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On Quantum Duality of Group Amenability

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Abstract: In this paper, we investigate the co-amenability of compact quantum groups. Combining with some properties of regular C*-norms on algebraic compact quantum groups, we show that the quantum double of co-amenable compact quantum groups is unique. Based on this, this paper proves that co-amenability is preserved under formulation of the quantum double construction of compact quantum groups, which exhibits a type of nice symmetry between the co-amenability of quantum groups and the amenability of groups.

Keywords: compact quantum group; quantum duality; amenability; co-amenability; quantum double construction; Haar integral

MSC: 46L05; 46L65; 46L89

1. Introduction

Given a compact group G, denoted by $C(G)$ the C*-algebra of continuous functions on G, one can define a morphism

$$\Delta : C(G) \to C(G) \otimes C(G),$$

by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$, where $f \in C(G), g_1, g_2 \in G$, and $C(G) \otimes C(G)$ is naturally identified with $C(G \times G)$, which satisfies the co-associativity

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta.$$

The morphism Δ is called a co-multiplication on $C(G)$, under which the pair $(C(G), \Delta)$ comes into being a compact quantum group defined in the sense of Woronowicz [1].

Definition 1 ([1])**.** *Assume that A is a C*-algebra with an identity and $\Delta : A \to A \otimes A$ is a unital *-homomorphism satisfying the following two relationships,*

(i) $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$,
(ii) *the linear spans of $(1 \otimes A)\Delta(A)$ and $(A \otimes 1)\Delta(A)$ are each equal to $A \otimes A$.*

Then, the pair (A, Δ) is called a compact quantum group (CQG).

For an arbitrary CQG (A, Δ), by [2], there exists a unique state h_A on A so that for all $a \in A$,

$$(id \otimes h_A)\Delta(a) = (h_A \otimes id)\Delta(a) = h_A(a)1,$$

which is called the Haar integral of (A, Δ). For the commutative CQG $(C(G), \Delta)$ associated to a classical compact group G described as above, the Haar integral $h_{C(G)}$ is the integral with respect to the Haar measure on G, which has full support and, therefore, is faithful. However, the Haar integral on an arbitrary CQG (A, Δ) needs not be always faithful. Each CQG (A, Δ) has a canonical dense Hopf *-subalgebra (A_0, Δ_0) linearly spanned by matrix entries of all finite dimensional co-representations of (A, Δ), where Δ_0 is given by restricting the co-multiplication Δ from A to A_0. In the article, we call (A_0, Δ_0) the associated algebraic CQG of (A, Δ) ($algCQG$).

Let Γ be a discrete group, and let $C_r^*(\Gamma)$ and $C^*(\Gamma)$ be its reduced and full group C*-algebras. Γ is called amenable if there exists an invariant mean on $L^\infty(\Gamma)$. Endowed with co-multiplications Δ_r and Δ, $(C_r^*(\Gamma), \Delta_r)$ and $(C^*(\Gamma), \Delta)$ come into being $CQGs$, which are called reduced and universal CQG, respectively. The Haar integral of $(C_r^*(\Gamma), \Delta_r)$ is faithful, but that of $(C^*(\Gamma), \Delta)$ may not be; the co-unit of $(C^*(\Gamma), \Delta)$ is norm-bounded, but that of $(C_r^*(\Gamma), \Delta_r)$ may not be. From [3], the Haar integral of $(C^*(\Gamma), \Delta)$ is faithful if and only if the co-unit of $(C^*(\Gamma), \Delta)$ is norm-bounded if and only if Γ is amenable. Under what conditions is the Haar integral on a CQG faithful and the co-unit norm-bounded? In [3], Bédos, Murphy, and Tuset defined the co-amenability of CQG, which can induce the faithfulness of its Haar integral and the norm-boundness of its co-unit. As the quantum dual of group amenability, $(C_r^*(\Gamma), \Delta_r)$ is co-amenable if and only if Γ is amenable. Denote $C[\Gamma]$ the group algebra of Γ equipped with its canonical Hopf *-algebra structure. By [3], $C_r^*(\Gamma)$ and $C^*(\Gamma)$ are the CQG completions of $C[\Gamma]$. Under what conditions, for an arbitrary $algCQG$ (A_0, Δ_0), is the CQG completion of (A_0, Δ_0) unique? Generally, it is not unique. However, in the co-amenable case, the answer is affirmative [3]. Moreover, in [4,5], Bédos, Murphy, and Tuset studied the amenability and co-amenability of algebraic quantum groups, a sufficient large quantum group class including $CQGs$ and discrete quantum groups($DQGs$), which admits a dual that is also an algebraic quantum group.

In the group case, a product of two discrete amenable groups is amenable; as a quantum counterpart, co-amenability is preserved under formulation of the tensor product of two $CQGs$ [3]. In [6], we constructed the reduced and universal quantum double of two dually paired $CQGs$. Since the tensor product of two $CQGs$ is a special case of quantum double of $CQGs$ when the pairing is trivial, inspired by the underlying stability of co-amenability of $CQGs$ and the symmetrical idea, in the article, we will focus on studying the stability of the co-amenability in the process of quantum double constructions. In Section 2, we first recall the definition of co-amenability of compact quantum groups, as well as some related properties, and then briefly present the quantum double construction procedure. By symmetric calculations, as used in the case of the group amenability, in Section 3, we show that the quantum double of $CQGs$ is unique when the paired $CQGs$ are both co-amenable and that co-amenability is preserved under formulation of the quantum double constructions of $CQGs$. Using this result, one can yield a co-amenable new CQG from a pair of co-amenable $CQGs$.

In the article, all algebras are considered over the complex field $\mathbb{C}$. For the details on $CQGs$ and C*-norms, we refer to [6–13]; and for the general conclusions for pairing and quantum double, we refer to [2,6,14–17]. In our proofs, we make use of a large quantity of calculations by the standard Sweedler notation.

2. Preliminaries

In this section, we first recall the definition of co-amenability of $CQGs$ and some of its properties.

Let (A, Δ) be a CQG, (A_0, Δ_0) be the associated $algCQG$ of (A, Δ), and h the Haar integral of (A, Δ). As is well known, h is faithful on (A_0, Δ_0) but need not be faithful on the C*-algebra (A, Δ). Set

$$A_r = A/N_h,$$

where N_h is the left kernel of h. Then, A_r becomes a CQG, where its co-multiplication Δ_r is defined as

$$\Delta_r(\eta(a)) = (\eta \otimes \eta)\Delta(a),$$

for all $a \in A$, where $\eta : A \to A_r$ is the canonical map. (A_r, Δ_r) is called the reduced quantum group of (A, Δ), where its co-unit ε_r, antipode S_r, and Haar state h_r are determined by

$$\varepsilon = \varepsilon_r \circ \eta, \quad \eta \circ S = S_r \circ \eta, \quad h = h_r \circ \eta,$$

respectively. What needs to be pointed out is that the co-unit ε_r of (A_r, Δ_r) is faithful. However, generally, the co-unit ε_r needs not be norm-bounded.

Definition 2 ([3]). *A CQG (A, Δ) is called co-amenable if the co-unit ε_r of (A_r, Δ_r) is norm-bounded, where (A_r, Δ_r) is the reduced quantum group of (A, Δ).*

With the following proposition, one can obtain the co-amenability of (A, Δ) without reference to the reduced quantum group (A_r, Δ_r).

Proposition 1 ([3]). *Let (A, Δ) be a CQG, and h and ε be its Haar integral and co-unit, respectively. Then, (A, Δ) is co-amenable if and only h is faithful and ε is norm-bounded.*

Assume that (A, Δ) and (A_0, Δ_0) are described as above. Let $\|\cdot\|_c$ be a C*-norm on (A_0, Δ_0), and let (A_c, Δ_c) be a compact quantum group completion of (A_0, Δ_0). $\|\cdot\|_c$ is called regular on A_0, if it is the restriction to A_0 of the C*-norm on (A_c, Δ_c). Define $\|\cdot\|_u$ on A_0 as

$$\|a\|_u := \sup_{\pi} \|\pi(a)\|,$$

where the variable π travels over all unital *-representations π of A_0. It is not difficult to find that $\|\cdot\|_u$ is the greatest regular C*-norm on A_0. Denote A_u as the C*-algebra completion of A_0 with respect to $\|\cdot\|_u$ and Δ_u the extension to A_u of Δ. Then, (A_u, Δ_u) is a *CQG*, which is called the universal quantum group of (A, Δ). Define $\|\cdot\|_r$ on A_0 as

$$\|a\|_r := \|\eta(a)\|,$$

for all $a \in A_0$, which is the least regular C*-norm on A_0. Then, the underlying A_r is the C*-algebraic completion of A_0 with respect to $\|\cdot\|_r$.

Proposition 2 ([3]). *Let (A, Δ) be a CQG, (A_0, Δ_0) be the associated algCQG of (A, Δ), and $\|\cdot\|_c$ a regular C*-norm on A_0. Then,*

(i) *For all $a \in A_0$,*

$$\|a\|_r \leq \|a\|_c \leq \|a\|_u.$$

(ii) *(A, Δ) is co-amenable if and only if*

$$(A, \Delta) = (A_u, \Delta_u) = (A_r, \Delta_r).$$

Now, we recall the procedure of quantum double construction for *CQGs* simply exhibited in [11].

Definition 3. *Let (A, Δ_A) and (B, Δ_B) be two dully paired CQGs, and let (A_0, Δ_{A_0}) and (B_0, Δ_{B_0}) be the associated algCQGs.*

(1) *Let A_0 and B_0 be two algCQGs, and $< \cdot, \cdot >: A_0 \otimes B_0 \longrightarrow \mathbb{C}$ be a bilinear form. Assume that they satisfy the relations*

$$\langle \Delta(a), b_1 \otimes b_2 \rangle = \langle a, b_1 b_2 \rangle, \quad \langle a_1 \otimes a_2, \Delta(b) \rangle = \langle a_1 a_2, b \rangle, \quad \langle a^*, b \rangle = \overline{\langle a, S_{B_0}(b)^* \rangle},$$

$$\langle a, 1_{B_0} \rangle = \varepsilon_{A_0}(a), \quad \langle 1_{A_0}, b \rangle = \varepsilon_{B_0}(b), \quad \langle S_{A_0}(a), b \rangle = \langle a, S_{B_0}(b) \rangle,$$

for all $a_1, a_2, a \in A_0, b_1, b_2, b \in B_0$, where $\varepsilon_{A_0}, S_{A_0}$(resp.$\varepsilon_{B_0}, S_{B_0}$) denote the co-unit and antipode on A_0(resp.B_0), respectively. Then, $(A_0, B_0, \langle\cdot,\cdot\rangle)$ is called an algebraic compact quantum group pairing.

(2) *Let $\langle\cdot,\cdot\rangle : A \otimes B \longrightarrow \mathbb{C}$ is a bilinear form. If $(A_0, B_0, \langle\cdot,\cdot\rangle|_{A_0 \otimes B_0})$ is an algebraic compact quantum group pairing, then the bilinear form is called a compact quantum group pairing, denoted by $(A, B, \langle\cdot,\cdot\rangle)$.*

Let (A, Δ_A) and (B, Δ_B) be two dually paired $CQGs$, and let (A_0, Δ_{A_0}) and (B_0, Δ_{B_0}) be described as above. Denote by $A_0 \odot B_0$. It is well known that $A_0 \odot B_0$, the algebraic tensor product of A_0 and B_0, can be made into a linear space in a natural way. Under the multiplication map, m_D and involution $*_D$ on $A_0 \odot B_0$ defined as the following:

$$m_D((a,b)(a',b')) := \sum_{(a')(b)} (aa'_{(2)}, b_{(2)}b')\langle a'_{(1)}, S_{B_0}^{-1}(b_{(3)})\rangle\langle a'_{(3)}, b_{(1)}\rangle,$$

$$*_D(a,b) := \sum_{(a)(b)} (a^*_{(2)}, b^*_{(2)})\langle a^*_{(3)}, b_{(1)}\rangle\langle a^*_{(1)}, S^*_{B_0} b_{(3)}\rangle \triangleq (a,b)^*,$$

where $(a,b), (a',b') \in A_0 \odot B_0$, $A_0 \odot B_0$ turn into a non-degenerate associative $*-$algebra, which is similar to the classical Drinfeld's quantum double [18] in the pure algebra level, and then we denote it by $D(A_0, B_0)$. To avoid using too many brackets, we will simplify $m_D((a,b)(a',b'))$ as $(a,b)(a',b')$ and simplify $S(a)$ as Sa in sequel.

Under the structure maps,

$$\Delta_{D_0}(a,b) := \sum_{(a)(b)} (a_{(1)}, b_{(1)}) \otimes (a_{(2)}, b_{(2)}),\ \varepsilon_{D_0}(a,b) := \varepsilon_{A_0}(a)\varepsilon_{B_0}(b),$$

$$S_{D_0}(a,b) := \sum_{(a)(b)} (S_{A_0}a_{(2)}, S_{B_0}b_{(2)}) < a_{(1)}, S_{B_0}b_{(3)} >< a_{(3)}, b_{(1)} > .$$

$D(A_0, B_0)$ forms a Hopf $*$-algebra. Furthermore, we have:

Proposition 3. *$(D(A_0, B_0), \Delta_{D_0})$ is an algCQG.*

Define

$$D_u(A,B) := \overline{D(A_0,B_0)}^{\|\cdot\|_u},$$

where for any $(a,b) \in D(A_0, B_0)$,

$$\|(a,b)\|_u := \sup_\pi \|(a,b)\|.$$

By Theorem 5.4.3 in [19], $(D_u(A,B), \Delta_{D_u})$ is the universal compact quantum group of $D(A_0, B_0)$, where Δ_{D_u} is the extension to $D_u(A,B)$ of Δ_D. Let h_{D_u} be the Haar state on $D_u(A,B)$ and (H, Λ, π) be the GNS- representation of $(D_u(A,B), \Delta_{D_u})$ for the Haar integral h_{D_u}. Define

$$D_r(A,B) := \pi(D_u(A,B)).$$

Denote Δ_{D_r} the extension to $D_r(A,B)$ of Δ_{D_0}. Then, $(D_r(A,B), \Delta_{D_r})$ is the reduced quantum group of $D(A_0, B_0)$, and its Haar integral h_{D_r} is faithful naturally.

Proposition 4. *$(D_u(A,B), \Delta_{D_u})$ and $(D_r(A,B), \Delta_{D_r})$ are both CQGs.*

Definition 4. *$(D_u(A,B), \Delta_{D_u})$ and $(D_r(A,B), \Delta_{D_r})$ are called the universal and reduced quantum double of A and B, respectively.*

3. The Main Results

Theorem 1. *Let $(A, B, \langle\cdot,\cdot\rangle)$ be a non-degenerate compact quantum group pairing. If (A, Δ_A) and (B, Δ_B) are two co-amenable CQGs, then $(D_u(A, B), \Delta_{D_u}) = (D_r(A, B), \Delta_{D_r})$.*

Proof. Suppose that (A_0, Δ_{A_0}) and (B_0, Δ_{B_0}) are the associated $algCQGs$, respectively. Let $\|\cdot\|_c$ be a regular C*-norm and A_c be the CQG completion of (A_0, Δ_{A_0}). As described in Section 2, A_u and A_r are both CQG completions of (A_0, Δ_{A_0}). Because A is co-amenable, by Proposition 2 (ii), there is a unique CQG completion for the associated $algCQGs$ (A_0, Δ_{A_0}). Hence,

$$A_r = A_c = A_u.$$

Analogously,

$$B_r = B_c = B_u.$$

By Proposition 2 (i),

$$\|a\|_r \le \|a\|_c \le \|a\|_u, \ \|b\|_r \le \|a\|_c \le \|b\|_u,$$

for all $a \in A_0$ and $b \in B_0$. Combining with the equations $A_r = A_u$ and $B_r = B_u$, one can symmetrically obtain that

$$\|a\|_r = \|a\|_u, \ \|b\|_r = \|b\|_u.$$

So,

$$\|\cdot\|_r = \|\cdot\|_c = \|\cdot\|_u \tag{1}$$

on A_0 and B_0. Moreover, Equation (1) also holds on $A_0 \odot B_0$. In fact, for any C*-norm $\|\cdot\|$ on $A_0 \odot B_0$, we have

$$\|a \otimes b\| = \|a\|\|b\|,$$

for all $a \in A_0, b \in B_0$. Then,

$$\|a \otimes b\|_u = \|a\|_u\|b\|_u = \|a\|_r\|b\|_r = \|a \otimes b\|_r.$$

From Proposition 2 (i),

$$\|a \otimes b\|_r = \|a \otimes b\|_c = \|a \otimes b\|_u$$

for all $a \otimes b \in A_0 \odot B_0$.

Considering the multiplication rule on the quantum double $D(A_0, B_0)$ ([6]), for any $(a, b) \in D(A_0, B_0)$,

$$(a, b) = \sum_{(a)(b)} \langle a_{(1)}, S_{B_0} b_{(1)}\rangle \langle a_{(3)}, b_{(3)}\rangle (a_{(2)} \otimes b_{(2)}). \tag{2}$$

From the above expression Equation (2), one can find that each element (a, b) in $D(A_0, B_0)$ is a linear combination of elements as $c \otimes d \in A_0 \odot B_0$. By the discussion in the underlying paragraph, we have

$$\|a_{(2)} \otimes b_{(2)}\|_u = \|a_{(2)} \otimes b_{(2)}\|_r,$$

where $a_{(2)}$ and $b_{(2)}$ are as presented in Equation (2), which induces that

$$\|(a, b)\|_u = \|(a, b)\|_r,$$

i.e., Equation (1) holds on $D(A_0, B_0)$. Hence, $D(A_0, B_0)$ has a unique CQG completion. Therefore, $(D_u(A, B), \Delta_{D_u})$ coincides with $(D_r(A, B), \Delta_{D_r})$, i.e.,

$$(D_u(A, B), \Delta_{D_u}) = (D_r(A, B), \Delta_{D_r}).$$

□

In sequel, $(D_u(A,B),\Delta_{D_u})$ and $(D_r(A,B),\Delta_{D_r})$ will be denoted by $(D(A,B),\Delta_D)$.

Theorem 2. *Let $(D(A,B),\Delta_D)$ be the quantum double of (A,Δ_A) and (B,Δ_B) based on a non-degenerate compact quantum group pairing $(A,B,\langle\cdot,\cdot\rangle)$. Assume that (A,Δ_A) and (B,Δ_B) are both co-amenable. Then, $(D(A,B),\Delta_D)$ is co-amenable.*

Proof. By Proposition 1, we have to prove that the following two conditions hold.

(i) The Haar integral of $D(A,B)$ is faithful.

Above all, we show that there exists a Haar integral h_{D_0} on it. For all $(a,b)\in D(A_0,B_0)$, we define

$$h_{D_0}(a,b) := h_{A_0}(a)h_{B_0}(b).$$

Denote bb^* by k; then, we can obtain that

$$\begin{aligned}
h_{D_0}((a,b)(a,b)^*) &= h_{D_0}((a,k)(a^*,1_{B_0}))\\
&= \sum_{(a)(k)} h((aa^*_{(2)},c_{(2)}))\langle a^*_{(1)},S_{B_0}k_{(3)}\rangle\langle a^*_{(3)},k_{(1)}\rangle\\
&= \sum_{(a)(k)} h_{A_0}(aa^*_{(2)})h_{B_0}(c_{(2)})\langle a^*_{(1)},S_{B_0}k_{(3)}\rangle\langle a^*_{(3)},k_{(1)}\rangle\\
&= \sum_{(a)(k)} h_{A_0}(aa^*_{(2)})\langle a^*_{(1)},S_{B_0}k_{(3)}\rangle\langle a^*_{(3)},h_{B_0}(k_{(2)})k_{(1)}\rangle\\
&= \sum_{(a)(k)} h_{A_0}(aa^*_{(2)})\langle a^*_{(1)},S_{B_0}k_{(2)}\rangle\langle a^*_{(3)},1_{B_0}\rangle h_{B_0}(k_{(1)})\\
&= \sum_{(a)(c)} h_{A_0}(a\varepsilon_{A_0}(a^*_{(3)})a^*_{(2)})\langle a^*_{(1)},S_{B_0}k_{(2)}\rangle h_{B_0}(k_{(1)})\\
&= \sum_{(a)(k)} h_{A_0}(aa^*_{(2)})\langle a^*_{(1)},S_{B_0}k_{(2)}\rangle h_{B_0}(k_{(1)}).
\end{aligned}$$

Considering $h_{B_0}\circ S_{B_0}=h_{B_0}$, we have

$$\begin{aligned}
h_{D_0}((a,b)(a,b)^*) &= \sum_{(a)(k)} h_{A_0}(aa^*_{(2)})\langle a^*_{(1)},h_{B_0}\circ S_{B_0}k_{(1)}S_{B_0}k_{(2)}\rangle\\
&= \sum_{(a)(k)} h_{A_0}(aa^*_{(2)})\langle a^*_{(1)},1_{B_0}\rangle h_{B_0}(k)\\
&= \sum_{(a)(k)} h_{A_0}(aa^*_{(2)})\varepsilon_{A_0}(a^*_{(1)})h_{B_0}(k)\\
&= h_{A_0}(aa^*)h_{B_0}(bb^*)\geq 0.
\end{aligned}$$

Again, for all $(c,d)\in D(A_0,B_0)$, one can get

$$\begin{aligned}
& h_{D_0}((a,b)(c,d)^*)\\
= {}& h_{D_0}((\sum_{(a)(b)}\langle a_{(1)},S_{B_0}b_{(1)}\rangle\langle a_{(3)},b_{(3)}\rangle(a_{(2)}\otimes b_{(2)}))(\sum_{(c)(d)}\langle c_{(1)},S_{B_0}d_{(1)}\rangle\langle c_{(3)},d_{(3)}\rangle(c_{(2)}\otimes d_{(2)}))^*)\\
= {}& h_{D_0}((\sum_{(a)(b)}\langle a_{(1)},S_{B_0}b_{(1)}\rangle\langle a_{(3)},b_{(3)}\rangle(a_{(2)}\otimes b_{(2)}))\sum_{(c)(d)}\langle c^*_{(1)},S^*_{B_0}d^*_{(1)}\rangle\langle c^*_{(3)},d_{(3)}\rangle(c^*_{(2)}\otimes d^*_{(2)}))\\
= {}& \sum_{(a)(b)(c)(d)}\langle a_{(1)},S_{B_0}b_{(1)}\rangle\langle a_{(3)},b_{(3)}\rangle\langle a^*_{(1)},S^*_{B_0}b^*_{(1)}\rangle\langle a^*_{(3)},b_{(3)}\rangle h_{D_0}(a_{(2)}\otimes b_{(2)})h_{D_0}(c^*_{(2)}\otimes d^*_{(2)})\\
= {}& \sum_{(a)(b)(c)(d)}\langle a_{(1)},S_{B_0}b_{(1)}\rangle\langle a_{(3)},b_{(3)}\rangle\langle c^*_{(1)},S^*_{B_0}d^*_{(1)}\rangle\langle c^*_{(3)},d_{(3)}\rangle h_{A_0}(a_{(2)}c^*_{(2)})h_{B_0}(b_{(2)}d^*_{(2)})
\end{aligned}$$

and

$$\begin{aligned}
& h_{D_0}((a,b)^*(c,d)) \\
= \ & h_{D_0}((\sum_{(a)(b)} \langle a_{(1)}, S_{B_0} b_{(1)} \rangle \langle a_{(3)}, b_{(3)} \rangle (a_{(2)} \otimes b_{(2)})^*)(\sum_{(c)(d)} \langle c_{(1)}, S_{B_0} d_{(1)} \rangle \langle c_{(3)}, d_{(3)} \rangle (c_{(2)} \otimes d_{(2)}))) \\
= \ & h_{D_0}((\sum_{(a)(b)} \langle a^*_{(1)}, S^*_{B_0} b^*_{(1)} \rangle \langle a^*_{(3)}, b_{(3)} \rangle (a^*_{(2)} \otimes b^*_{(2)})) \sum_{(c)(d)} \langle c_{(1)}, S_{B_0} d_{(1)} \rangle \langle c_{(3)}, d_{(3)} \rangle (c_{(2)} \otimes d_{(2)})) \\
= \ & \sum_{(a)(b)(c)(d)} \langle a^*_{(1)}, S^*_{B_0} b^*_{(1)} \rangle \langle a^*_{(3)}, b_{(3)} \rangle \langle c_{(1)}, S_{B_0} d_{(1)} \rangle \langle c_{(3)}, d_{(3)} \rangle h_{D_0}(a^*_{(2)} \otimes b^*_{(2)}) h_{D_0}(c_{(2)} \otimes d_{(2)}) \\
= \ & \sum_{(a)(b)(c)(d)} \langle a^*_{(1)}, S^*_{B_0} b^*_{(1)} \rangle \langle a^*_{(3)}, b_{(3)} \rangle \langle c_{(1)}, S_{B_0} d_{(1)} \rangle \langle c_{(3)}, d_{(3)} \rangle h_{A_0}(a^*_{(2)} c_{(2)}) h_{B_0}(b^*_{(2)} d_{(2)}),
\end{aligned}$$

which implies that

$$h_{D_0}((a,b)(c,d)^* + (a,b)^*(c,d)) \geq 0.$$

Therefore, h_{D_0} is positive on $D(A_0, B_0)$. From the underlying formula, $h_{D_0}((a,b)(a,b)^*) = 0$ if and only if $(a,b) = 0$. Thus, h_{D_0} is a positive faithful linear functional on $D(A_0, B_0)$. Considering the invariance of h_{A_0} and h_{B_0}, we can get

$$(\iota \otimes h_{D_0})\Delta_{D_0}((a,b)) = (h_{D_0} \otimes \iota)\Delta_{D_0}((a,b)) = h_{D_0}((a,b))1,$$

for all $(a,b) \in D(A_0, B_0)$.

Define h_D is the extension to $(D(A,B), \Delta_D)$ of h_{D_0}. It is easy to see that h_D is a Haar state on $(D(A,B), \Delta_D)$ by the fact h_{D_0} is a Haar integral on $D(A_0, B_0)$. Denote by h_A and h_B the Haar integrals on A and B, respectively. Then, one can get that

$$h_D = h_A \otimes h_B.$$

To prove h_D is faithful, it suffices to show that the Haar integral h_{D_u} of $D_u(A,B)$ is faithful, since the Haar integral of $D_r(A,B)$ is always faithful. Moreover, we just need to check the faithfulness of h_{D_u} on $D_u(A,B) \setminus D(A_0, B_0)$.

Let $(a', b') \in D_u(A,B) \setminus D(A_0, B_0)$. From the definitions of $D_u(A,B)$ and $D(A_0, B_0)$, we have that

$$(a', b') = \lim_{\alpha} (a,b)_\alpha, \tag{3}$$

$$(a,b) = \sum_{(a)(b)} \langle a_{(1)}, S_{B_0} b_{(1)} \rangle \langle a_{(3)}, b_{(3)} \rangle (a_{(2)} \otimes b_{(2)}), \tag{4}$$

where $(a,b) \in D(A_0, B_0)$, $\alpha's$ are in some index set, and the limit is taken with respect to the universal C*-norm $\| \cdot \|_u$ on $D(A_0, B_0)$. Thus, (a', b') can be rewritten as the following:

$$(a', b') = \sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)}), \tag{5}$$

where $a'_{(2)}$ is in $A_u \setminus A_0$ or $b'_{(2)}$ is in $B_u \setminus B_0$. If $h_{D_u}((a,b)(a,b)^*) = 0$, then

$$\begin{aligned}
& h_{D_u}((a',b')(a',b')^*) \\
= \ & h_{D_u}((\sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)}))(\sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)}))^*) \\
= \ & h_{D_u}((\sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)})) \sum_{(a')(b')} \langle a'^*_{(1)}, S^*_{B_0} b'^*_{(1)} \rangle \langle a'^*_{(3)}, b'_{(3)} \rangle (a'^*_{(2)} \otimes b'^*_{(2)})) \\
= \ & \sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle \langle a'^*_{(1)}, S^*_{B_0} b'^*_{(1)} \rangle \langle a'^*_{(3)}, b'_{(3)} \rangle h_{D_u}(a'_{(2)} \otimes b'_{(2)}) h_{D_u}(a'^*_{(2)} \otimes b'^*_{(2)}) \\
= \ & \sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle \langle a'^*_{(1)}, S^*_{B_0} b'^*_{(1)} \rangle \langle a'^*_{(3)}, b'_{(3)} \rangle h_{A_u}(a'_{(2)} a'^*_{(2)}) h_{B_u}(b'_{(2)} b'^*_{(2)}) \\
= \ & 0.
\end{aligned}$$

Because (A, Δ_A) and (B, Δ_B) are both co-amenable, by Proposition 1, h_A and h_B are both faithful. Hence, h_{A_u} and h_{B_u} are also faithful. Combining with the underlying equation, we obtain that $a'_{(2)} = 0$ and $b'_{(2)} = 0$; thus, by (5), we get

$$(a', b') = 0,$$

which states that h_{D_u} is faithful on $D_u(A, B)$.

(ii) The co-unit of $D(A, B)$ is norm-bounded.

First, we show that ε_{D_0} defined as before Proposition 3 is a *-homomorphism. Using the definition of ε_{D_0}, we have

$$\begin{aligned} & \varepsilon_{D_0}((a,b)^*) \\ = \; & \varepsilon_{D_0}((1_{A_0}, b^*)(a^*, 1_{B_0})) \\ = \; & \varepsilon_{D_0}(1_{A_0}, b^*)\varepsilon_{D_0}(a^*, 1_{B_0}) \\ = \; & \varepsilon_{B_0}(b^*)\varepsilon_{A_0}(a^*) \\ = \; & (\varepsilon_{A_0}(a)\varepsilon_{B_0}(b))^* \\ = \; & (\varepsilon_{D_0}(a,b))^*. \end{aligned}$$

Let ε_A and ε_B be the co-units on A and B, respectively. For all $(a, b) \in D(A, B)$, we define

$$\varepsilon_D(a, b) := \varepsilon_A(a)\varepsilon_B(b),$$

i.e.,

$$\varepsilon_D = \varepsilon_A \otimes \varepsilon_B,$$

which can be regarded as the extension to $(D(A, B), \Delta_D)$ of ε_{D_0}.

Considering the continuity of extension of ε_{D_0} from $D(A_0, B_0)$ to $D(A, B)$, ε_D is a *-homomorphism and then the co-unit on $D(A, B)$.

To prove that the co-unit ε_D on $D(A, B)$ is norm-bounded, it suffices to show that the Haar integral ε_{D_r} of $D_r(A, B)$ is norm-bounded with respect to the supremum norm, since the co-unit of $D_u(A, B)$ is always norm-bounded. Moreover, we just need to check the norm-bounded-ness of ε_{D_r} on $D_r(A, B) \setminus D(A_0, B_0)$. Let $(a', b') \in D_r(A, B) \setminus D(A_0, B_0)$. By a similar discussion, in Equations (3)–(5), we have

$$(a', b') = \sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)}), \tag{6}$$

where $a'_{(2)}$ is in $A_r \setminus A_0$ or $b'_{(2)}$ is in $B_r \setminus B_0$. Since A and B are co-amenable, by Proposition 1, ε_A and ε_B are both norm-bounded. Hence, ε_{A_r} and ε_{B_r} are norm-bounded, i.e., there exist two positive number M_A and M_B such that

$$\|\varepsilon_{A_r}\| = \sup_{\|a'\|_r = 1} | \varepsilon_{A_r}(a') | \leq M_A,$$

and

$$\|\varepsilon_{B_r}\| = \sup_{\|b'\|_r = 1} | \varepsilon_{B_r}(b') | \leq M_B.$$

Thus,

$$\begin{aligned} \|\varepsilon_{D_r}\| &= \sup\nolimits_{\|(a',b')\|_r=1} | \varepsilon_{D_r}((a', b')) | \\ &= \sup\nolimits_{\|(a',b')\|_r=1} | \varepsilon_{D_r}(\sum_{(a')(b')} \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle (a'_{(2)} \otimes b'_{(2)})) | \\ &= \sup\nolimits_{\|(a',b')\|_r=1} \sum_{(a')(b')} | \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle || \varepsilon_{A_r}(a'_{(2)}) || \varepsilon_{B_r}(b'_{(2)}) | \\ &\leq \sup\nolimits_{\|(a',b')\|_r=1} \sum_{(a')(b')} | \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle | M_A M_B \\ &\leq K M_A M_B, \end{aligned}$$

where K represents the supremum of $\{ \sum_{(a')(b')} \mid \langle a'_{(1)}, S_{B_0} b'_{(1)} \rangle \langle a'_{(3)}, b'_{(3)} \rangle \mid \mid \|(a', b')\|_r = 1\}$ and is a finite positive real number, which states that ε_{D_r} is norm-bounded. □

Remark 1. *Consider the trivial case where $A = B = C(S^1)$, the C*-algebra of continuous functions on the circle group S^1. Clearly, $D(A, B) = C(S^1 \times S^1) = C(T)$, where T represents the 2-torus. It is easy to know that in this case A, B and $D(A, B)$ are all co-amenable CQGs for their commutativity. In fact, we can also get the co-amenability of $D(A, B)$ by Theorem 2. The Haar integral h_D on $C(T)$ is the integral with respect to the Haar measure μ on T. For all $f \in C(T)$, $t = (g_1, g_2) \in S^1 \times S^1 = T$, we have*

$$\begin{aligned} h_D(f)(t) &= \int_T f(t) d\mu(t) \\ &= \int_{S^1 \times S^1} f(g_1, g_2) d\mu_1(g_1) d\mu_2(g_2) \\ &= \int_{S^1} f(g_1) d\mu_1(g_1) \int_{S^1} f(g_2) d\mu_2(g_2) \\ &= h_A(f_1)(g_1) h_B(f_2)(g_2), \end{aligned}$$

where f_1, μ_1 and f_2, μ_2 are the restrictions of f and μ on A and B, respectively. From the formula, since h_A and h_B are both faithful, h_D is also faithful.

The co-unit ε_D on $C(T)$ is the evaluation map on the unit of T, i.e., for all $f \in C(T)$,

$$\varepsilon_D(f) = f(e) = f(e_1, e_2),$$

where e_1 and e are the units of S^1 and T, respectively. Thus, we have

$$\begin{aligned} \|\varepsilon_D\| &= \sup_{\|f\|=1} \mid \varepsilon_D(f) \mid \\ &= \sup_{\|f\|=1} \mid f(e_1, e_2) \mid \\ &= \sup_{\|f\|=1} \mid (f_1 \otimes f_2)(e_1, e_2) \mid \\ &= \sup_{\|(f_1, f_2)\|=1} \mid f_1(e_1) \mid\mid f_2(e_2) \mid \\ &= \sup_{\|(f_1, f_2)\|=1} \mid f_1(e_1) \mid\mid f_2(e_2) \mid \\ &\leq 1. \end{aligned}$$

By the formula, we have ε_D is norm-bounded.

4. Conclusions

Based on the research for quantum double construction arising from co-amenable compact quantum groups and the C*-norms on quantum groups, in the article, using the C*-norm inequality and norm-bounded-ness of the co-unit on algebraic compact quantum groups, we prove that co-amenability is preserved under formulation of the quantum double construction of compact quantum groups. The result not only presents the stability of the co-amenability of quantum groups in the quantum double construction process but also exhibits the nice quantum symmetry between the co-amenability of quantum groups and the amenability of group.

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Abbreviations

The following abbreviations are used in this manuscript:

CQG	Compact Quantum Group
CQGs	Compact Quantum Groups
algCQG	algebraic Compact Quantum Group
DQG	Discrete Quantum Group

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