

Article

On Application Oriented Fuzzy Numbers for Imprecise Investment Recommendations

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Abstract: The subtraction of fuzzy numbers (FNs) is not an inverse operator to FNs addition. The family of all oriented FNs (OFNs) may be considered as symmetrical closure of all the FNs family in that the subtraction is an inverse operation to addition. An imprecise present value is modelled by a trapezoidal oriented FN (TrOFN). Then, the expected discount factor (EDF) is a TrOFFN too. This factor may be applied as a premise for invest-making. Proposed decision strategies are dependent on a comparison of an oriented fuzzy profit index and the specific profitability threshold. This way we get an investment recommendation described as a fuzzy subset on the fixed rating scale. Risk premium measure is a special case of profit index. Further in the paper, the Sharpe's ratio, the Jensen's ratio, the Treynor's ratio, the Sortino's ratio, Roy's criterion and the Modiglianis' coefficient are generalised for the case when an EDF is given as a TrOFN. In this way, we get many different imprecise recommendations. For this reason, an imprecise recommendation management module is described. Obtained results show that the proposed theory can be used as a theoretical background for financial robo-advisers. All theoretical considerations are illustrated by means of a simple empirical case study.

Keywords: behavioural finance; investment recommendations; oriented fuzzy number

1. Introduction

Imprecision is a natural feature of financial market information. A widely accepted way of representing an imprecise number is a fuzzy number (FN). The notion of an ordered FN was intuitively introduced by Kosiński et al. [1]. It was defined as an FN supplemented by its orientation. A significant drawback of Kosiński's theory is that there exists such ordered FNs that cannot be considered as FN [2]. This caused the original Kosiński's theory to be revised by Piasecki [3]. At present, the ordered FNs defined within Kosiński's original theory are called Kosiński's numbers [4–7]. If ordered FNs are linked to the revised theory, then they are called Oriented FNs (OFNs) [6,7].

The family of all OFNs has a symmetry axis that is equal to the family \mathbb{R} of all real numbers. In Section 2, this axial symmetry is described in detail. The family of all OFNs may be defined equivalently with the use of the discussed axial symmetry as the symmetrical closure of all of the FNs family. Symmetry allows us to avoid problems related to the fact that FNs subtraction is not an inverse operator to FNs addition.

A robo-adviser is an internet platform providing an automated financial planning service. This service is always algorithm-driven. Therefore, no robo-adviser requires any human involvement. It implies a minimal operating cost for any robo-advisor. Due to using robo-advisers we can apply different finance models to develop algorithms editing financial advice. Implemented algorithms can inform investors of any change in the market within a short period of time. In this way, robo-advisers efficiently implement any investment strategy by using their built-in automated algorithms [8].

With the development of Fintech, robo-adviser becomes more popular. The fundamental theoretical background of robo-advisers is the classical mean-variance optimisation developed by Markowitz [9]. Nonetheless, this well-known approach is not good [10–12]. To obtain algorithms generating more profitable portfolios, researchers take into account an investors' risk-aversion. Then they construct optimal portfolios by dealing with both conflicting objectives of minimising the risk and maximising the return together in a static manner [11,13,14] and dynamic case [12,15,16]. Since the financial data are imprecise, we can adapt robo-advisers using fuzzy logic [17,18] to financial practice. To the best of my knowledge, no financial robo-advisers using OFNs have been described so far.

To deal with information imprecision, researches develop portfolio selection models with fuzzy theory [19–40]. In all of the above-mentioned fuzzy models, an imprecision is included by the assumption of fuzzy rates of return from financial instruments given a priori. Then, return rate (RR) from portfolio can only be defined ex cathedra as a weighted sum of RRs assigned to its components. This approach is only justified by the mechanical generalisation of Markowitz's portfolio theory to the fuzzy case. The proposed generalisation is not justified by a mathematical deduction. This greatly reduces the reliability of performed analyses. It is a significant drawback of all fuzzy portfolio publications stated above.

In general, a present value (PV) is equal to a current equivalent of a payment due at a fixed point in time [41]. PV understood in this way is an imprecise value. For this reason, we can also point out the PV imprecision as the cause of imprecision in the financial analysis. This approach is presented in papers [42–50]. Then, with the help of mathematical deduction we can prove that the RR from a portfolio is a linear combination of return rates of their components. This increases the credibility of the performed financial analysis.

Fuzzy PV was proposed by Ward [51] as a discounted fuzzy prediction of a future payment. Buckley [52] proposed fuzzy financial arithmetic. Greenhut [53], Sheen [54] and Huang [22] generalised the definition of Ward. They expanded this definition to the case of a future payment described by a fuzzy variable. In 2005, Tsao [44] generalised a fuzzy PV definition. He assumed that the future payment may be considered as a fuzzy probabilistic set. Those authors described a present value as a discounted, imprecisely evaluated future payment. Piasecki [45,46] proposed a different approach. In this case, PV was imprecisely assessed on the basis of a current asset price. Buckley [52], Gutierrez [55], Kuchta [56] and Lesage [57] showed the purposefulness of applying a trapezoidal FN (TrFN) for fuzzy arithmetic. For this reason, we determine an imprecise PV by means of TrFN. Piasecki [45] showed that if the PV is an FN, then its RR is a fuzzy probabilistic set [58]. Additionally, then the expected RR is an FN. It is a theoretical background for investment-making models described in [46]. Moreover, in [48] it is shown that the fuzzy expected discount factor (EDF) is a better tool for appraising the considered securities than the fuzzy expected RR. Therefore, we use an EDF as a premise for invest-making.

OFNs have already been used by many scientists to describe and analyse many decision-making [59–64], financial [65–70] and economic [71,72] problems. Among other things, there it is shown that

- The use of FNs in financial analysis only leads to averaging the imprecision risk,
- The application of OFNs in financial analysis may minimise imprecision risk.

Therefore, the main aim of this paper is an extension of the investment-making models described in [46] to the case of imprecise PV estimated by trapezoidal OFNs (TrOFNs). The first attempt of this subject was presented in [70]. Here, we use our experience gathered during our work on the other criteria. Therefore, here we present a revised approach to the considered extension.

The paper is drafted as follows. Section 2 presents OFNs with their basic properties and describes the imprecision evaluation by an energy and entropy measure. In Section 3, PV is assessed by TrOFNs. The oriented fuzzy EDF is determined in Section 4. Investment recommendations dependent on the oriented fuzzy EDF are discussed in Section 5. Profitability criteria for investments are extended in

Section 6. In Section 7 we explore the management of a set of investment recommendations. Section 8 concludes the article and proposes some future research directions. In Appendix A, the optimisation algorithm used is described in detail.

2. Fuzzy Sets—Selected Facts

2.1. Fuzzy Sets

Fuzzy sets (FSs) [73] are a suitable tool that allows to describe and process imprecise values and information. In a given space \mathbb{X} an FS \mathcal{A} is distinguished by its membership function $\mu_A \in [0, 1]^{\mathbb{X}}$ in the following manner

$$\mathcal{A} = \{(x, \mu_A(x)); x \in \mathbb{X}\} \tag{1}$$

By $\mathcal{F}(\mathbb{X})$ we denote the family of all FSs in a space \mathbb{X} .

Multi-valued operations in the FSs family are defined by means of the following identities:

$$\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x) = \max \{\mu_A(x), \mu_B(x)\} \tag{2}$$

$$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x) = \min \{\mu_A(x), \mu_B(x)\} \tag{3}$$

$$\mu_{A^c}(x) = 1 - \mu_A(x) \tag{4}$$

FSs are widely used for modelling imprecise information. Following the work in [74], the imprecision is understood as a composition of ambiguity and indistinctness of. Ambiguity is defined a lack of a clear indication of one alternative among others. Indistinctness is defined as a lack of an explicit distinction between distinguished and not distinguished alternatives. More imprecise information is less useful. For this reason, it is sensible to assess the imprecision.

For the finite space $\mathbb{X} = \{x_1, x_2, \dots, x_f\}$, the suitable tool for assessing the ambiguity of an FS $A \in \mathcal{F}(\mathbb{X})$ is the energy measure $d : \mathcal{F}(\mathbb{X}) \rightarrow \mathbb{R}_0^+$ [75] given as follows:

$$d(A) = m(A) = \sum_{i=1}^f \mu_A(x_i) \tag{5}$$

The proper tool for measuring the indistinctness is the entropy measure $e : \mathcal{F}(\mathbb{X}) \rightarrow \mathbb{R}_0^+$ [76,77] determined by the identity

$$e(A) = \frac{m(A \cap A^c)}{m(A \cup A^c)} \tag{6}$$

2.2. Fuzzy Numbers

The fuzzy number (FN) may be intuitively defined as FS in the real line \mathbb{R} . The most general FN definition is proposed by Dubois and Prade [78]. Any FN may be defined in an equivalent way as follows [79]:

Theorem 1. For any FN \mathcal{L} there exists such a non-decreasing sequence $(a, b, c, d) \subset \mathbb{R}$ that $\mathcal{L}(a, b, c, d, L_L, R_L) = \mathcal{L} \in \mathcal{F}(\mathbb{R})$ is determined by its membership function $\mu_{\mathcal{L}}(\cdot|a, b, c, d, L_L, R_L) \in [0, 1]^{\mathbb{R}}$ described by the identity

$$\mu_{\mathcal{L}}(x|a, b, c, d, L_L, R_L) = \begin{cases} 0, & x \notin [a, d], \\ L_L(x), & x \in [a, b], \\ 1, & x \in [b, c], \\ R_L(x), & x \in [c, d]. \end{cases} \tag{7}$$

where the left reference function $L_L \in [0, 1]^{[a,b]}$ and the right reference function $R_L \in [0, 1]^{[c,d]}$ are upper semi-continuous monotonic ones meeting the conditions:

$$L_L(b) = R_L(c) = 1 \tag{8}$$

The family of all FNs is denoted by the symbol \mathbb{F} .

The symbol $*$ denotes any arithmetic operation defined on \mathbb{R} . By the symbol \otimes we denote such extension of operation $*$ to \mathbb{F} that it is coherent with Zadeh’s Extension Principle [80]. It means that, for each pair $(\mathcal{K}, \mathcal{L}) \in \mathbb{F}^2$ described by their membership functions $\mu_{\mathcal{K}}, \mu_{\mathcal{L}} \in [0, 1]^{\mathbb{R}}$, the FN

$$\mathcal{M} = \mathcal{K} \otimes \mathcal{L} \tag{9}$$

is described by membership function $\mu_{\mathcal{M}} \in [0, 1]^{\mathbb{R}}$ determined by the identity:

$$\mu_{\mathcal{M}}(z) = \sup\{\min\{\mu_{\mathcal{K}}(x), \mu_{\mathcal{L}}(y)\} : z = x * y, (x, y) \in \mathbb{R}\}. \tag{10}$$

A special case of FNs is trapezoidal FNs (TrFNs). Due to their simplicity and ease of performing operations on them, they are often used in applications. A suitable definition of trapezoidal fuzzy numbers is given in [81]:

Definition 1. For any non-decreasing sequence $(a, b, c, d) \subset \mathbb{R}$, a trapezoidal FN (TrFN) is the FN $\mathcal{T} = Tr(a, b, c, d) \in \mathbb{F}$ defined by its membership functions $\mu_{\mathcal{T}} \in [0, 1]^{\mathbb{R}}$ in the following way

$$\mu_{\mathcal{T}}(x) = \mu_{Tr}(x|a, b, c, d) = \begin{cases} 0, & x \notin [a, d], \\ \frac{x-a}{b-a}, & x \in [a, b], \\ 1, & x \in [b, c], \\ \frac{x-d}{c-d}, & x \in [c, d]. \end{cases} \tag{11}$$

2.3. Oriented Fuzzy Number

Ordered FN was defined by Kosiński et al. [1] as an extension of the FN concept. An important disadvantage of Kosiński’s theory is that there exists such ordered FNs that cannot be represented by a membership function [2]. On the other hand, ordered FNs’ usefulness is a result of their interpretation as FN supplemented by its orientation. The ordered FN orientation describes a forecast of the nearest future changes of FN. This caused Kosiński’s theory to be revised by Piasecki [3]. An ordered FN linked to the revised theory is called Oriented FN (OFN) [6,7]. In a general case, OFNs are defined as follows:

Definition 2 [3]. For any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$, the oriented fuzzy number OFN $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L) = \overleftrightarrow{\mathcal{L}}$ is a pair of an orientation $\overrightarrow{a, d} = (a, d)$ and a fuzzy set $\mathcal{L} \in \mathcal{F}(\mathbb{R})$ described by a membership function $\mu_{\mathcal{L}}(\cdot|a, b, c, d, S_L, E_L) \in [0, 1]^{\mathbb{R}}$ given by the identity

$$\mu_{\mathcal{L}}(x|a, b, c, d, S_L, E_L) = \begin{cases} 0, & x \notin [a, d] \equiv [d, a], \\ S_L(x), & x \in [a, b] \equiv [b, a], \\ 1, & x \in [b, c] \equiv [c, b], \\ E_L(x), & x \in [c, d] \equiv [d, c], \end{cases} \tag{12}$$

where the starting function $S_L \in [0, 1]^{[a, b]}$ and the ending function $E_L \in [0, 1]^{[c, d]}$ are upper semi-continuous monotonic ones meeting the condition

$$\forall_{x \in [a, d]} : \mu_{\mathcal{L}}(x|a, b, c, d, S_L, E_L) \tag{13}$$

The symbol \mathbb{K} denotes the space of all OFNs. If $a < d$, then any $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is a positively oriented OFN. Any positively oriented OFN may be interpreted as such FN, which can increase in the near future. The symbol \mathbb{K}^+ denotes the space of all positively oriented OFN. If $a > d$, then any $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)$ is a negatively oriented OFN, which may be interpreted as decreasing FN. The symbol \mathbb{K}^- denotes the space of all negatively oriented OFN. For $a = d$, $\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L) = \llbracket a \rrbracket$ represents the unoriented crisp number $a \in \mathbb{R}$. Summing up, we see that

$$\mathbb{K}^+ \cup \mathbb{R} \cup \mathbb{K}^- = \mathbb{K} \tag{14}$$

Let us consider the mapping $\mathcal{U} : \mathbb{K} \rightarrow \mathbb{K}$ given by identity

$$\mathcal{U}\left(\overleftrightarrow{\mathcal{L}}(a, b, c, d, S_L, E_L)\right) = \overleftrightarrow{\mathcal{L}}(d, c, b, a, E_L, S_L) \tag{15}$$

This mapping meets following conditions:

$$\overleftrightarrow{\mathcal{L}} \in \mathbb{K}^+ \Rightarrow \mathcal{U}\left(\overleftrightarrow{\mathcal{L}}\right) \in \mathbb{K}^- \tag{16}$$

$$\overleftrightarrow{\mathcal{L}} \in \mathbb{K}^- \Rightarrow \mathcal{U}\left(\overleftrightarrow{\mathcal{L}}\right) \in \mathbb{K}^+ \tag{17}$$

$$\overleftrightarrow{\mathcal{L}} \in \mathbb{R} \Rightarrow \mathcal{U}\left(\overleftrightarrow{\mathcal{L}}\right) = \overleftrightarrow{\mathcal{L}} \tag{18}$$

It shows that the mapping (15) is axial symmetry on the space \mathbb{K} of all OFNs. Then the symmetry axis is identical with family \mathbb{R} of all real numbers. Moreover, Theorem 1 together with Definition 2 implies that the space \mathbb{F} of FNs and the space \mathbb{K}^+ of all positively oriented OFNs are isomorphic. Therefore, we can say that the space \mathbb{K} may be determined as symmetry closure of the space \mathbb{F} .

In the studies planned here, we limit discussion to a special kind of OFNs defined as follows.

Definition 3 [3]. For any monotonic sequence $(a, b, c, d) \subset \mathbb{R}$, the trapezoidal OFN (TrOFN) $\overleftrightarrow{Tr}(a, b, c, d) = \overleftrightarrow{\mathcal{T}}$ is the pair of the orientation $\overrightarrow{a, d} = (a, d)$ and a fuzzy set $\mathcal{T} \in \mathcal{F}(\mathbb{R})$ determined by membership functions $\mu_{\mathcal{T}} \in [0, 1]^{\mathbb{R}}$ as follows

$$\mu_{\mathcal{T}}(x) = \mu_{Tr}(x|a, b, c, d) = \begin{cases} 0, & x \notin [a, d] \equiv [d, a], \\ \frac{x-a}{b-a}, & x \in [a, b] \equiv [b, a], \\ 1, & x \in [b, c] \equiv [c, b], \\ \frac{x-d}{c-d}, & x \in [c, d] \equiv [d, c]. \end{cases} \tag{19}$$

The symbol \mathbb{K}_{Tr} denotes the space of all TrOFNs. On the space \mathbb{K}_{Tr} , a relation $\overleftrightarrow{\mathcal{K}}.\widetilde{GE}.\overleftrightarrow{\mathcal{L}}$ was defined as follows

$$\overleftrightarrow{\mathcal{K}}.\widetilde{GE}.\overleftrightarrow{\mathcal{L}} \Leftrightarrow \text{“TrOFN } \overleftrightarrow{\mathcal{K}} \text{ is greater than or equal to TrOFN } \overleftrightarrow{\mathcal{L}} \text{”} \tag{20}$$

This is a fuzzy preorder $\widetilde{GE} \in \mathcal{F}(\mathbb{K}_{Tr} \times \mathbb{K}_{Tr})$ described by membership function $\nu_{GE} \in [0, 1]^{\mathbb{K}_{Tr} \times \mathbb{K}_{Tr}}$ described in detail in [6]. Due to these results, for any pair $(\overleftrightarrow{Tr}(a, b, c, d), h) \in \mathbb{K}_{Tr} \times \mathbb{R}$ we have:

$$\nu_{GE}\left(\overleftrightarrow{Tr}(a, b, c, d), \llbracket h \rrbracket\right) = \begin{cases} 0, & h > \max\{a, d\}, \\ \frac{h - \max\{a, d\}}{\max\{b, c\} - \max\{a, d\}}, & \max\{a, d\} \geq h > \max\{b, c\}, \\ 1, & \max\{b, c\} \geq h, \end{cases} \tag{21}$$

$$v_{GE}(\llbracket h \rrbracket, \overset{\leftrightarrow}{Tr}(a, b, c, d)) = \begin{cases} 0, & h < \min\{a, d\}, \\ \frac{h - \min\{a, d\}}{\min\{b, c\} - \min\{a, d\}}, & \min\{a, d\} \leq h < \min\{b, c\}, \\ 1, & \min\{b, c\} \leq h. \end{cases} \quad (22)$$

3. Oriented Present Value

The present value (PV) is defined as a current equivalent of a payment due at fixed point in time [41]. Therefore, we commonly accept that PV of future payments may be imprecise. This means that PV should be assessed with FNs. Such PV is called a fuzzy one. Buckley [52], Gutierrez [55], Kuchta [56] and Lesage [57] show the soundness of using TrFNs as an imprecise financial arithmetic tool. Moreover, PV estimation should be supplemented by a forecast of PV closest price changes. These price changes may be subjectively predicted. Moreover, closest price changes may be predicted with the help of the prediction tables presented in [82]. For these reasons, an imprecise PV should be evaluated by OFN [7,70]. Such PV is called an oriented PV (O-PV). Any O-PV is estimated by TrOFN

$$\overset{\leftrightarrow}{PV} = \overset{\leftrightarrow}{Tr}(V_s, V_f, V_l, V_e) \quad (23)$$

where the monotonic sequence $(V_s, V_f, \check{P}, V_l, V_e)$ is defined as follows

- \check{P} is a quoted price,
- $[V_s, V_e] \subset \mathbb{R}^+$ is the set of all possible values of PV,
- $[V_f, V_l] \subset [V_s, V_e]$ is the set of all values that do not noticeably differ from the quoted price \check{P} .

If we predict a rise in price then O-PV is described by a positively oriented TrOFN. If we predict a fall in price, then O-PV is described by a negatively oriented OFN.

Example 1. We observe the portfolio π composed of company shares included in WIG20 quoted on the Warsaw Stock Exchange (WSE). Based on a session closing on the WSE on 28 January 2020, for each observed share we assess its O-PV equal to TrOFN describing its Japanese candle [83]. Shares' O-PVs, obtained in such a manner, are presented in Table 1. For each portfolio component \hat{S} , we determine its quoted price \check{P}_s as an initial price on 29.01.2020.

Table 1. Recorded values of the portfolio π components.

Stock Company	Present Value $\overset{\leftrightarrow}{PV}_s$	Quoted Price \check{P}_s
ALR	$\overset{\leftrightarrow}{Tr}(27.42; 27.30; 27.00; 26.84)$	27.00
CCC	$\overset{\leftrightarrow}{Tr}(83.35; 88.00; 88.00; 89.65)$	88.00
CDR	$\overset{\leftrightarrow}{Tr}(271.50; 271.50; 276.30; 276.30)$	277.00
CPS	$\overset{\leftrightarrow}{Tr}(26.42; 26.60; 27.04; 27.34)$	27.20
DNP	$\overset{\leftrightarrow}{Tr}(155.00; 155.00; 155.10; 157.30)$	155.30
JSW	$\overset{\leftrightarrow}{Tr}(18.60; 19.36; 20.14; 20.14)$	20.32
KGH	$\overset{\leftrightarrow}{Tr}(91.78; 93.60; 93.70; 94.90)$	94.24
LTS	$\overset{\leftrightarrow}{Tr}(83.88; 83.40; 81.16; 80.26)$	81.44
LPP	$\overset{\leftrightarrow}{Tr}(8205.00; 8380.00; 8395.00; 8460.00)$	8385.00
MBK	$\overset{\leftrightarrow}{Tr}(367.00; 366.00; 359.80; 357.00)$	359.00
OPL	$\overset{\leftrightarrow}{Tr}(7.01; 7.05; 7.20; 7.35)$	7.17
PEO	$\overset{\leftrightarrow}{Tr}(97.22; 97.70; 98.20; 98.66)$	98.20
PGE	$\overset{\leftrightarrow}{Tr}(7.08; 7.15; 7.30; 7.40)$	7.30
PGN	$\overset{\leftrightarrow}{Tr}(3.91; 3.88; 3.86; 3.82)$	3.87

Table 1. Cont.

Stock Company	Present Value $\overset{\leftrightarrow}{PV}_s$	Quoted Price \check{P}_s
PKN	$\overset{\leftrightarrow}{Tr}(83.22; 83.00; 81.62; 81.18)$	81.90
PKO	$\overset{\leftrightarrow}{Tr}(34.59; 34.68; 34.90; 35.26)$	34.93
PLY	$\overset{\leftrightarrow}{Tr}(35.82; 35.94; 36.76; 37.20)$	36.70
PZU	$\overset{\leftrightarrow}{Tr}(40.72; 40.73; 40.89; 41.11)$	40.88
SPL	$\overset{\leftrightarrow}{Tr}(276.20; 278.00; 281.80; 283.80)$	287.00
TPE	$\overset{\leftrightarrow}{Tr}(1.51; 1.53; 1.56; 1.56)$	1.56

CCC, CDR, CPS, DNP, JWS, KGH, LPP, OPL, PEO, PGE, PKO, PLY, PZU, SPL and TPE are evaluated by a positively oriented PV, which predicts a rise in a quoted price. Similarly, the stock companies ALR, LTS, MBK, PGN and PKN are evaluated by a negatively oriented PV, which predicts a fall in the quoted price.

4. Oriented Expected Discount Factor

We assume that duration $t > 0$ of an investment is fixed. Then, the considered security is determined by two values: a foreseen $FV = V_t$ and an estimated $PV = V_0$. The benefits from owning this security are characterised by the simple return rate (RR) defined by the identity

$$r_t = \frac{V_t - V_0}{V_0} \tag{24}$$

Variable FV is described by a relation

$$\tilde{V}_t(\omega) = \check{P} \cdot (1 + \tilde{r}_t(\omega)) \tag{25}$$

where the simple RR $\tilde{r}_t : \Omega \rightarrow \mathbb{R}$ is determined for PV assessed as the quoted price \check{P} . After Markowitz [9] we assume that the RR \tilde{r} rate is the gaussian probability distribution $N(\bar{r}, \sigma)$.

In our case PV is determined as O-PV $\overset{\leftrightarrow}{PV} = \overset{\leftrightarrow}{Tr}(V_s, V_f, V_l, V_e)$ represented by its membership function $\mu_{\overset{\leftrightarrow}{PV}} \in [0; 1]^{\mathbb{R}}$ given by the identity

$$\mu_{\overset{\leftrightarrow}{PV}}(x) = \mu_{\overset{\leftrightarrow}{Tr}}(x | V_s, V_f, V_l, V_e) \tag{26}$$

According to (10), the simple RR calculated for the O-PV is a fuzzy probabilistic set represented by membership function $\tilde{\rho} \in [0; 1]^{\mathbb{R} \times \Omega}$ given as follows

$$\tilde{\rho}(r, \omega) = \sup \left\{ \mu_{\overset{\leftrightarrow}{PV}}(x) : r = \frac{V_t(\omega) - x}{x}, x \in \mathbb{R} \right\} = \mu_{\overset{\leftrightarrow}{PV}} \left(\frac{V_t(\omega)}{1+r} \right) = \mu_{\overset{\leftrightarrow}{PV}} \left(\check{P} \cdot \frac{1+r(\omega)}{1+r} \right) \tag{27}$$

Then, the membership function $\rho \in [0; 1]^{\mathbb{R}}$ of the expected RR is computed in the following manner

$$\rho(r) = \int_{-\infty}^{+\infty} \mu_{\overset{\leftrightarrow}{PV}} \left(\check{P} \cdot \frac{1+y}{1+r} \right) dF_r(y) = \mu_{\overset{\leftrightarrow}{PV}} \left(\check{P} \cdot \frac{1+\bar{r}}{1+r} \right) \tag{28}$$

In [48] it is shown that the fuzzy expected discount factor (EDF) is a better tool for appraising any securities than the expected fuzzy RR. Therefore, we determine EDF for the case of O-PV. In general, for a given RR r_t , the discount factor v_t is explicitly determined by the identity

$$v_t = \frac{1}{1 + r_t} \tag{29}$$

We consider the EDF $\bar{v} \in \mathbb{R}$ defined by the identity:

$$\bar{v} = \frac{1}{1 + \bar{r}} \tag{30}$$

In line with (28), the membership function $\delta \in [0, 1]^{\mathbb{R}}$ of an oriented fuzzy EDF (O-EDF) $\overleftrightarrow{\mathcal{V}} \in \mathbb{K}$ is given by the identity:

$$\begin{aligned} \delta(v) = \rho\left(\frac{1}{v} - 1\right) &= \mu_{PV}^{\leftrightarrow}\left(\check{P} \cdot \frac{1+\bar{r}}{1+\frac{1}{v}-1}\right) = \mu_{PV}^{\leftrightarrow}\left(\check{P} \cdot \frac{v}{\bar{v}}\right) \\ &= \mu_{Tr}^{\leftrightarrow}\left(v \left| \frac{V_s \cdot \bar{v}}{\check{P}}, \frac{V_f \cdot \bar{v}}{\check{P}}, \frac{V_l \cdot \bar{v}}{\check{P}}, \frac{V_e \cdot \bar{v}}{\check{P}} \right.\right) \end{aligned} \tag{31}$$

Then, O-EDF is given as follows:

$$\overleftrightarrow{\mathcal{V}} = \overleftrightarrow{Tr}\left(\frac{V_s}{\check{P}} \cdot \bar{v}, \frac{V_f}{\check{P}} \cdot \bar{v}, \frac{V_l}{\check{P}} \cdot \bar{v}, \frac{V_e}{\check{P}} \cdot \bar{v}\right) \tag{32}$$

Example 2. All considerations in the paper are run for the quarterly period of the investment time $t = 1$ quarter. We research the components of the portfolio π presented in Table 1. Using the one-year time series of quotations, for each portfolio component \hat{S} we calculate the following parameters:

- Expected RR \bar{r}_s ,
- CAPM directional factor β_s ,
- Variance σ_s^2 ,
- Downside semi variance ζ_s^2 .

With the application of (30) and (32), we calculated quarterly O-EDF for each component of the portfolio π . All evaluations obtained in this way are presented in Table 2.

The O-EDF of a security described in this way is a TrOFN with the identical orientation as the O-PV used for estimation. It is worth stressing that the maximum criterion of the expected RR can be equivalently replaced by the minimum criterion of the EDF.

Table 2. Characteristic of portfolio π components.

Stock Company	Expected Return Rate \bar{r}_s	CAPM Factor β_s	Variance σ_s^2	Downside Semi Variance ζ_s^2	EDF \bar{v}_s	OEDF \overleftrightarrow{V}_s
ALR	0.0263	1.706	0.000031	0.000018	0.9744	$\overleftrightarrow{Tr}(0.9896; 0.9852; 0.9744; 0.9686)$
CCC	0.0367	2.160	0.000048	0.000026	0.9646	$\overleftrightarrow{Tr}(0.9136; 0.9646; 0.9646; 0.9827)$
CDR	0.2490	9.925	0.000311	0.000189	0.8006	$\overleftrightarrow{Tr}(0.7847; 0.7847; 0.7986; 0.7986)$
CPS	0.0594	3.852	0.000093	0.000050	0.9439	$\overleftrightarrow{Tr}(0.9168; 0.9231; 0.9384; 0.9488)$
DNP	0.0672	3.465	0.000011	0.000006	0.9370	$\overleftrightarrow{Tr}(0.9352; 0.9352; 0.9358; 0.9491)$
JSW	0.0199	-0.598	0.000016	0.000010	0.9805	$\overleftrightarrow{Tr}(0.8975; 0.9342; 0.9718; 0.9718)$
KGH	0.0567	2.699	0.000063	0.000039	0.9463	$\overleftrightarrow{Tr}(0.9216; 0.9399; 0.9409; 0.9529)$
LTS	0.1054	3.643	0.000161	0.000092	0.9047	$\overleftrightarrow{Tr}(0.9318; 0.9265; 0.9016; 0.8916)$
LPP	0.0872	1.958	0.000126	0.000071	0.9198	$\overleftrightarrow{Tr}(0.9001; 0.9193; 0.9209; 0.9280)$
MBK	0.0674	6.243	0.000097	0.000059	0.9369	$\overleftrightarrow{Tr}(0.9578; 0.9552; 0.9390; 0.9317)$
OPL	0.0278	1.9406	0.000028	0.000017	0.9730	$\overleftrightarrow{Tr}(0.9513; 0.9567; 0.9771; 0.9974)$
PEO	0.0459	1.348	0.000068	0.000036	0.9561	$\overleftrightarrow{Tr}(0.9466; 0.9512; 0.9561; 0.9606)$
PGE	0.0674	4.392	0.000099	0.000071	0.9369	$\overleftrightarrow{Tr}(0.9087; 0.9177; 0.9369; 0.9497)$

Table 2. Cont.

Stock Company	Expected Return Rate \bar{r}_s	CAPM Factor β_s	Variance σ_s^2	Downside Semi Variance ζ_s^2	EDF \bar{v}_s	OEDF \vec{V}_s
PGN	0.0751	3.976	0.000109	0.000065	0.9302	$\vec{Tr}(0.9398; 0.9326; 0.9278; 0.9182)$
PKN	0.0408	2.674	0.000049	0.000029	0.9608	$\vec{Tr}(0.9763; 0.9737; 0.9575; 0.9524)$
PKO	0.1974	5.463	0.000216	0.000099	0.8351	$\vec{Tr}(0.8270; 0.8291; 0.8344; 0.8430)$
PLY	0.2607	6.2156	0.000589	0.000312	0.7932	$\vec{Tr}(0.7742; 0.7768; 0.7945; 0.8040)$
PZU	0.1952	5.541	0.000301	0.000181	0.8367	$\vec{Tr}(0.8334; 0.8336; 0.8369; 0.8414)$
SPL	0.3001	8.867	0.000563	0.000391	0.7692	$\vec{Tr}(0.7403; 0.7451; 0.7553; 0.7606)$
TPE	0.0432	2.991	0.000056	0.000035	0.9586	$\vec{Tr}(0.9279; 0.9402; 0.9586; 0.9586)$

5. Investment Recommendations

We understand an investment recommendation as a counsel given by the advisors to the investor. After evaluating the stocks, the advisor compares the obtained assessment with the current market value of the stocks. The difference between those values determines the potential of the investment return rate. Advisors give various recommendations depending on the volume of the return rate potential and its direction. Experts also define the potential of the return rate in different ways. We will here consider the collection of standardised recommendations, which are applied in [46]. The rating scale is given as the set $\mathbb{A} = \{A^{++}, A^+, A^0, A^-, A^{--}\}$, where

- A^{++} denotes the advice Buy suggesting that the expected price is well above the current quoted price,
- A^+ denotes the advice Accumulate suggesting that the expected price is above the current quoted price,
- A^0 denotes the advice Hold suggesting that the expected price is similar to the current quoted price,
- A^- denotes the advice Reduce suggesting that the expected price is below the current quoted price,
- A^{--} denotes the advice Sell suggesting that the expected price is well below the current quoted price.

The investor attributes each recommendation with the appropriate way of entering the transaction and the value of its volume. The way of entering the transaction describes the investment strategy. Investors can differ among one another by the implemented strategies.

Let fixed security \check{S} be represented by the pair (\bar{r}_s, ω_s) , where \bar{r}_s is an expected RR on \check{S} and ω_s is other parameter characterising \check{S} . The symbol \mathbb{S} denotes the set of all considered securities. Adviser’s counsel depends on the expected RR. The criterion for a competent choice of advice can be presented as a comparison of the profit index $g(\bar{r}_s|\omega_s)$ and the profitability threshold (PT) \check{G} , where $g(\cdot|\omega_s) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function of the expected RR. The advice choice function $\Lambda : \mathbb{S} \times \mathbb{R} \rightarrow 2^{\mathbb{A}}$ was given in the following way [46]

$$A^{++} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\omega_s) > \check{G} \Leftrightarrow \neg g(\bar{r}_s|\omega_s) \leq \check{G} \tag{33}$$

$$A^+ \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\omega_s) \geq \check{G} \tag{34}$$

$$A^0 \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\omega_s) = \check{G} \Leftrightarrow \wedge g(\bar{r}_s|\omega_s) \geq \check{G} \wedge g(\bar{r}_s|\omega_s) \leq \check{G} \tag{35}$$

$$A^- \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\omega_s) \leq \check{G} \tag{36}$$

$$A^{--} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow g(\bar{r}_s|\omega_s) < \check{G} \Leftrightarrow \neg g(\bar{r}_s|\omega_s) \geq \check{G} \tag{37}$$

This way, the advice set $\Lambda(\check{S}, \check{G}) \subset \mathbb{A}$ was assigned. We interpret the advice set $\Lambda(\check{S}, \check{G})$ as the investment recommendation given for the security \check{S} .

The security \check{S} may be equivalently represented by the ordered pair (\bar{v}_s, ω_s) , where \bar{v}_s is the EDF determined by (30). Then the identity (30) implies

$$g(\bar{r}_s|\omega_s) \geq \check{G} \Leftrightarrow \bar{v}_s \leq \frac{1}{1 + g^{-1}(\check{G}|\omega_s)} = H_s(\check{G}) \tag{38}$$

$$g(\bar{r}_s|\omega_s) \leq \check{G} \Leftrightarrow \bar{v}_s \geq H_s(\check{G}) \tag{39}$$

The value H_s is used as a specific profitability threshold (SPT) appointed for the security \check{S} . Then, the advice choice function $\Lambda : \mathbb{S} \times \mathbb{R} \rightarrow 2^{\mathbb{A}}$ is equivalently described in the following way

$$A^{++} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow -\bar{v}_s \geq H_s(\check{G}) \tag{40}$$

$$A^+ \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \leq H_s(\check{G}) \tag{41}$$

$$A^0 \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \leq H_s(\check{G}) \wedge \bar{v}_s \geq H_s(\check{G}) \tag{42}$$

$$A^- \in \Lambda(\check{S}, \check{G}) \Leftrightarrow \bar{v}_s \geq H_s(\check{G}) \tag{43}$$

$$A^{--} \in \Lambda(\check{S}, \check{G}) \Leftrightarrow -\bar{v}_s \leq H_s(\check{G}) \tag{44}$$

We consider the case when the security \check{S} is characterised by the ordered pair $(\overset{\leftrightarrow}{V}_s, \omega_s)$ where $\overset{\leftrightarrow}{V}_s \in \mathbb{K}_{Tr}$ is O-EDF calculated with use (32). Then the advice choice function $\tilde{\Lambda}(\check{S}, \check{G})$ is FS described by membership function $\lambda(\cdot|\check{S}, \check{G}) : \mathbb{A} \rightarrow [0, 1]$ determined in line with (40)–(44) in the following way:

$$\lambda(A^{++}|\check{S}, \check{G}) = 1 - \nu_{GE}(\overset{\leftrightarrow}{V}_s, \llbracket H_s(\check{G}) \rrbracket) \tag{45}$$

$$\lambda(A^+|\check{S}, \check{G}) = \nu_{GE}(\llbracket H_s(\check{G}) \rrbracket, \overset{\leftrightarrow}{V}_s) \tag{46}$$

$$\lambda(A^0|\check{S}, \check{G}) = \min\{\nu_{GE}(\llbracket H_s(\check{G}) \rrbracket, \overset{\leftrightarrow}{V}_s), \nu_{GE}(\overset{\leftrightarrow}{V}_s, \llbracket H_s(\check{G}) \rrbracket)\} \tag{47}$$

$$\lambda(A^-|\check{S}, \check{G}) = \nu_{GE}(\overset{\leftrightarrow}{V}_s, \llbracket H_s(\check{G}) \rrbracket) \tag{48}$$

$$\lambda(A^{--}|\check{S}, \check{G}) = 1 - \nu_{GE}(\llbracket H_s(\check{G}) \rrbracket, \overset{\leftrightarrow}{V}_s) \tag{49}$$

where $\nu_{GE} : \mathbb{K}_{Tr} \times \mathbb{K}_{Tr} \rightarrow [0, 1]$ is membership function of relation “less than or equal” (20). The required values of this function are computed with the use of (21) and (22).

From the point of view of invest-making, the value $\lambda(A|\check{S}, \check{G})$ is understood as a recommendation degree of the advice $A \in \mathbb{A}$, i.e., a declared participation of the advisor’s responsibility in the case of a final invest-made according to the advice $A \in \mathbb{A}$. It implies that the investment recommendation $\tilde{\Lambda}(\check{S}, \check{G})$ is emphasised as a FS in the rating scale \mathbb{A} .

In turn, the final decision is taken by the investors. Their personal responsibility for taking this investment decision decreases along with the increase in the recommendation degree related to the decision taken.

The increase in the ambiguity of the recommendation $\tilde{\Lambda}(\check{S}, \check{G}) \in \mathcal{F}(\mathbb{A})$ suggests a higher number of alternative recommendations to choose from. This is an increase in the risk of choosing an incorrect decision from recommended ones. This may result in obtaining a profit lower than maximal, that is with a loss of chance. Such risk is called an ambiguity risk. The ambiguity risk burdening the recommendation $\tilde{\Lambda}(\check{S}, \check{G}) \in \mathcal{F}(\mathbb{A})$ is assessed with an energy measure $d(\tilde{\Lambda}(\check{S}, \check{G}))$ computed with the use of (5).

An increase in the indistinctness of the recommendation $\tilde{\Lambda}(\check{S}, \check{G}) \in \mathcal{F}(\mathbb{A})$ suggests that the explicit distinction between recommended and not recommended decisions is more difficult. This causes an increase in the indistinctness risk understood as risk of choosing a not recommended decision. The indistinctness risk burdening the recommendation $\tilde{\Lambda}(\check{S}, \check{G}) \in \mathcal{F}(\mathbb{A})$ is measured by the entropy measure $e(\tilde{\Lambda}(\check{S}, \check{G}))$ computed with the use of (6).

An imprecision risk is always determined as a combination of indistinctness and ambiguity risks combined.

6. The Profitability Criteria for Investments

We evaluate chosen securities traded on a fixed capital market. We always assume that there exists a risk-free bond instrument represented by the pair $(r_0, 0)$. Moreover, we distinguish the market portfolio represented by the pair (r_M, σ_M) .

Example 3. We focus on the WSE. We take into account a risk-free bond instrument determined as quarterly treasury bonds with a risk-free RR $r_0 = 0.0075$. The market portfolio is determined as the portfolio determining a stock exchange index WIG. The RR from WIG has the normal distribution $N(r_M, \sigma_M^2) = N(0.0200, 0.000025)$.

6.1. Sharpe Ratio

The profit index is defined as Sharpe’s ratio estimating the amount of the premium per overall risk unit. Then Sharpe’s PT is equal to the unit premium of the market portfolio risk [84].

If the security \check{S} is represented by the pair (\bar{r}_s, σ_s) , then, in line to Sharpe, the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the PT \check{G} are defined as follows:

$$g(\bar{r}_s|\sigma_s) = \frac{r_s - r_0}{\sigma_s} \tag{50}$$

$$\check{G} = \frac{r_M - r_0}{\sigma_M} \tag{51}$$

We compute SPT H_s with the use of (38) in the following manner:

$$H_s = \frac{\sigma_M}{\sigma_s \cdot (r_M - r_0) + \sigma_M \cdot (r_0 + 1)} \tag{52}$$

Example 4. Using Equation (52), we compute an SPT H_s for all components of the portfolio π described in Examples 1 and 2. Obtained SPT values are compared with O-EDFs in Table 3.

If we estimate PV by TrOFN presented in Table 1, then using the Sharpe criterion is simply comparing an imprecise O-EDF with the precise SPT. By means of Equations (45)–(49), we compute the values of the recommendation choice function presented in Table 4. Table 4 also presents information on the imprecision risk burdening individual recommendations. That information will be used to choose the recommendation.

Investment recommendations for ALR, CCC, OPL and PKN are burdened with the increased ambiguity risk. Moreover, the recommendations for CCC, OPL and PKN carry the indistinctness risk. For that reason, those recommendations are rejected. Eventually, only the following stocks are attributed with “Buy” or “Accumulate” recommendation: CDR, CPS, DNP, JSW, KGH, LTS, LPP, MBK, PEO, PGE, PGN, PKO, PLY, PZU, SPL and TPE. Thus, the disclosure of imprecision of PV estimations allows rejecting riskier recommendations.

Table 3. Specific profitability threshold (SPT) H_s for the Sharpe criterion.

Stock Company	OEDF $\overset{\leftrightarrow}{V}_s$	SPT H_s
ALR	$\overset{\leftrightarrow}{Tr}(0.9896; 0.9852; 0.9744; 0.9686)$	0.9790
CCC	$\overset{\leftrightarrow}{Tr}(0.9136; 0.9646; 0.9646; 0.9827)$	0.9758
CDR	$\overset{\leftrightarrow}{Tr}(0.7847; 0.7847; 0.7986; 0.7986)$	0.9509
CPS	$\overset{\leftrightarrow}{Tr}(0.9168; 0.9231; 0.9384; 0.9488)$	0.9694
DNP	$\overset{\leftrightarrow}{Tr}(0.9352; 0.9352; 0.9358; 0.9491)$	0.9845
JSW	$\overset{\leftrightarrow}{Tr}(0.8975; 0.9342; 0.9718; 0.9718)$	0.9828
KGH	$\overset{\leftrightarrow}{Tr}(0.9216; 0.9399; 0.9409; 0.9529)$	0.9734
LTS	$\overset{\leftrightarrow}{Tr}(0.9318; 0.9265; 0.9016; 0.8916)$	0.9623
LPP	$\overset{\leftrightarrow}{Tr}(0.9001; 0.9193; 0.9209; 0.9280)$	0.9657
MBK	$\overset{\leftrightarrow}{Tr}(0.9578; 0.9552; 0.9390; 0.9317)$	0.9689
OPL	$\overset{\leftrightarrow}{Tr}(0.9513; 0.9567; 0.9771; 0.9974)$	0.9797
PEO	$\overset{\leftrightarrow}{Tr}(0.9466; 0.9512; 0.9561; 0.9606)$	0.9727
PGE	$\overset{\leftrightarrow}{Tr}(0.9087; 0.9177; 0.9369; 0.9497)$	0.9686
PGN	$\overset{\leftrightarrow}{Tr}(0.9398; 0.9326; 0.9278; 0.9182)$	0.9675
PKN	$\overset{\leftrightarrow}{Tr}(0.9763; 0.9737; 0.9575; 0.9524)$	0.9756
PKO	$\overset{\leftrightarrow}{Tr}(0.8270; 0.8291; 0.8344; 0.8430)$	0.9576
PLY	$\overset{\leftrightarrow}{Tr}(0.7742; 0.7768; 0.7945; 0.8040)$	0.9362
PZU	$\overset{\leftrightarrow}{Tr}(0.8334; 0.8336; 0.8369; 0.8414)$	0.9516
SPL	$\overset{\leftrightarrow}{Tr}(0.7403; 0.7451; 0.7553; 0.7606)$	0.9461
TPE	$\overset{\leftrightarrow}{Tr}(0.9279; 0.9402; 0.9586; 0.9586)$	0.9745

Table 4. Imprecise recommendations determined with the use of the Sharpe ratio.

Stock Company	Recommendation Choice Function					Energy Measure	Entropy Measure
	A^{--}	A^-	A^0	A^+	A^{++}		
ALR	0	1	1	1	0	3	0
CCC	0	0.3812	0.3812	1	0.6188	2.3812	0.2966
CDR	0	0	0	1	1	2	0
CPS	0	0	0	1	1	2	0
DNP	0	0	0	1	1	2	0
JSW	0	0	0	1	1	2	0
KGH	0	0	0	1	1	2	0
LTS	0	0	0	1	1	2	0
LPP	0	0	0	1	1	2	0
MBK	0	0	0	1	1	2	0
OPL	0	0.8719	0.8719	1	0.1281	2.8719	0.0833
PEO	0	0	0	1	1	2	0
PGE	0	0	0	1	1	2	0
PGN	0	0	0	1	1	2	0
PKN	0	0.2692	0.2692	1	0.7308	2.2692	0.1926
PKO	0	0	0	1	1	2	0
PLY	0	0	0	1	1	2	0
PZU	0	0	0	1	1	2	0
SPL	0	0	0	1	1	2	0
TPE	0	0	0	1	1	2	0

6.2. Jensen’s Alpha

The profit index is defined as Jensen’s alpha [85], estimating the amount of the premium for market risk. The security \check{S} is represented by the pair (r_s, β_s) , where β_s is the directional factor of the

CAPM model assigned to this instrument. Then, the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the PT \check{G} are defined as follows:

$$g(\bar{r}_s|\beta_s) = r_s - \beta_s(r_M - r_0) \tag{53}$$

$$\check{G} = r_0 \tag{54}$$

We calculate SPT H_s with the use of (38) in the following manner

$$H_s = \frac{1}{1 + r_0 + \beta_s(r_M - r_0)} \tag{55}$$

Example 5. Using (55), we calculate a specific profitability threshold SPT H_s for all components of the portfolio π described in Examples 1 and 2. The CAPM directional factors for each portfolio component are presented in Table 2. Evaluations obtained in this way are presented in Table 5.

Table 5. SPT H_s for the Jensen’s alpha.

Stock Company	OEDF $\overset{\leftrightarrow}{V}_s$	SPT H_s
ALR	$\overset{\leftrightarrow}{Tr}(0.9896; 0.9852; 0.9744; 0.9686)$	0.9720
CCC	$\overset{\leftrightarrow}{Tr}(0.9136; 0.9646; 0.9646; 0.9827)$	0.9667
CDR	$\overset{\leftrightarrow}{Tr}(0.7847; 0.7847; 0.7986; 0.7986)$	0.8837
CPS	$\overset{\leftrightarrow}{Tr}(0.9168; 0.9231; 0.9384; 0.9488)$	0.9473
DNP	$\overset{\leftrightarrow}{Tr}(0.9352; 0.9352; 0.9358; 0.9491)$	0.9517
JSW	$\overset{\leftrightarrow}{Tr}(0.8975; 0.9342; 0.9718; 0.9718)$	1.0000
KGH	$\overset{\leftrightarrow}{Tr}(0.9216; 0.9399; 0.9409; 0.9529)$	0.9604
LTS	$\overset{\leftrightarrow}{Tr}(0.9318; 0.9265; 0.9016; 0.8916)$	0.9496
LPP	$\overset{\leftrightarrow}{Tr}(0.9001; 0.9193; 0.9209; 0.9280)$	0.9690
MBK	$\overset{\leftrightarrow}{Tr}(0.9578; 0.9552; 0.9390; 0.9317)$	0.9212
OPL	$\overset{\leftrightarrow}{Tr}(0.9513; 0.9567; 0.9771; 0.9974)$	0.9692
PEO	$\overset{\leftrightarrow}{Tr}(0.9466; 0.9512; 0.9561; 0.9606)$	0.9762
PGE	$\overset{\leftrightarrow}{Tr}(0.9087; 0.9177; 0.9369; 0.9497)$	0.9413
PGN	$\overset{\leftrightarrow}{Tr}(0.9398; 0.9326; 0.9278; 0.9182)$	0.9459
PKN	$\overset{\leftrightarrow}{Tr}(0.9763; 0.9737; 0.9575; 0.9524)$	0.9607
PKO	$\overset{\leftrightarrow}{Tr}(0.8270; 0.8291; 0.8344; 0.8430)$	0.9296
PLY	$\overset{\leftrightarrow}{Tr}(0.7742; 0.7768; 0.7945; 0.8040)$	0.9215
PZU	$\overset{\leftrightarrow}{Tr}(0.8334; 0.8336; 0.8369; 0.8414)$	0.9287
SPL	$\overset{\leftrightarrow}{Tr}(0.7403; 0.7451; 0.7553; 0.7606)$	0.8942
TPE	$\overset{\leftrightarrow}{Tr}(0.9279; 0.9402; 0.9586; 0.9586)$	0.9570

If now we estimate PV with the use of TrOFN presented in Table 1 then using the Jensen’s alpha goes down to the comparison of an imprecise O-EDF with the precise SPT. By means of (45)–(49) we estimate the values of a recommendation choice function presented in Table 6.

Investment recommendations for ALR, CCC, CPS, OPL, PGE, PKN and TPE are burdened with the increased ambiguity risk. Moreover, the recommendations for ALR, CCC, CPS and PGE carry an indistinctness risk. For that reason, those recommendations are rejected. Eventually, only the following stocks are attributed with “Buy” or “Accumulate” advice: CDR, DNP, JSW, KGH, LTS, LPP, PEO, PGN, PKO, PLY, PZU and SPL. Advice “Sell” or “Reduce” were associated with MBK. Thus, the disclosure of imprecision of PV estimations allows rejecting riskier recommendations.

Table 6. Imprecise recommendations determined with the use of Jensen’s alpha.

Stock Company	Recommendation Choice Function					Energy Measure	Entropy Measure
	A^{--}	A^{-}	A^0	A^{+}	A^{++}		
ALR	0.4138	1	0.5862	0.5862	0	2.5862	0.3303
CCC	0	0.8840	0.8840	1	0.1160	2.8840	0.0748
CDR	0	0	0	1	1	2	0
CPS	0	0.1442	0.1442	1	0.8558	2.1442	0.0947
DNP	0	0	0	1	1	2	0
JSW	0	0	0	1	1	2	0
KGH	0	0	0	1	1	2	0
LTS	0	0	0	1	1	2	0
LPP	0	0	0	1	1	2	0
MBK	1	1	0	0	0	2	0
OPL	0	1	1	1	0	3	0
PEO	0	0	0	1	1	2	0
PGE	0	0.6563	0.6563	1	0.3437	2.6563	0.2598
PGN	0	0	0	1	1	2	0
PKN	0	1	1	1	0	3	0
PKO	0	0	0	1	1	2	0
PLY	0	0	0	1	1	2	0
PZU	0	0	0	1	1	2	0
SPL	0	0	0	1	1	2	0
TPE	0	1	1	1	0	3	0

6.3. Treynor Ratio

The profit index is defined as the Treynor ratio [86], which estimates the amount of premium for the market risk. The security \check{S} is represented by the pair (r_s, β_s) , where β_s is the directional factor of the CAPM model assigned to this instrument. Then the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the PT \check{G} are defined as follows:

$$g(\bar{r}_s|\beta_s) = \frac{r_s - r_0}{\beta_s} \quad (56)$$

$$\check{G} = r_M - r_0 \quad (57)$$

We compute SPT H_s with the use of (38) in the following manner

$$H_s = \frac{1}{1 + r_s + \beta_s(r_M - r_0)} \quad (58)$$

Example 6. Using (58), we calculate SPT for all components of the portfolio π described in Examples 1 and 2. Evaluations obtained in this way are presented in Table 7.

If we estimate PV with the use of TrOFN presented in Table 1 then using the Treynor ratio criterion goes down to the comparison of an imprecise O-EDF with the precise SPT. By means of (45)–(49) we estimate the values of the recommendation choice function presented in Table 8.

Investment recommendation for CCC is burdened with an increased ambiguity risk and carries an indistinctness risk. For that reason, this recommendation is rejected. Eventually, only the following stocks are attributed with “Sell” or “Reduce” advice: ALR, CDR, CPS, DNP, KGH, LTS, LPP, MBK, OPL, PEO, PGE, PGN, PKN, PKO, PLY, PZU, SPL and TPE. Advice “Buy” or “Accumulate” were associated just with the stock of JSW. Thus, the disclosure of imprecision of PV estimations allows rejecting riskier recommendations.

Table 7. SPT H_s for the Treynor ratio.

Stock Company	OEDF $\overset{\leftrightarrow}{V}_s$	SPT H_s
ALR	$\overset{\leftrightarrow}{Tr}(0.9896; 0.9852; 0.9744; 0.9686)$	0.9545
CCC	$\overset{\leftrightarrow}{Tr}(0.9136; 0.9646; 0.9646; 0.9827)$	0.9401
CDR	$\overset{\leftrightarrow}{Tr}(0.7847; 0.7847; 0.7986; 0.7986)$	0.7283
CPS	$\overset{\leftrightarrow}{Tr}(0.9168; 0.9231; 0.9384; 0.9488)$	0.9029
DNP	$\overset{\leftrightarrow}{Tr}(0.9352; 0.9352; 0.9358; 0.9491)$	0.9005
JSW	$\overset{\leftrightarrow}{Tr}(0.8975; 0.9342; 0.9718; 0.9718)$	0.9877
KGH	$\overset{\leftrightarrow}{Tr}(0.9216; 0.9399; 0.9409; 0.9529)$	0.9171
LTS	$\overset{\leftrightarrow}{Tr}(0.9318; 0.9265; 0.9016; 0.8916)$	0.8689
LPP	$\overset{\leftrightarrow}{Tr}(0.9001; 0.9193; 0.9209; 0.9280)$	0.8995
MBK	$\overset{\leftrightarrow}{Tr}(0.9578; 0.9552; 0.9390; 0.9317)$	0.8730
OPL	$\overset{\leftrightarrow}{Tr}(0.9513; 0.9567; 0.9771; 0.9974)$	0.9505
PEO	$\overset{\leftrightarrow}{Tr}(0.9466; 0.9512; 0.9561; 0.9606)$	0.9410
PGE	$\overset{\leftrightarrow}{Tr}(0.9087; 0.9177; 0.9369; 0.9497)$	0.8910
PGN	$\overset{\leftrightarrow}{Tr}(0.9398; 0.9326; 0.9278; 0.9182)$	0.8891
PKN	$\overset{\leftrightarrow}{Tr}(0.9763; 0.9737; 0.9575; 0.9524)$	0.9309
PKO	$\overset{\leftrightarrow}{Tr}(0.8270; 0.8291; 0.8344; 0.8430)$	0.7901
PLY	$\overset{\leftrightarrow}{Tr}(0.7742; 0.7768; 0.7945; 0.8040)$	0.7472
PZU	$\overset{\leftrightarrow}{Tr}(0.8334; 0.8336; 0.8369; 0.8414)$	0.7909
SPL	$\overset{\leftrightarrow}{Tr}(0.7403; 0.7451; 0.7553; 0.7606)$	0.7088
TPE	$\overset{\leftrightarrow}{Tr}(0.9279; 0.9402; 0.9586; 0.9586)$	0.9254

Table 8. Imprecise recommendations determined with use Treynor ratio.

Stock Company	Recommendation Choice Function					Energy Measure	Entropy Measure
	A^{--}	A^-	A^0	A^+	A^{++}		
ALR	1	1	0	0	0	2	0
CCC	0.4804	1	0.5196	0.5196	0	2.5196	0.4050
CDR	1	1	0	0	0	2	0
CPS	1	1	0	0	0	2	0
DNP	1	1	0	0	0	2	0
JSW	0	0	0	1	1	2	0
KGH	1	1	0	0	0	2	0
LTS	1	1	0	0	0	2	0
LPP	1	1	0	0	0	2	0
MBK	1	1	0	0	0	2	0
OPL	1	1	0	0	0	2	0
PEO	1	1	0	0	0	2	0
PGE	1	1	0	0	0	2	0
PGN	1	1	0	0	0	2	0
PKN	1	1	0	0	0	2	0
PKO	1	1	0	0	0	2	0
PLY	1	1	0	0	0	2	0
PZU	1	1	0	0	0	2	0
SPL	1	1	0	0	0	2	0
TPE	1	1	0	0	0	2	0

6.4. Sortino Ratio

The Sortino ratio [87] is a tool for risk management under a financial equilibrium. In this model we compare the expected RR \bar{r}_s from considered security and the expected return rate \bar{r}_M from the

distinguished market portfolio. We consider the advice choice function where the profit index and the limit value are determined by the Sortino ratio. Then, the profit index evaluates the amount of a specific unit premium for the loss risk. Moreover, the limit value evaluates an amount of the market unit premium for the loss risk. The benchmark of our assessment is a market portfolio represented by such an ordered pair $(\bar{r}_M, \varsigma_M^2)$, where the downside semi variance ς_M^2 evaluates the market loss risk. The reference point is a risk-free bond instrument represented by the ordered pair $(r_0, 0)$, where r_0 is a risk-free return rate.

The considered security \check{S} is represented by the ordered pair $(\bar{r}_s, \varsigma_s^2)$, where downside semi variance ς_s^2 evaluates the loss risk. Then, Sortino and Price (1997) define the profit index $g(\cdot|\varsigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the limit value PT \check{G} as follows:

$$g(\bar{r}_s|\varsigma_s) = \frac{r_s - r_0}{\varsigma_s} \tag{59}$$

$$\check{G} = \frac{r_M - r_0}{\varsigma_M} \tag{60}$$

We compute SPT $H_s(\check{G})$ with the use of (38) in the following manner

$$H_s(\check{G}) = \frac{\varsigma_M}{\varsigma_s \cdot (r_M - r_0) + \varsigma_M \cdot (r_0 + 1)} \tag{61}$$

Example 7. The market portfolio is represented by the ordered pair $(r_M, \varsigma_M^2) = (0.0200, 0.000015)$. Using (61), we calculate SPT for all securities belonging to the portfolio π described in Examples 1 and 2. Evaluations obtained in this way are presented in Table 9.

Table 9. SPT H_s for the Sortino ratio.

Stock Company	OEDF $\overset{\leftrightarrow}{V}_s$	SPT H_s
ALR	$\overset{\leftrightarrow}{Tr}(0.9896; 0.9852; 0.9744; 0.9686)$	0.9793
CCC	$\overset{\leftrightarrow}{Tr}(0.9136; 0.9646; 0.9646; 0.9827)$	0.9766
CDR	$\overset{\leftrightarrow}{Tr}(0.7847; 0.7847; 0.7986; 0.7986)$	0.9507
CPS	$\overset{\leftrightarrow}{Tr}(0.9168; 0.9231; 0.9384; 0.9488)$	0.9706
DNP	$\overset{\leftrightarrow}{Tr}(0.9352; 0.9352; 0.9358; 0.9491)$	0.9848
JSW	$\overset{\leftrightarrow}{Tr}(0.8975; 0.9342; 0.9718; 0.9718)$	0.9826
KGH	$\overset{\leftrightarrow}{Tr}(0.9216; 0.9399; 0.9409; 0.9529)$	0.9731
LTS	$\overset{\leftrightarrow}{Tr}(0.9318; 0.9265; 0.9016; 0.8916)$	0.9630
LPP	$\overset{\leftrightarrow}{Tr}(0.9001; 0.9193; 0.9209; 0.9280)$	0.9665
MBK	$\overset{\leftrightarrow}{Tr}(0.9578; 0.9552; 0.9390; 0.9317)$	0.9687
OPL	$\overset{\leftrightarrow}{Tr}(0.9513; 0.9567; 0.9771; 0.9974)$	0.9796
PEO	$\overset{\leftrightarrow}{Tr}(0.9466; 0.9512; 0.9561; 0.9606)$	0.9738
PGE	$\overset{\leftrightarrow}{Tr}(0.9087; 0.9177; 0.9369; 0.9497)$	0.9665
PGN	$\overset{\leftrightarrow}{Tr}(0.9398; 0.9326; 0.9278; 0.9182)$	0.9676
PKN	$\overset{\leftrightarrow}{Tr}(0.9763; 0.9737; 0.9575; 0.9524)$	0.9757
PKO	$\overset{\leftrightarrow}{Tr}(0.8270; 0.8291; 0.8344; 0.8430)$	0.9619
PLY	$\overset{\leftrightarrow}{Tr}(0.7742; 0.7768; 0.7945; 0.8040)$	0.9394
PZU	$\overset{\leftrightarrow}{Tr}(0.8334; 0.8336; 0.8369; 0.8414)$	0.9516
SPL	$\overset{\leftrightarrow}{Tr}(0.7403; 0.7451; 0.7553; 0.7606)$	0.9334
TPE	$\overset{\leftrightarrow}{Tr}(0.9279; 0.9402; 0.9586; 0.9586)$	0.9741

For each considered security, by means of (45)–(49) we calculate membership functions of investment recommendations presented in Table 10.

Table 10. Imprecise recommendations determined with the use of the Sortino ratio.

Stock Company	Recommendation Choice Function					Energy Measure	Entropy Measure
	A^{--}	A^{-}	A^0	A^{+}	A^{++}		
ALR	0	1	1	1	0	3	0
CCC	0	0.3370	0.3370	1	0.663	2.3370	0.0228
CDR	0	0	0	1	1	2	0
CPS	0	0	0	1	1	2	0
DNP	0	0	0	1	1	2	0
JSW	0	0	0	1	1	2	0
KGH	0	0	0	1	1	2	0
LTS	0	0	0	1	1	2	0
LPP	0	0	0	1	1	2	0
MBK	0	0	0	1	1	2	0
OPL	0	0.8769	0.8769	1	0.1231	2.8769	0.0798
PEO	0	0	0	1	1	2	0
PGE	0	0	0	1	1	2	0
PGN	0	0	0	1	1	2	0
PKN	0	0.2308	0.2308	1	0.7692	2.2308	0.1607
PKO	0	0	0	1	1	2	0
PLY	0	0	0	1	1	2	0
PZU	0	0	0	1	1	2	0
SPL	0	0	0	1	1	2	0
TPE	0	0	0	1	1	2	0

Investment recommendations for ALR, CCC, OPL and PKN are burdened with the increased ambiguity risk. Moreover, the recommendations for CCC, OPL and PKN carry the indistinctness risk. For that reason, those recommendations are rejected. Eventually, only the following stocks are attributed with “Buy” or “Accumulate” advice: CDR, CPS, DNP, JSW, KGH, LTS, LPP, MBK, PEO, PGE, PGN, PKO, PLY, PZU, SPL and TPE. Thus, the disclosure of the imprecision of PV estimations allows rejecting riskier recommendations.

6.5. Modiglianis’ Coefficient

In the crisp case, the Modiglianis’ Coefficient Criterion is equivalent to Sharpe Ratio Criterion. In this model, the compared values are the expected RR on a security and the expected RR on the market portfolio. Modiglianis’ profit coefficient estimates the bonus over market profits. Modiglianis’ limit value equals zero.

If the security \check{S} is represented by the pair (\bar{r}_s, σ_s^2) , then Modigliani [88] defines the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the PT \check{G} as follows:

$$g(\bar{r}_s|\sigma_s) = r_0 - r_M + \frac{r_s - r_0}{\sigma_s} \cdot \sigma_M \quad (62)$$

$$\check{G} = 0 \quad (63)$$

We compute SPT H_s with the use of (38) in the following manner

$$H_s = \frac{\sigma_M}{\sigma_s \cdot (r_M - r_0) + \sigma_M \cdot (r_0 + 1)} \quad (64)$$

We see that in a fuzzy case, the Modiglianis’ Coefficient Criterion is also equivalent to the Sharpe Ratio Criterion. In this case, the recommendations obtained with the use of Modiglianis’ Coefficient can be found in Table 4.

6.6. Roy's Criterion

Roy [89] has considered a fixed security \check{S} , represented by the pair (\bar{r}_s, σ_s) , where \bar{r}_s is an expected return on \check{S} and σ_s^2 is the variance of a return rate of the considered financial instrument. After Markowitz [9] we assume that the considered security \check{S} has a simple return rate with Gaussian distribution $N(\bar{r}_s, \sigma_s)$. This distribution is described by its increasing and continuous cumulative distribution function $F(\cdot|\bar{r}_s, \sigma_s) : \mathbb{R} \rightarrow [0; 1]$ given by the identity

$$F(x|\bar{r}_s, \sigma_s) = \Phi\left(\frac{x - \bar{r}_s}{\sigma_s}\right) \quad (65)$$

where the function $\Phi : \mathbb{R} \rightarrow [0; 1]$ is the cumulative distribution function of the Gaussian distribution $N(0, 1)$. The Safety Condition [89] is given as follows:

$$F(L|\bar{r}_s, \sigma_s) = \varepsilon \quad (66)$$

where

- L —a minimum acceptable RR,
- ε —the probability of RR realisation below the minimum acceptable rate.

The RR realisation below the minimum acceptable rate is identified with a loss. The Roy's criterion minimises the probability of a loss for a set minimum acceptable rate of return [46]. Additionally, the investor assumes the maximum level ε^* of the loss probability. Then the Roy's criterion is described by the inequality

$$F(L|\bar{r}_s, \sigma_s) = \Phi\left(\frac{L - \bar{r}_s}{\sigma_s}\right) \leq \varepsilon^* < \frac{1}{2} \quad (67)$$

It implies that

$$\bar{r}_s \geq L - \sigma_s \cdot \Phi^{-1}(\varepsilon^*) \quad (68)$$

In line with (38), SPT is given as follows

$$H_s = \frac{1}{1 + L - \sigma_s \cdot \Phi^{-1}(\varepsilon^*)} \quad (69)$$

Example 8. We study recommendations implied by Roy's criterion all components of portfolio π described in Example 1. The investor assumes the minimal acceptable RR $L = 0.0075$. Additionally, the investor assumes the maximum level of a loss probability $\varepsilon^* = 0.05$. Then, we have $\Phi^{-1}(0.05) = -1.64$. Table 2 lists the values of O-EDF. Using (69), we compute SPT for all components of the portfolio π described in Examples 1 and 2. Evaluations obtained in this way are presented in Table 11.

If we estimate PV with the use of TrOFN presented in Table 1 then using the Roy's criterion goes down to the comparison of an imprecise OEF with the precise SPT [70]. By means of (45)–(49) we then estimate the values of a recommendation choice function presented in Table 12.

Investment recommendations for ALR and CCC are burdened with an increased ambiguity risk. Moreover, the recommendations for CCC carry the indistinctness risk. For that reason, those recommendations are rejected. Eventually, only the following stocks are attributed with "Buy" or "Accumulate" advice: CDR, CPS, DNP, JSW, KGH, LTS, LPP, MBK, OPL, PEO, PGE, PGN, PKN, PKO, PLY, PZU, SPL and TPE. Thus, the disclosure of imprecision of PV estimations allows rejecting riskier recommendations.

Table 11. SPT H_s for the Roy’s criterion.

Stock Company	OEDF $\overset{\leftrightarrow}{V}_s$	SPT H_s
ALR	$\overset{\leftrightarrow}{Tr}(0.9896; 0.9852; 0.9744; 0.9686)$	0.9836
CCC	$\overset{\leftrightarrow}{Tr}(0.9136; 0.9646; 0.9646; 0.9827)$	0.9815
CDR	$\overset{\leftrightarrow}{Tr}(0.7847; 0.7847; 0.7986; 0.7986)$	0.9649
CPS	$\overset{\leftrightarrow}{Tr}(0.9168; 0.9231; 0.9384; 0.9488)$	0.9772
DNP	$\overset{\leftrightarrow}{Tr}(0.9352; 0.9352; 0.9358; 0.9491)$	0.9873
JSW	$\overset{\leftrightarrow}{Tr}(0.8975; 0.9342; 0.9718; 0.9718)$	0.9861
KGH	$\overset{\leftrightarrow}{Tr}(0.9216; 0.9399; 0.9409; 0.9529)$	0.9799
LTS	$\overset{\leftrightarrow}{Tr}(0.9318; 0.9265; 0.9016; 0.8916)$	0.9725
LPP	$\overset{\leftrightarrow}{Tr}(0.9001; 0.9193; 0.9209; 0.9280)$	0.9748
MBK	$\overset{\leftrightarrow}{Tr}(0.9578; 0.9552; 0.9390; 0.9317)$	0.9769
OPL	$\overset{\leftrightarrow}{Tr}(0.9513; 0.9567; 0.9771; 0.9974)$	0.9841
PEO	$\overset{\leftrightarrow}{Tr}(0.9466; 0.9512; 0.9561; 0.9606)$	0.9794
PGE	$\overset{\leftrightarrow}{Tr}(0.9087; 0.9177; 0.9369; 0.9497)$	0.9767
PGN	$\overset{\leftrightarrow}{Tr}(0.9398; 0.9326; 0.9278; 0.9182)$	0.9760
PKN	$\overset{\leftrightarrow}{Tr}(0.9763; 0.9737; 0.9575; 0.9524)$	0.9814
PKO	$\overset{\leftrightarrow}{Tr}(0.8270; 0.8291; 0.8344; 0.8430)$	0.9694
PLY	$\overset{\leftrightarrow}{Tr}(0.7742; 0.7768; 0.7945; 0.8040)$	0.9548
PZU	$\overset{\leftrightarrow}{Tr}(0.8334; 0.8336; 0.8369; 0.8414)$	0.9653
SPL	$\overset{\leftrightarrow}{Tr}(0.7403; 0.7451; 0.7553; 0.7606)$	0.9557
TPE	$\overset{\leftrightarrow}{Tr}(0.9279; 0.9402; 0.9586; 0.9586)$	0.9806

Table 12. Imprecise recommendations.

Stock Company	Recommendation Choice Function					Energy Measure	Entropy Measure
	A^{--}	A^-	A^0	A^+	A^{++}		
ALR	0	1	1	1	0	3	0
CCC	0	0.0663	0.0663	1	0.9337	2.0663	0.0414
CDR	0	0	0	1	1	2	0
CPS	0	0	0	1	1	2	0
DNP	0	0	0	1	1	2	0
JSW	0	0	0	1	1	2	0
KGH	0	0	0	1	1	2	0
LTS	0	0	0	1	1	2	0
LPP	0	0	0	1	1	2	0
MBK	0	0	0	1	1	2	0
OPL	0	1	1	1	0	3	0
PEO	0	0	0	1	1	2	0
PGE	0	0	0	1	1	2	0
PGN	0	0	0	1	1	2	0
PKN	0	0	0	1	1	2	0
PKO	0	0	0	1	1	2	0
PLY	0	0	0	1	1	2	0
PZU	0	0	0	1	1	2	0
SPL	0	0	0	1	1	2	0
TPE	0	0	0	1	1	2	0

6.7. Discussions

This chapter presented the recommendations obtained by means of ratios representing various criteria of assessment of the current financial efficiency of a considered asset. Here we have

- Sharpe ratio and Sortino ratio used to maximise the premium per overall risk unit,
- Jensen's alpha and Treynor ratio used to maximise the premium for market risk.
- Roy's criterion used to minimise the probability of bearing the unacceptable loss.

This opulence of the used criteria explains to some extent the variety of recommendations attributed by the mentioned criteria to the same financial instrument. However, this is not the only reason of the differentiation between those recommendations. We should pay attention to a big differentiation of the recommendations established by Jensen's alpha and Treynor ratio used to maximise the premium for risk. This phenomenon is difficult to explain substantively. Hence, we deduce that while managing the chosen financial instruments, we should take into account a fixed set of recommendations that attributed to them. The next chapter will be dedicated to the issue of managing the fixed set of investment recommendations.

7. Management of Investment Recommendation Set

In Sections 5 and 6, the proposed procedure for recommendations was always considered in the case of one established criterion. Due to that we could mark all recommendations with a single symbol. In this chapter we will consider the relations between recommendations with various criteria attributed to them. For a bigger transparency of those considerations we will introduce a modified system of recommendation labels.

Any FS $\tilde{\Lambda}_\gamma \in \mathcal{F}(\mathbb{A})$ is called a recommendation. The subscript γ means any set of symbols identifying the kind of distinguished recommendation. Any recommendation $\tilde{\Lambda}_\gamma$ is represented by its membership function $\lambda_\gamma : \mathbb{A} \rightarrow [0, 1]$. Also, each recommendation can be noted as

$$\tilde{\Lambda}_\gamma = \frac{\lambda_\gamma(A^{--})}{A^{--}} + \frac{\lambda_\gamma(A^-)}{A^-} + \frac{\lambda_\gamma(A^0)}{A^0} + \frac{\lambda_\gamma(A^+)}{A^+} + \frac{\lambda_\gamma(A^{++})}{A^{++}} \quad (70)$$

In special cases we have

$$\tilde{\Lambda}_\gamma = \frac{0}{A^{--}} + \frac{0}{A^-} + \frac{0}{A^0} + \frac{0}{A^+} + \frac{0}{A^{++}} = \emptyset \quad (71)$$

$$\tilde{\Lambda}_\gamma = \frac{1}{A^{--}} + \frac{1}{A^-} + \frac{1}{A^0} + \frac{1}{A^+} + \frac{1}{A^{++}} = \mathbb{A} \quad (72)$$

Moreover, in notation (70) of recommendation $\tilde{\Lambda}_\gamma$, we can omit every advice $A \in \mathbb{A}$ satisfying the condition $\mu_\gamma(A) = 0$. Each security \check{S} is assigned a recommendation $\tilde{\Lambda}_{S,1}, \tilde{\Lambda}_{S,2}, \dots, \tilde{\Lambda}_{S,5} \in \mathcal{F}(\mathbb{A})$, where

- $\tilde{\Lambda}_{S,1}$ —recommendations obtained with the use of the Sharpe ratio,
- $\tilde{\Lambda}_{S,2}$ —recommendations obtained with the use of Jensen's alpha,
- $\tilde{\Lambda}_{S,3}$ —recommendations obtained with the use of the Treynor ratio,
- $\tilde{\Lambda}_{S,4}$ —recommendations obtained with the use of the Sortino ratio,
- $\tilde{\Lambda}_{S,5}$ —recommendations obtained with the use of Roy's criterion.

Example 9. Table 12 presents the recommendation

$$\tilde{\Lambda}_{CCC5} = \frac{0}{A^{--}} + \frac{0.0663}{A^-} + \frac{0.0663}{A^0} + \frac{1}{A^+} + \frac{0.9337}{A^{++}} = \frac{0.0663}{A^-} + \frac{0.0663}{A^0} + \frac{1}{A^+} + \frac{0.9337}{A^{++}}$$

Let us note that various criteria assign different recommendations to the same security. Each recommendation can bear a different imprecision risk. We propose to limit the acceptable recommendations to those that are characterised by the minimal risk of imprecision. However,

imprecision is evaluated by the means of two indices, which should be minimised. In this case, to minimise the risk, a multicriterial approach was implemented.

Each recommendation $\tilde{\Lambda}_{S,i}$ is given a pair $(d(\tilde{\Lambda}_{S,i}), e(\tilde{\Lambda}_{S,i}))$ where $d(\tilde{\Lambda}_{S,i})$ and $e(\tilde{\Lambda}_{S,i})$ respectively mean energy and entropy measures. On the recommendation set we define two preorders " $\tilde{\Lambda}_{S,i}$ is more acceptable than $\tilde{\Lambda}_{S,j}$ ":

$$\tilde{\Lambda}_{S,i} Q_1 \tilde{\Lambda}_{S,j} \Leftrightarrow d(\tilde{\Lambda}_{S,i}) \leq d(\tilde{\Lambda}_{S,j}) \tag{73}$$

$$\tilde{\Lambda}_{S,i} Q_2 \tilde{\Lambda}_{S,j} \Leftrightarrow e(\tilde{\Lambda}_{S,i}) \leq e(\tilde{\Lambda}_{S,j}) \tag{74}$$

Those preorders are formal models of ambiguity and indistinctness of information minimisation criterion. A multicriterial comparison defined by the preorders Q_1 and Q_2 is a model of satisfying the postulate of minimisation of both factors.

Using the multicriterial comparison (73) and (74) for each security \hat{S} we determine the Pareto optimum \mathbb{O}_s which includes all acceptable recommendations. To solve this optimisation task, we use an algorithm described in Appendix A.

Example 10. For each security \hat{S} described in Example 1, using respectively Sharpe ratio, Jensen’s alpha, Treynor ratio, Sortino ratio, and Roy’s criterion, we determined recommendations $\tilde{\Lambda}_{S,1}, \tilde{\Lambda}_{S,2}, \tilde{\Lambda}_{S,3}, \tilde{\Lambda}_{S,4}, \tilde{\Lambda}_{S,5}$. Those recommendations are presented in Tables 4, 6, 8, 10 and 12. Using the multidimensional comparison (73) and (74), for each security \hat{S} we determine the Pareto optimum \mathbb{O}_s containing the information of a minimum risk. Those optima are shown in Table 13.

Table 13. Pareto optimum of portfolio π components.

Stock Company	Pareto Optimum
ALR	$\{\tilde{\Lambda}_{ALR,3}\}$
CCC	$\{\tilde{\Lambda}_{CCC,4}, \tilde{\Lambda}_{CCC,5}\}$
CDR	$\{\tilde{\Lambda}_{CDR,1}, \tilde{\Lambda}_{CDR,2}, \tilde{\Lambda}_{CDR,3}, \tilde{\Lambda}_{CDR,4}, \tilde{\Lambda}_{CDR,5}\}$
CPS	$\{\tilde{\Lambda}_{CPS,1}, \tilde{\Lambda}_{CPS,3}, \tilde{\Lambda}_{CPS,4}, \tilde{\Lambda}_{CPS,5}\}$
DNP	$\{\tilde{\Lambda}_{DNP,1}, \tilde{\Lambda}_{DNP,2}, \tilde{\Lambda}_{DNP,3}, \tilde{\Lambda}_{DNP,4}, \tilde{\Lambda}_{DNP,5}\}$
JSW	$\{\tilde{\Lambda}_{JSW,1}, \tilde{\Lambda}_{JSW,2}, \tilde{\Lambda}_{JSW,3}, \tilde{\Lambda}_{JSW,4}, \tilde{\Lambda}_{JSW,5}\}$
KGH	$\{\tilde{\Lambda}_{KGH,1}, \tilde{\Lambda}_{KGH,2}, \tilde{\Lambda}_{KGH,3}, \tilde{\Lambda}_{KGH,4}, \tilde{\Lambda}_{KGH,5}\}$
LTS	$\{\tilde{\Lambda}_{LTS,1}, \tilde{\Lambda}_{LTS,2}, \tilde{\Lambda}_{LTS,3}, \tilde{\Lambda}_{LTS,4}, \tilde{\Lambda}_{LTS,5}\}$
LPP	$\{\tilde{\Lambda}_{LPP,1}, \tilde{\Lambda}_{LPP,2}, \tilde{\Lambda}_{LPP,3}, \tilde{\Lambda}_{LPP,4}, \tilde{\Lambda}_{LPP,5}\}$
MBK	$\{\tilde{\Lambda}_{MBK,1}, \tilde{\Lambda}_{MBK,2}, \tilde{\Lambda}_{MBK,3}, \tilde{\Lambda}_{MBK,4}, \tilde{\Lambda}_{MBK,5}\}$
OPL	$\{\tilde{\Lambda}_{OPL,3}\}$
PEO	$\{\tilde{\Lambda}_{PEO,1}, \tilde{\Lambda}_{PEO,2}, \tilde{\Lambda}_{PEO,3}, \tilde{\Lambda}_{PEO,4}, \tilde{\Lambda}_{PEO,5}\}$
PGE	$\{\tilde{\Lambda}_{PGE,1}, \tilde{\Lambda}_{PGE,3}, \tilde{\Lambda}_{PGE,4}, \tilde{\Lambda}_{PGE,5}\}$
PGN	$\{\tilde{\Lambda}_{PGN,1}, \tilde{\Lambda}_{PGN,2}, \tilde{\Lambda}_{PGN,3}, \tilde{\Lambda}_{PGN,4}, \tilde{\Lambda}_{PGN,5}\}$
PKN	$\{\tilde{\Lambda}_{PKN,3}, \tilde{\Lambda}_{PKN,5}\}$
PKO	$\{\tilde{\Lambda}_{PKO,1}, \tilde{\Lambda}_{PKO,2}, \tilde{\Lambda}_{PKO,3}, \tilde{\Lambda}_{PKO,4}, \tilde{\Lambda}_{PKO,5}\}$
PLY	$\{\tilde{\Lambda}_{PLY,1}, \tilde{\Lambda}_{PLY,2}, \tilde{\Lambda}_{PLY,3}, \tilde{\Lambda}_{PLY,4}, \tilde{\Lambda}_{PLY,5}\}$
PZU	$\{\tilde{\Lambda}_{PZU,1}, \tilde{\Lambda}_{PZU,2}, \tilde{\Lambda}_{PZU,3}, \tilde{\Lambda}_{PZU,4}, \tilde{\Lambda}_{PZU,5}\}$
SPL	$\{\tilde{\Lambda}_{SPL,1}, \tilde{\Lambda}_{SPL,2}, \tilde{\Lambda}_{SPL,3}, \tilde{\Lambda}_{SPL,4}, \tilde{\Lambda}_{SPL,5}\}$
TPE	$\{\tilde{\Lambda}_{TPE,1}, \tilde{\Lambda}_{TPE,2}, \tilde{\Lambda}_{TPE,3}, \tilde{\Lambda}_{TPE,4}, \tilde{\Lambda}_{TPE,5}\}$

The results obtained in Example 10 show that in the case of many securities there is a big variety in the sets of optimum recommendations.

To unify the final recommendations for each security \hat{S} we determine:

- A weakly justified recommendation (WJR) $\tilde{\Lambda}_{S,WJR}$ defined as the union of such Pareto optimal recommendations, which are linked to the security \hat{S} ;
- A strongly justified recommendation (SJR) $\tilde{\Lambda}_{S,SJR}$ defined as the intersection of such Pareto optimal recommendations, which are linked to the security \hat{S} .

The WJR $\tilde{\Lambda}_{S,WJR}$ and the SJR $\tilde{\Lambda}_{S,SJR}$ are determined respectively by their membership functions given as follows

$$\lambda_{S,WJR}(A) = \max\{\lambda_{S,i}(A) : \tilde{\Lambda}_{S,i} \in \mathbb{O}_S\} \quad (75)$$

$$\lambda_{S,SJR}(A) = \min\{\lambda_{S,i}(A) : \tilde{\Lambda}_{S,i} \in \mathbb{O}_S\} \quad (76)$$

Example 11. Separately for each security \hat{S} described in Example 1, imprecise recommendations $\tilde{\Lambda}_{S,1}, \tilde{\Lambda}_{S,2}, \tilde{\Lambda}_{S,3}, \tilde{\Lambda}_{S,4}, \tilde{\Lambda}_{S,5}$ are compared in more detail in Tables 14–33. In the two bottom rows of Tables 14–33 WJRs and SJRs are given along with their imprecision estimates. All Tables 14–33 are linked to the comments by using the names of discussed stock companies.

Table 14. Imprecise and Pareto optimal recommendations for ALR.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{ALR,i})$	$e(\tilde{\Lambda}_{ALR,i})$
$\tilde{\Lambda}_{ALR,1}$	0	1	1	1	0	3	0
$\tilde{\Lambda}_{ALR,2}$	0.4138	1	0.5862	0.5862	0	2.5862	0.3303
$\tilde{\Lambda}_{ALR,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{ALR,4}$	0	1	1	1	0	3	0
$\tilde{\Lambda}_{ALR,5}$	0	1	1	1	0	3	0
$\tilde{\Lambda}_{ALR,WJR}$	1	1	0	0	0		
$\tilde{\Lambda}_{ALR,SJR}$	1	1	0	0	0		

For the ALR shares only the following recommendation is Pareto optimal

$$\tilde{\Lambda}_{ALR,3} = \frac{1}{A^{--}} + \frac{1}{A^{-}} = \tilde{\Lambda}_{ALR,WJR} = \tilde{\Lambda}_{ALR,SJR} \quad (77)$$

which was obtained by means of the Treynor ratio. In such situation, for this recommendation, WJR and SJR are identical. The advice Sell and Reduce is recommended by the advisor with the degree that equals 1. It means that the advisor is prepared to take full responsibility for making the investment decisions resulting from the suggested advice. The other recommendations are rejected by the advisor. In such case it is the investor who takes full responsibility for making any other decision resulting from the rejected recommendations of Buy, Accumulate and Hold.

Table 15. Imprecise and Pareto optimal recommendations for CCC.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{CCC,i})$	$e(\tilde{\Lambda}_{CCC,i})$
$\tilde{\Lambda}_{CCC,1}$	0	0.3812	0.3812	1	0.6188	2.3812	0.2966
$\tilde{\Lambda}_{CCC,2}$	0	0.8840	0.8840	1	0.1160	2.8840	0.0748
$\tilde{\Lambda}_{CCC,3}$	0.4804	1	0.5196	0.5196	0	2.5196	0.4050
$\tilde{\Lambda}_{CCC,4}$	0	0.3370	0.3370	1	0.6630	2.3370	0.0228
$\tilde{\Lambda}_{CCC,5}$	0	0.0663	0.0663	1	0.9337	2.0663	0.0414
$\tilde{\Lambda}_{CCC,WJR}$	0	0.3370	0.3370	1	0.9337		
$\tilde{\Lambda}_{CCC,SJR}$	0	0.0663	0.0663	1	0.6630		

For the CCC shares, WJR and SJR are as follows

$$\tilde{\Lambda}_{CCC,WJR} = \frac{0.337}{A^-} + \frac{0.337}{A^0} + \frac{1}{A^+} + \frac{0.9337}{A^{++}}, \quad (78)$$

$$\tilde{\Lambda}_{CCC,SJR} = \frac{0.0663}{A^-} + \frac{0.0663}{A^0} + \frac{1}{A^+} + \frac{0.663}{A^{++}}. \quad (79)$$

From the distribution of a recommendation degree represented by WJR it shows that the advisor definitely rejects the Sell recommendation. WJR also tells us that the Accumulate and Buy recommendations can be taken into consideration. Additional information on the distribution of the responsibility for the decisions taken is reflected in SJR. It supplements the picture with the following information:

- The investor bears almost all the responsibility for making an investment decision resulting from the Reduce and Hold recommendations,
- The advisor bears full responsibility for making an investment decision resulting from the Accumulate recommendations,
- The investor and the advisor share the responsibility among themselves for making the investment decision based on the Buy recommendation, however, the advisor bear approximately two-thirds of that responsibility.

After analysing the information and interpretations, the investor takes the decision. We can suspect that the investor characterised by risk-aversion will choose the Accumulate recommendation while the investor who is a risk-taker will choose the Buy advice.

Table 16. Imprecise and Pareto optimal recommendations for CDR.

Criterion	A^{--}	A^-	A^0	A^+	A^{++}	$d(\tilde{\Lambda}_{CDR,i})$	$e(\tilde{\Lambda}_{CDR,i})$
$\tilde{\Lambda}_{CDR,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{CDR,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{CDR,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{CDR,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{CDR,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{CDR,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{CDR,SJR}$	0	0	0	0	0		

For the CDR shares the following WJR and SJR were determined

$$\tilde{\Lambda}_{CDR,WJR} = \frac{1}{A^{--}} + \frac{1}{A^-} + \frac{1}{A^+} + \frac{1}{A^{++}}, \quad (80)$$

$$\tilde{\Lambda}_{CDR,SJR} = \emptyset \quad (81)$$

From the distribution of a recommendation degree represented by WJR it shows that the advisor definitely rejects the Hold recommendation. It means that the advisor recommends an investment activity without defining its kind. SJR shows that it is the investor who bears full responsibility for any decisions made. It is obvious that such a recommendation is not useful so in such a situation we state that there is no useful recommendation.

Table 17. Imprecise and Pareto optimal recommendations for CPS.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{CPS,i})$	$e(\tilde{\Lambda}_{CPS,i})$
$\tilde{\Lambda}_{CPS,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{CPS,2}$	0	0.1442	0.1442	1	0.8558	2.1442	0.0947
$\tilde{\Lambda}_{CPS,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{CPS,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{CPS,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{CPS,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{CPS,SJR}$	0	0	0	0	0		

For the CPS shares, WJR and SJR are determined by (80) and (81). In this situation we state that there is no useful recommendation.

Table 18. Imprecise and Pareto optimal recommendations for DNP.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{DNP,i})$	$e(\tilde{\Lambda}_{DNP,i})$
$\tilde{\Lambda}_{DNP,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{DNP,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{DNP,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{DNP,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{DNP,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{DNP,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{DNP,SJR}$	0	0	0	0	0		

For the DNP shares, WJR and SJR are determined by (80) and (81). In this situation we state that there is no useful recommendation.

Table 19. Imprecise and Pareto optimal recommendations for JSW.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{JSW,i})$	$e(\tilde{\Lambda}_{JSW,i})$
$\tilde{\Lambda}_{JSW,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{JSW,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{JSW,3}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{JSW,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{JSW,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{JSW,WJR}$	0	0	0	1	1		
$\tilde{\Lambda}_{JSW,SJR}$	0	0	0	1	1		

For the JSW shares, the following WJR and SJR were determined

$$\tilde{\Lambda}_{JSW,WJR} = \frac{1}{A^{+}} + \frac{1}{A^{++}} \tag{82}$$

$$\tilde{\Lambda}_{JSW,SJR} = \frac{1}{A^{+}} + \frac{1}{A^{++}} \tag{83}$$

From the distribution of a recommendation degree represented by WJR it shows that the advisor definitely rejects the Sell, Reduce and Hold recommendations. The advisor strongly recommends Accumulate or Buy. SJR shows that it is the advisor who is willing to take full responsibility for taking the investment decisions resulting from the advised recommendations.

Table 20. Imprecise and Pareto optimal recommendations for KGH.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{KGH,i})$	$e(\tilde{\Lambda}_{KGH,i})$
$\tilde{\Lambda}_{KGH,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{KGH,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{KGH,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{KGH,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{KGH,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{KGH,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{KGH,SJR}$	0	0	0	0	0		

For the KGH shares, WJR and SJR are determined by (80) and (81). In this situation we state that there is no useful recommendation.

Table 21. Imprecise and Pareto optimal recommendations for LTS.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{LTS,i})$	$e(\tilde{\Lambda}_{LTS,i})$
$\tilde{\Lambda}_{LTS,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LTS,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LTS,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{LTS,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LTS,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LTS,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{LTS,SJR}$	0	0	0	0	0		

For the LTS shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 22. Imprecise and Pareto optimal recommendations for LPP.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{LPP,i})$	$e(\tilde{\Lambda}_{LPP,i})$
$\tilde{\Lambda}_{LPP,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LPP,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LPP,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{LPP,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LPP,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{LPP,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{LPP,SJR}$	0	0	0	0	0		

For the LPP shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 23. Imprecise and Pareto optimal recommendations for MBK.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{MBK,i})$	$e(\tilde{\Lambda}_{MBK,i})$
$\tilde{\Lambda}_{MBK,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{MBK,2}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{MBK,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{MBK,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{MBK,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{MBK,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{MBK,SJR}$	0	0	0	0	0		

For the MBK shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 24. Imprecise and Pareto optimal recommendations for OPL.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{OPL,i})$	$e(\tilde{\Lambda}_{OPL,i})$
$\tilde{\Lambda}_{OPL,1}$	0	0.8719	0.8719	1	0.1281	2.8719	0.0833
$\tilde{\Lambda}_{OPL,2}$	0	1	1	1	0	3	0
$\tilde{\Lambda}_{OPL,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{OPL,4}$	0	0.8769	0.8769	1	0.1231	2.8769	0.0798
$\tilde{\Lambda}_{OPL,5}$	0	1	1	1	0	3	0
$\tilde{\Lambda}_{OPL,WJR}$	1	1	0	0	0		
$\tilde{\Lambda}_{OPL,SJR}$	1	1	0	0	0		

For the OPL shares, Pareto optimal recommendation (77) is only the one determined by the Treynor ratio. From the distribution of a recommendation degree represented by WJR it shows that the advisor definitely rejects the Hold, Accumulate and Buy recommendations. The advisor strongly recommends Sell or Reduce. SJR shows that the advisor is willing to take full responsibility for taking the investment decisions resulting from the advised recommendations.

Table 25. Imprecise and Pareto optimal recommendations for PEO.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{PEO,i})$	$e(\tilde{\Lambda}_{PEO,i})$
$\tilde{\Lambda}_{PEO,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PEO,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PEO,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{PEO,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PEO,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PEO,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{PEO,SJR}$	0	0	0	0	0		

For the PEO shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 26. Imprecise and Pareto optimal recommendations for PGE.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{PGE,i})$	$e(\tilde{\Lambda}_{PGE,i})$
$\tilde{\Lambda}_{PGE,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PGE,2}$	0	0.6563	0.6563	1	0.3437	2.6563	0.2598
$\tilde{\Lambda}_{PGE,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{PGE,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PGE,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PGE,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{PGE,SJR}$	0	0	0	0	0		

For the PGE shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 27. Imprecise and Pareto optimal recommendations for PGN.

Kryterium	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{PGN,i})$	$e(\tilde{\Lambda}_{PGN,i})$
$\tilde{\Lambda}_{PGN,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PGN,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PGN,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{PGN,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PGN,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PGN,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{PGN,SJR}$	0	0	0	0	0		

For the PGN shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 28. Imprecise and Pareto optimal recommendations for PKN.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{PKN,i})$	$e(\tilde{\Lambda}_{PKN,i})$
$\tilde{\Lambda}_{PKN,1}$	0	0.2692	0.2692	1	0.7308	2.2692	0.1926
$\tilde{\Lambda}_{PKN,2}$	0	1	1	1	0	3	0
$\tilde{\Lambda}_{PKN,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{PKN,4}$	0	0.2308	0.2308	1	0.7692	2.2308	0.1607
$\tilde{\Lambda}_{PKN,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PKN,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{PKN,SJR}$	0	0	0	0	0		

For the PKN shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 29. Imprecise and Pareto optimal recommendations for PKO.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{PKO,i})$	$e(\tilde{\Lambda}_{PKO,i})$
$\tilde{\Lambda}_{PKO,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PKO,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PKO,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{PKO,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PKO,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PKO,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{PKO,SJR}$	0	0	0	0	0		

For the PKO shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 30. Imprecise and Pareto optimal recommendations for PLY.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{PLY,i})$	$e(\tilde{\Lambda}_{PLY,i})$
$\tilde{\Lambda}_{PLY,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PLY,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PLY,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{PLY,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PLY,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PLY,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{PLY,SJR}$	0	0	0	0	0		

For the PLY shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 31. Imprecise and Pareto optimal recommendations for PZU.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{PZU,i})$	$e(\tilde{\Lambda}_{PZU,i})$
$\tilde{\Lambda}_{PZU,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PZU,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PZU,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{PZU,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PZU,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{PZU,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{PZU,SJR}$	0	0	0	0	0		

For the PZU shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 32. Imprecise and Pareto optimal recommendations for SPL.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{SPL,i})$	$e(\tilde{\Lambda}_{SPL,i})$
$\tilde{\Lambda}_{SPL,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{SPL,2}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{SPL,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{SPL,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{SPL,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{SPL,WJR}$	1	1	0	1	1		
$\tilde{\Lambda}_{SPL,SJR}$	0	0	0	0	0		

For the SPL shares, WJR and SJR are determined by (80) and (81). In this situation there is no useful recommendation.

Table 33. Imprecise and Pareto optimal recommendations for TPE.

Criterion	A^{--}	A^{-}	A^0	A^{+}	A^{++}	$d(\tilde{\Lambda}_{TPE,i})$	$e(\tilde{\Lambda}_{TPE,i})$
$\tilde{\Lambda}_{TPE,1}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{TPE,2}$	0	1	1	1	0	3	0
$\tilde{\Lambda}_{TPE,3}$	1	1	0	0	0	2	0
$\tilde{\Lambda}_{TPE,4}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{TPE,5}$	0	0	0	1	1	2	0
$\tilde{\Lambda}_{TPE,WJR}$	1	1	1	1	1		
$\tilde{\Lambda}_{TPE,SJR}$	0	0	0	0	0		

For the TPE shares, WJR and SJR were determined as follows

$$\tilde{\Lambda}_{TPE,WJR} = \mathbb{A} \quad (84)$$

and (81). WJR informs us that the advisor does not exclude any recommendation. SJR shows that full responsibility for taking any investment decision goes to the investor. Therefore, there is no useful recommendation.

Summing up, for the public companies considered in the examples, in most cases there was no useful recommendation. Such a situation occurred in the case of CDR, CPS, DNP, KGH, LTS, LPP, MBK, PEO, PGE, PGN, PKN, PKO, PLY, PZU, SPL and TPE. Only for three following companies: ALR, CCC

and OPL the recommendations could be considered useful. This situation does not differ from the real phenomena in financial markets. The number of useless recommendations can be decreased by limiting the number of assessment criteria. Also, another set of criteria can be implemented. The solution to those problems should be searched based on finance.

An observation can be useful that each pair of WJR and SJR might be presented as an intuitionistic fuzzy set [90] representing a justified recommendation (JR). Then any JR is defined by its membership function equal to the SJR membership function and by its non-membership function equal to the membership function of WJR complement.

8. Conclusions

In the subject literature it is shown that OFNs are a more convenient tool for financial analysis than FNs. Therefore, the most important achievement of this work is the implementation of OFNs into the algorithmic system supporting investment decisions. In my best knowledge, the obtained algorithmic system is the only one that applies any set of profitability criteria evaluated with the use of OFNs. Until now, only an analogous system was known to be linked to the Sharpe's criterion. For any security, this simple system assigns exactly one imprecise recommendation. The algorithmic system described in Sections 5 and 6 assigns each security many different imprecise recommendations. For this reason, in Section 7, the proposed system is equipped with an imprecise recommendation management module.

Obtained results may provide theoretical foundations for constructing a robo-advice system supporting investment decisions. Then, we can use determined recommendations as behavioural premises for investment decisions. The attempt to use chosen recommendations multiple times leads to establishing an investing strategy. In Example 11, the interpretation of determined recommendations was presented. The shown case study is the reflection of only a little share of the set of all possible recommendations. Therefore, taking up research on a wider spectrum of recommendations established by the described algorithms seems justified. It should be an empirical research leading to establishing the heuristic investment strategy.

In financial practice, we can meet with the situation when part of PV securities is imprecisely evaluated without a subjective forecast of future quotation changes. Such PV should be evaluated by unoriented FNs. Against imprecisely evaluated PV, other securities may be equipped with a subjective forecast of rise in quotation. Such PV should be evaluated by positively oriented OFNs. In both of these cases the membership functions are identical. This results in the impossibility of a simultaneous comparison of oriented PV and unoriented PV. This is a significant disadvantage of the proposed algorithmic system supporting invest-making. The intention to deal with this inconvenience points to another direction of research into the OFNs theory.

The obtained results may as well be a starting point for future research on the impact of the PV imprecision and orientation on the investment recommendation determined with the use of algorithms presented in this paper. The implementation of intuitionistic fuzzy sets should be preceded by a theoretical and empirical research of the expediency of such approach for a representation of the justified recommendations mentioned in Section 7.

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Appendix A

Given security \check{S} is assigned recommendations $\tilde{\Lambda}_{S,\gamma} \in \mathcal{F}(\mathbb{A})$, where the subscript $\gamma = 1, 2, \dots, n$ identifies the method used to derive this recommendation. The acceptability of any recommendation increases along with the decrease of its ambiguity and with the decrease of indistinctness. Therefore,

the subset of the most acceptable recommendation systems is distinguished as the Pareto's optimum, determined as a two-criteria comparison of minimisation recommendation ambiguity and minimisation recommendation indistinctness.

The ambiguity of recommendation $\tilde{\Lambda}_{S,\gamma}$ is valued by energy measure $d(\tilde{\Lambda}_{S,\gamma})$ calculated with the use of (5). The indistinctness of recommendation $\tilde{\Lambda}_{S,\gamma}$ is valued by energy measure $e(\tilde{\Lambda}_{S,\gamma})$ determined by (6). Therefore, we represent each recommendation by the pair

$$(d(\tilde{\Lambda}_{S,\gamma}), e(\tilde{\Lambda}_{S,\gamma})) = (d_{S,\gamma}, e_{S,\gamma}) \quad (\text{A1})$$

On the recommendation set we define two preorders " $\tilde{\Lambda}_{S,i}$ is more than $\tilde{\Lambda}_{S,j}$ ":

$$\tilde{\Lambda}_{S,i} Q_1 \tilde{\Lambda}_{S,j} \Leftrightarrow d_{S,i} \leq d_{S,j} \quad (\text{A2})$$

$$\tilde{\Lambda}_{S,i} Q_2 \tilde{\Lambda}_{S,j} \Leftrightarrow e_{S,i} \leq e_{S,j} \quad (\text{A3})$$

The set of all acceptable recommendations we appoint as Pareto's optimum \mathbb{O}_S determined by multi-criterial comparison $Q_1 \cap Q_2$. To solve this optimisation task, we adapt an analogous algorithm presented in [91]. In order to determine the Pareto optimum \mathbb{O}_S , we execute the following algorithm:

STEP 1:

$$\mathbb{O}_S := \{\tilde{\Lambda}_{S,1}\} \quad (\text{A4})$$

STEP 2:

$$i := 2 \quad (\text{A5})$$

STEP 3:

$$\begin{aligned} & \left(\forall_{\tilde{\Lambda}_{S,j} \in \mathbb{O}_S} : (d_{S,j} \leq d_{S,i} \wedge e_{S,j} \geq e_{S,i}) \vee (d_{S,j} \geq d_{S,i} \wedge e_{S,j} \leq e_{S,i}) \right) \\ & \quad \vee (d_{S,j} \geq d_{S,i} \wedge e_{S,j} \geq e_{S,i}) \\ & \quad \Downarrow \\ & \mathbb{O}_S := \mathbb{O}_S \cup \{\tilde{\Lambda}_{S,i}\} \end{aligned} \quad (\text{A6})$$

STEP 4:

$$\forall_{\tilde{\Lambda}_{S,j} \in \mathbb{O}_S} : ((d_{S,j} < d_{S,i} \wedge e_{S,j} > e_{S,i}) \Rightarrow \mathbb{O}_S := \mathbb{O}_S / \{\tilde{\Lambda}_{S,j}\}) \quad (\text{A7})$$

STEP 5:

$$i := i + 1 \quad (\text{A8})$$

STEP 6:

$$i > n \Rightarrow \text{go to STOP} \quad (\text{A9})$$

STEP 7:

$$\text{go to STEP 3} \quad (\text{A10})$$

STOP.

In this way, we obtain the sequence \mathbb{O}_S of partial optima of Pareto.

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