}\tau^k q^{(n)}(t).$$

The same holds for momentum:

$$P = p(t) + \dot{p}(t)\tau + \Delta p(t),$$
$$\Delta p(t) = \sum_{k=2}^{n}(-1)^k \frac{1}{k!}\tau^k p^{(n)}(t)$$

$$\langle P \rangle = \frac{1}{2}[p(t+\tau) + p(t-\tau)]. \tag{2}$$

Here, $\Delta q(t)$ and $\Delta p(t)$ are remainder terms of the Taylor expansion. Remainder terms $\Delta q(t)$ and $\Delta p(t)$ in a noninertial reference frame may be interpreted as coordinate and momentum uncertainties of a point particle in this reference system. In quantum mechanics, coordinate and momentum uncertainties of a microparticle obey rule

$$\Delta q(t)\Delta p(t) \geq \hbar/2. \tag{3}$$

In noninertial physics a general uncertainty relation can be introduced , as there always exist random small fields and forces influencing either the very system to be described or an observer, that is,

$$[\sum_{k=2}^{n}(-1)^k \frac{1}{k!}\tau^k x^{(k)}(t)][\sum_{k=2}^{n}(-1)^k \frac{1}{k!}\tau^k p^{(k)}(t)] \leq K/2$$

The supremum of the difference of the action function in noninertial reference frames (with higher time derivatives of the generalized coordinate) from the classical mechanics action functions (without higher derivatives) is that, in this case, higher derivatives are nonlocal additional variables that disclose the sense of the classical analogue $K$ of Planck's constant. $K$ constant defines the supremum of the influence of random fields onto the physical system and the observer. We analyze this case in terms of a noninertial reference frame. In this case, $K$ defines the supremum of the difference between a noninertial and an inertial reference frame:

$$\sup \left| S(q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)}, ...) - S(q, \dot{q}) \right| = K. \tag{4}$$

Action functions in higher derivatives of a noninertial reference frame describe physical systems dynamics and differ from the action function neglecting random fields, which are accounted for via

noninertial reference frame. In our case, a classical space is featured by an infinite number of variables, same as one of Hilbert's. In the search for a unified axiomatic, classical constant $K$ coincides with the quantum constant, i.e., Planck's constant $\hbar$. In this approach, the estimate of the Planck constant may be determined by higher derivatives playing the role of nonlocal hidden variables.

In this case the state of quantum object, we can describe

$$|\psi(t)\rangle = \left|q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)}\right\rangle = |Q(t)\rangle.$$

The transfer object from point 1 to point 2 is

$$\langle Q_1, t_1 | Q_2, t_2 \rangle = \int_{t_1}^{t_2} DQ \exp(\tfrac{i}{\hbar} L(Q)) dt.$$

The introduced function can be represented as

$$A_{Q(R)} = \left\langle q(t), \dot{q}(t), \ddot{q}(t), ..., q^{(n)}(t) | [i\hbar \tfrac{\partial}{\partial t} - H] | q(t), \dot{q}(t), \ddot{q}(t), ..., q^{(n)}(t) \right\rangle dt.$$

## 3. Stability Principle

Classical mechanics describes a stable trajectory and noninertial mechanics to add instability to random trajectories with high-order-derivative variables. The stability condition in calculations of mechanical trajectories was put forward in publications by Chetayev [1]. According to him, "stability is probably an essentially general phenomenon that has to manifest itself in the principal laws of Nature." In his opinion, stability is not a mere casualness, but rather a consequence of a system being affected by persistent infinitesimal perturbations that, no matter how small, affect the state of a mechanical system. The condition of stability usually used in mechanics can be extended to other areas of physics. In this case, the condition of stability can be named the stability principle. The stability principle is a generalization of basic fundamental physical laws, such as the least-action principle, Newton's laws, Euler–Lagrange equations, and Schrödinger's equation. Our definition of the stability condition was extended to other areas of physics. Let us define a stable state of a physical system through the stability principle.

Stability principle: State $A$ of a physical system is considered stable if it returns to the initial state after finishing the action of external factors, and the variance of the variable with itself is zero $Var(A) = \sigma_A = 0$.

We considered noninertial reference frames due the influence of the background of random fields and waves because the variance of the action function for an unstable trajectory with itself can be represented as $Var(S_r) = \sigma_{Sr} = K$, and the complex variance can be defined in the form

$$Var(S_c) = Var(S(q, \dot{q})) + iVar(S_r(q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)}, ...)) = \sigma_C = \sigma_S + i\sigma_{Sr}.$$

where variance for the classical stable trajectory is $\sigma_S = 0$; for an unstable trajectory, it is $\sigma_{Sr} = K$.

## 4. Quantum Correlations and Illusion of Superluminal Interaction

By discussing the nonlocality of entangled-state quantum correlations for observers Alice and Bob, we see: The emerging illusion of transfer from $A$ to $B$, or the interaction of entangled quantum objects in $A$ and $B$ follows from experimentally observed correlation of their states. So, it would be correct to not only negate faster-than-light interaction or transfer, but the very fact of any interaction or transfer. The existence of quantum correlations and the nonlocality of micro-object quantum states may be described by the noninertial nature of a noninertial reference. In other words, the existence of quantum nonlocality and quantum correlations means an illusion rather than the realness of any transfer or faster-than-light interaction of these objects.

Let us perform an imaginary experiment of the classical analogue of teleportation of quantum polarization states of biphotons.

For this purpose, let us consider the classical analogue of teleportation of biphoton-polarization-state quantum entanglement. A classical analogue of this situation may be considered on the example of newspapers with news printed in city $O$ and sent to cities $A$ and $B$.

If a reader in city $A$ reads the news, then the coincidence of their information with that in $B$ may be described with a nonzero correlation factor. This is so because news information in A and B correlates with a nonzero factor.

The complete match of news information could only occur provided that readers $A$ and $B$ read newspapers with the same title and of the same date.

If the newspapers are different but both of the same date, then the correlation factor is not unity, but at the same time, it is not zero. To achieve the complete match of news information with correlation-factor unity, reader $A$ advises reader $B$ on both the title and date of the newspaper.

To provide teleportation of biphoton quantum states from $A$ to $B$, we may consider a primary photon that, with the aid of a nonlinear crystal (e.g., $BBO$), is split into two photons in point $O$ with vertical $H$ and horizontal $V$ polarizations. Photon $B$ may be compared with photon $C$, entangled with photon $D$. Therefore, in points $A$ and $D$, measurements of polarizations of the photons always coincide.

Let us repeat the proof of Bell's theorem incorporating influences of any random fields, waves, or forces onto both particles $A$ and $B$, and the observers. We consider here a noninertial reference frame. We may consider that, in the inertial reference frame, these particles are influenced by random inertia forces that, due to the equivalence principle, can be described by random metrics.

## 5. Quantum Correction to Newton's Second Law

Ostrogradsky formalism [2] using a Lagrange function is

$$L = L(q, \dot{q}, \ddot{q}, ..., q^{(n)}, ...),$$

but not

$$L = L(q, \dot{q}).$$

The Euler–Lagrange equation in this case follows from the least-action principle [3–6]:

$$\delta S = \delta \int L(q, \dot{q}, \ddot{q}, \dddot{q}, ..., q^{(n)}) dt = \int \sum_{n=0}^{N} (-1)^n \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q^{(n)}} \right) \delta q^{(n)} dt = 0.$$

Alternatively,

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} - \frac{d^3}{dt^3} \frac{\partial L}{\partial \dddot{q}} + ... + (-1)^n \frac{d^n}{dt^n} \frac{\partial L}{\partial q^{(n)}} + ... = 0.$$

This equation can be written in the form of a corrected Newton's second law of motion in noninertial reference frames:

$$F - ma + f_0 = 0.$$

Here,

$$f_0 = mw = w(t) + \dot{w}(t)\tau + \sum_{k=2}^{n} (-1)^k \frac{1}{k!} \tau^k w^{(n)}(t)$$

is a random inertial force (1) that can be represented by Taylor expansion with high-order derivatives coordinates on time

$$F - ma + \tau m\dot{a} - \frac{1}{2}\tau^2 ma^{(2)} + ... + \frac{1}{n!}(-1)^n \tau^n ma^{(n)} + ... = 0$$

in inertial reference frame $w = 0$.

## 6. Dark Metric for Matter and Energy

From [7], it follows that the phase space of coordinates and high-order derivatives gives the corrected Newton's formula for gravitational potential

$$\varphi = GM \exp s/r,$$

where $\varphi$, potential; $G$, gravitational constant; $s = -2GM$, constant; and $M$, mass.

On the one hand, force $F$ is expressed using infinite Taylor expansion. On the other hand, gravitational force $F_g$ can also be represented as a series, as follows from the principle of equivalence. If this series is replaced by an exponential [7], then we can write metric

$$ds^2 = \exp(-\tfrac{r_0}{r})dt^2 - \exp(\tfrac{r_0}{r})dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,$$

which we call the dark metric [8], where $r_0 = 2GM$.

The dark metric is the asymptotic of the Schwarzschild metric for $r_0 < r$ [9,10]. The definition of dark metrics for matter and energy is presented to replace the standard notions of dark matter and dark energy. The dark metric can also be obtained from the standard metric:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2.$$

Conditions $A(r)B(r) = 1$ and $\lim A(r) = B(r) = 1$ for $r \to \infty$ must be satisfied for the standard metric. The dark metric also satisfies to these conditions. Gravitational forces are presented as a series with changing signs.

## 7. Macroexamples of Noninertial Mechanics

The behavior of macroscopic mechanical systems in noninertial reference frames can be described by higher-order differential equations. Here, we consider the case when the contribution of higher derivatives is small compared to lower ones. Therefore, at this stage, we restricted ourselves to only the third derivatives of the coordinates with respect to time. There are many examples of the description of mechanical systems in noninertial reference frames [3–6] due to the influence of the backgrounds of random fields and waves. Theoretical descriptions of such cases do not always fully describe the physical reality of processes occurring in this process. Such cases include Kapica's pendulum, the movement of bulk materials upwards, against the action of gravity, and Chalomey's pendulum [11]. For describing vibrating mechanical systems, the principle of least action is traditionally used to obtain critical states of mechanical systems. All such cases are described by second-order differential equations. In this case, the direction of the resultant force remains uncertain. This is the main disadvantage of this method of description. Using the extended Newton's second law [9]

$$F - ma + \tau m\dot{a} - \frac{1}{2}\tau^2 ma^{(2)} + ... + \frac{1}{n!}(-1)^n \tau^n ma^{(n)} + ... = 0, \tag{5}$$

where $\tau = 1/\omega$ is the averaging time during the transition from the micro- to the macroworld, which is inverse to the average cyclic frequency, we obtain the direction of the resultant force that coincides with the direction of the motion. In [9], the behavior of such systems is described by introducing experimental vibration forces. The introduction of vibration forces in these cases is not justified and is axiomatically introduced.

Here, we use a third-order differential equation. This allows to first obtain the correct direction of the resultant force. Second, it explains its occurrence and does not contradict already known descriptions.

Comparing the two descriptions: the differential equations of the second and third order can argue the consistency of these two descriptions. Indeed, in mathematics, there is a method of transition from

higher-order differential equations to lower ones by changing variables. In our case, from a third-order differential equation, we can go to two equations of an order not higher than the second.

For example, consider the description of Kapica's pendulum using differential Equation (6), limiting ourselves to the third order of the derivative of the coordinate with respect to time

$$F - ma + \tau m\dot{a} = 0. \tag{6}$$

Or

$$F = ma - mj\tau,$$

where $j = \dot{a} = \frac{d^3 q}{dt^3}$ is third-order derivative coordinate $q$ on the time named Jerk and $\tau = 1/\omega$ is the averaging time during the transition from the micro- to the macroworld, the opposite of average cyclic frequency.

Using the substitution, we get

$$F + V = ma, \tag{7}$$

where vibration force $V$ is equal to

$$V = mA\omega^2 \sin \omega t. \tag{8}$$

Thus, we showed that Equation (6) can be replaced by two others, Equations (7) and (8). In this case, the description with high-order derivatives of mechanical systems is more complete than the description with second-order derivatives [12–23].

## 8. Verifications of High-Order Derivatives as Nonlocal Hidden Variables

The role of high-order derivatives as hidden variables can be verified by using the equivalence principle when acceleration is equal to the gravitational field. Then, the correlation factor for entangled photons polarization measurements may be presented as

$$|M| = |\langle AB \rangle| = \left| \left\langle (\lambda^i A^k g_{ik})(\lambda^m A^n g_{mn}) \right\rangle \right|. \tag{9}$$

Here, the random-variable distribution function may be considered uniform, with photon polarization varying from 0 to $\pi$:

$$\frac{1}{\pi} \int_0^\pi \rho(\phi) d\phi = 1.$$

According to the definition,

$$\cos \phi = \frac{\lambda^i A^k g_{ik}}{\sqrt{\lambda^i \lambda_i} \sqrt{A^k A_k}},$$

$$\cos(\phi + \theta) = \frac{\lambda^n B^n g_{mn}}{\sqrt{\lambda^m \lambda_m} \sqrt{B^m B_n}}.$$

Hence, correlation factor is

$$|M| = \left| \frac{1}{\pi} \int_0^\pi \rho(\phi) \cos \phi \cos(\phi + \theta) d\phi + \frac{1}{\pi} \int_\pi^{2\pi} \rho(\phi) \cos \phi \cos(\phi + \theta) d\phi \right| = |\cos \theta|.$$

Bell's observable differs in our case from that calculated by Bell and does not contradict experiment data. Bell's inequality is not violated in either classical or quantum cases of accounting for random fields, forces, and waves.

## 9. Conclusions

Contemporary physics, both classical and quantum, requires a notion of inertial reference frames. However, to find a physical inertial frame in a reality where there always exist random weak forces, we suggest a description of the motion in noninertial reference frames by means of inclusion of higher time derivatives. They may play the role of nonlocal hidden variables in a more general description, and can be named noninertial mechanics, complementing both classical and quantum mechanics. In inertial reference frames, the derivatives of Lagrange function $L = L(q, \dot{q})$ with respect to time, coordinate, and angle are not equal to zero. This means a violation of conservation laws and symmetries in noninertial reference frames. To preserve the symmetries in noninertial reference frames, one may use generalized Lagrange function $L = L(q, \dot{q}, \ddot{q}, ..., q^{(n)}, ...)$. Then, conservation laws are satisfied for energy, momentum, and angular momentum that not only depend on coordinates and velocity, but also on acceleration and high time derivatives of coordinates. The dynamics of mechanical systems in noninertial reference frames can be described by differential equations above the second. In this particular case, for example, when using a third-order differential equation by the method of variable replacement, it can be represented by two second-order equations. In the general case, noninertial dynamics can be described by high order differential equations. From the principle of equivalence, it follows that the gravitational force also has to be represented as a series. The corresponding metric is called the dark metric. The dark metric describes gravitational interaction with additional terms that lead to the description of observable effects of dark matter and dark energy. This means that the correct calculation using the dark metric leads to an abandonment of notions of dark matter and dark energy. The correctness of the results presented in this work was confirmed by comparative analysis of the use of the results of mechanics of high-order derivatives and experiment results of Bell's observables.

**Conflicts of Interest:** The authors declare no conflict of interest.

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