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An Efficient Parallel Extragradient Method for Systems of Variational Inequalities Involving Fixed Points of Demicontractive Mappings

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Abstract: Herein, we present a new parallel extragradient method for solving systems of variational inequalities and common fixed point problems for demicontractive mappings in real Hilbert spaces. The algorithm determines the next iterate by computing a computationally inexpensive projection onto a sub-level set which is constructed using a convex combination of finite functions and an Armijo line-search procedure. A strong convergence result is proved without the need for the assumption of Lipschitz continuity on the cost operators of the variational inequalities. Finally, some numerical experiments are performed to illustrate the performance of the proposed method.

Keywords: extragradient method; variational inequalities; common solution; common fixed point; pseudomonotone; demicontractive

MSC: 65K15; 47J20; 65J15; 90C33

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1. Introduction

Let H be a real Hilbert space and C be a nonempty, closed, and convex subset of H. Let $A : C \to H$ be an operator. The Variational Inequalities (VI) is defined as finding $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1}$$

The solution set of the VI (1) is denoted by $VI(C, A)$. Mathematically, the VI is considered as a powerful tool for studying many nonlinear problems arising in mechanics, optimization, control network, equilibrium problems, etc.; see [1–3]. The problem of finding a common solution of a systems of VI has received a lot of attention by many authors recently, see, e.g., in [4–10] and references therein. This problem covers as special cases, convex feasibility problem, common equilibrium problem, etc. In this paper, we consider the following common problem.

Problem 1. *Find an element $x^* \in C$ such that*

$$x^* \in \left(\bigcap_{i=1}^{N} VI(C, A_i) \right) \cap \left(\bigcap_{j=1}^{M} Fix(T_j) \right), \tag{2}$$

where for $i = 1, 2, \ldots, N$, $A_i : H \to H$ are pseudomonotonotone operators, $j = 1, 2, \ldots, M$, $T_j : H \to H$ are k_j-demicontractive mappings, $Fix(T_j) = \{x \in H : T_j x = x\}$ denotes the fixed point set of T_j.

The motivation for considering Problem 1 lies in its possible applications to mathematical models whose constraints can be expressed as the common variational inequalities and common fixed point problems. This happen in particular, in network resource allocations, image processing, Nash equilibrium problem, etc., see, e.g., in [11–14].
The simplest method for solving the VI (1) is the projection method of Goldstein [15] which is a natural extension of the gradient projection method, and for $x_0 \in C$, $\lambda > 0$ it is given by

$$x_{n+1} = P_C(x_n - \lambda A x_n), \quad n \geq 0. \tag{3}$$

The projection method (3) converges weakly to a solution of VI (1) if and only if A satisfies some strong conditions such as α-strongly monotone and L-Lipschitz continuous. When this condition is relaxed, the method fails to convergence to any solution of the VI (1). Korpelevich [16] later introduced an Extragradient Method (EgM) for solving the VI when A is monotone and L-Lipschitz continuous as follows, for $x_0 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \quad n \geq 0, \end{cases}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$. The EgM has been extended to infinite-dimensional spaces by many authors, see, for instance, in [7,17–23]. More so, several modifications of the EgM have been introduced recently, see in [24–30]. For finding a common element in the set of solution of monotone variational inequalities and fixed point of k-demicontractive mapping, Mainge [14] introduced the following extragradient method, for $x_0 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = P_C(x_n - \lambda_n A y_n), \\ x_{n+1} = [(1-\alpha)I + \alpha T]u_n, \quad u_n = z_n - \gamma_n B z_n, \quad n \geq 0, \end{cases} \tag{4}$$

where $\{\lambda_n\}, \{\gamma_n\} \subset (0, \infty), w \in [0,1]$, $A : C \to H$ is a monotone and L-Lipschitz continuous, $T : H \to H$ is a k-demicontractive mapping and $B : H \to H$ is β-strongly monotone operator with $\beta > 0$. The author proved a strong convergence for the sequence generated by (4) provided the step size λ_n satisfies

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{L}. \tag{5}$$

Recently, Hieu et al. [31] modified (4) and introduced the following extragradient method for approximating a common solution of VI and fixed point problem; given $x_0 \in C$, compute x_{n+1} via

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = P_C(x_n - \rho_n A y_n), \\ w_n = P_C(x_n - \rho_n A z_n), \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \quad u_n = w_n - \gamma_n B w_n, \quad n \geq 0, \end{cases} \tag{6}$$

where $\{\rho_n\}, \{\lambda_n\} \subset (0, \infty)$ such that $0 \leq \lambda_n \leq \rho_n$, $\{\alpha_n\} \subset (0,1)$, A, T and B are as defined for Algorithm (4). They also proved a strong convergence for the sequence generated by (6) with the aid of (5). An obvious disadvantage in (4) and (6) which impedes their wide usage is the assumption that the Lipschitz constant of A admits a simple estimate. Moreover, in many practical problems, the cost operator may not satisfies Lipschitz condition.

On the other hand, for finding a common fixed point of quasi-nonexpansive mappings, Anh and Hieu [11,32] proposed a parallel hybrid algorithm as follows,

$$\begin{cases} x_0 \in C, \\ y_n^i = \alpha_n x_n + (1-\alpha_n) T_i x_n, \quad i = 1,2,\ldots,N, \\ i_n = Argmax\{\|y_n^i - x_n\| : i = 1,2,\ldots,N\}, \quad \bar{y}_n := y_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \|v - \bar{y}_n\| \le \|v - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{cases} \tag{7}$$

Furthermore, Censor et al. [6] proposed a parallel hybrid-extragradient method for finite family of variational inequalities as follows; choose $x_0 \in H$, compute

$$\begin{cases} y_n^i = P_{C_i}(x_n - \lambda_n^i A_i x_n), \\ z_n^i = P_{C_i}(x_n - \lambda_n^i A_i y_n^i), \\ C_n^i = \{z \in H : \langle x_n - z_n^i, z - x_n - \gamma_n^i (z_n^i - x_n)\rangle \le 0\}, \\ Q_n = \cap_{i=1}^N C_n^i, \\ W_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \le 0\}, \\ C_{n+1} = P_{Q_n \cap W_n} x_0. \end{cases} \tag{8}$$

Motivated by (7) and (8), Anh and Phuong [8] recently introduced the following Algorithm 1 parallel hybrid-extragradient method for solving variational inequalities and fixed point of nonexpansive mappings.

Algorithm 1: PHEM

Initialization: Given $x_0 \in H$, $\lambda_{n,i} \in \left(0, \frac{1-a}{L_i}\right)$, where L_i are the Lipschitz constant of A_i, $i = 1,2,\ldots,N$, $a \in (0,1)$, $\{\alpha_{n,i}\} \in (0,1)$, $\{\gamma_{n,i}\} \subset (0, \frac{1}{2})$, $n \ge 0$.

Iterative steps: Compute in parallel

$$\begin{cases} y_n^i = P_{C_i}(x_n - \lambda_n^i A_i x_n), \\ z_n^i = P_{C_i}(x_n - \lambda_n^i A_i y_n^i), \\ t_n^i = \alpha_{n,i} x_n + (1-\alpha_{n,i}) T_i z_n^i, \\ C_{n,i} = \{x \in C_i, \langle x_n - t_n^i, x - x_n - \gamma_{n,i}(t_n^i - x_n)\rangle \le 0\}, \\ Q_n = \cap_{i=1}^N C_{n,i}, \\ W_n = \{x \in H : \langle x_0 - x_n, x - x_n\rangle \le 0\}, \\ x_{n+1} = P_{Q_n \cap W_n} x_0, \quad n = n+1. \end{cases} \tag{9}$$

Meanwhile, Hieu [33] introduced a parallel hybrid-subgradient extragradient method which also required finding a farthest element from the iterate x_n as follows.

The author proved that the sequence generated by Algorithm 2 converges strongly to a solution of the systems of VI.

Algorithm 2: PHSEM

Initialization: Choose $x_0 \in H$, $0 < \lambda < \frac{1}{L}$. Set $n = 0$. **Step 1:** Find N projections z_n^i on C_i in parallel, i.e.,

$$y_n^i = P_C(x_n - \lambda A_i x_n), \quad i = 1, \dots, N.$$

Step 2: Find N projections z_n^i on half-spaces T_n^i in parallel, i.e.,

$$z_n^i = P_{T_n^i}(x_n - \lambda A_i y_n^i), \quad i = 1, \dots, N,$$

where $T_n^i = \{v \in H : \langle x_n - \lambda A_i x_n - y_n^i, v - y_n^i \rangle \leq 0\}$.

Step 3: Find the farthest element from x_n among z_n^i, i.e.,

$$i_n = argmax\{\|z_n^i - x_n\| : i = 1, \dots, N\}, \quad \bar{z}_n = z_n^{i_n}.$$

Step 4: Construct the half-spaces C_n and Q_n such that

$$C_n = \{w \in H : \|\bar{z}_n - w\| \leq \|x_n - w\|\},$$
$$Q_n = \{w \in H : \langle w - x_n, x_n - x_0 \rangle \geq 0\}.$$

Step 5: Find the next iterate via

$$x_{n+1} = P_{C_n \cap Q_n} x_0.$$

Set $n = n + 1$ and go to Step 1.

However, it should be observed that at each step in the parallel hybrid-extragradient methods mentioned above, one needs to calculate a projection onto the intersection of two sets Q_n and W_n. This can be computationally expensive when the feasible set is not simple. Moreover, the convergence of the algorithms require prior knowledge of the Lipschitz constants of A_i which are very difficult to estimate in practice.

Motivated by these results, in this paper, we introduce an efficient parallel-extragradient method which does not requires the computation of projection onto $Q_n \cap W_n$ and the prior estimates of the Lipschitz constants of A_i for $i = 1, 2, \dots, N$. In particular, we highlight some contributions in this paper as follows.

- In our method, the involved cost operators A_i do not need to satisfy the Lipschitz condition. Instead, we assumed that A_i are pseudomonotone and weakly sequentially continuous which is more general than the monotone and Lipschitz continuous used in previous results.
- The sequence generated by our method converges strongly to a solution of (2) without the aid of prior estimate of a Lipschitz constant.
- Furthermore, we performed only single projection onto C in parallel and our algorithm does not need to find the farthest element from the iterate x_n.
- More so, our algorithm does not require the computation of projection onto $Q_n \cap W_n$ which make it easier for computations.

2. Preliminaries

In this section, we give some Definitions and basic results that will be used in our subsequent analysis. Let H be a real Hilbert space. The weak and the strong convergence of $\{x_n\} \subset H$ to $x \in H$ is denoted by $x_n \rightharpoonup x$ and $x_n \to x$ as $n \to \infty$, respectively. Let C be a nonempty, closed, and convex subset of H. The metric projection of $x \in H$ onto C is defined as the necessary unique vector $P_C(x)$ satisfying

$$||x - P_C x|| \leq ||x - y|| \quad \forall\, y \in C.$$

It is well known that P_C has the following properties (see, e.g., in [34,35]).

(i) For each $x \in H$ and $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{10}$$

(ii) For any $x, y \in H$,

$$\langle P_C x - P_C y, x - y \rangle \geq ||P_C x - P_C y||^2.$$

(iii) For any $x \in H$ and $y \in C$,

$$||P_C x - y||^2 \leq ||x - y||^2 - ||x - P_C x||^2. \tag{11}$$

Next, we state some classes of functions that play essential roles in our convergence analysis.

Definition 1. *The operator $A : C \to H$ is said to be*

1. *β-strongly monotone if there exists $\beta > 0$ such that*

$$\langle Ax - Ay, x - y \rangle \geq \beta \|x - y\| \quad \forall x, y \in C;$$

2. *monotone if*

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall x, y \in C;$$

3. *η-strongly pseudomonotone if there exists $\eta > 0$ such that*

$$\langle Ax, y - x \rangle \geq 0 \Rightarrow \langle Ay, y - x \rangle \geq \gamma \|x - y\|^2,$$

for all $x, y \in C$;

4. *pseudomonotone if for all $x, y \in C$*

$$\langle Ax, y - x \rangle \geq 0 \Rightarrow \langle Ay, y - x \rangle \geq 0;$$

5. *L-Lipschitz continuous if there exists a constant $L > 0$ such that*

$$\|Ax - Ay\| \leq L \|x - y\|, \quad \forall x, y \in C.$$

When $L \in (0, 1)$, then A is called a contraction;

6. *weakly sequentially continuous if for any $\{x_n\} \subset H$ such that $x_n \rightharpoonup \bar{x}$ implies $Ax_n \rightharpoonup A\bar{x}$.*

It is easy to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4), but the converse implications do not hold in general, see, for instance, in [21].

Lemma 1 ([36] Lemma 2.1)**.** *Consider the VIP (1) with C being a nonempty closed convex subset of H and $A : C \to H$ is pseudomonotone and continuous. Then, $\bar{x} \in VIP(C, A)$ if and only if*

$$\langle Ay, y - \bar{x} \rangle \geq 0 \quad \forall y \in C.$$

Definition 2 ([37])**.** *A mapping $T : H \to H$ is called*

(i) *nonexpansive if*

$$||Tu - Tv|| \leq ||u - v||, \quad \forall\, u, v \in H;$$

(ii) *quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and*

$$||Tu - z|| \leq ||u - z||, \quad \forall\, u \in H, z \in F(T);$$

(iii) *μ-strictly pseudocontractive if there exists a constant $\mu \in [0,1)$ such that*

$$||Tu - Tv||^2 \leq ||u - v||^2 + \mu||(I - T)u - (I - T)v||^2 \quad \forall\, u, v \in H;$$

(iv) *κ-demicontractive mapping if there exists $\kappa \in [0,1)$ and $F(T) \neq \emptyset$ such that*

$$||Tu - z||^2 \leq ||u - z||^2 + \kappa||u - Tu||^2, \quad \forall\, u \in H, z \in F(T).$$

It is easy to see that the class of demicontractive mappings contains the class of quasi-nonexpansive and strictly pseudocontractive mappings. Due to this generality and its possible applications in applied analysis, the class of demicontractive mapping has continue to attracts the attention of many authors in this decade.

A bounded linear operator A on H is said to be strongly positive if there exists a constant $\gamma > 0$ such that

$$\langle x, Ax \rangle \geq \gamma \|x\|^2, \quad \forall x \in H.$$

It is known that when A is strongly positive bounded linear operator with $0 < \rho < \frac{1}{\|A\|}$, then

$$\|I - \rho A\| \leq 1 - \rho\gamma.$$

For any real Hilbert space H, it is known that the following identities hold (see, e.g., in [38]).

Lemma 2. *For all $x, y, z \in H$, then*

(i) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$,
(ii) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$,
(iii) $||\eta x + (1 - \eta)y||^2 = \eta||x||^2 + (1 - \eta)||y||^2 - \eta(1 - \eta)||x - y||^2, \quad \forall \eta \in [0,1]$.

Lemma 3 ([24])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and h be a real-valued function on H. Suppose $D = \{x \in C : h(x) \leq 0\}$ is nonempty and h is Lipschitz continuous on C with modulus $\vartheta > 0$, then*

$$dist(x, D) \geq \vartheta^{-1} \max\{h(x), 0\} \quad \forall x \in C.$$

Lemma 4 ([39])**.** *Let $\{\Gamma_n\}$ be a non-negative real sequence satisfying $\Gamma_{n+1} \leq (1 - \theta_n)\Gamma_n + \theta_n b_n$, where $\{\theta_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} \theta_n = \infty$ and $\{b_n\}$ is a sequence such that $\limsup_{n\to\infty} b_n \leq 0$. Then, $\lim_{n\to\infty} \Gamma_n = 0$.*

Lemma 5 ((Lemma 3.1) [37])**.** *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Consider the integer $\{m_k\}$ defined by*

$$m_k = \max\{j \leq k : a_j < a_{j+1}\}.$$

Then $\{m_k\}$ is a non-decreasing sequence verifying $\lim_{n\to\infty} m_n = \infty$, and for all $k \in \mathbb{N}$, the following estimate holds,

$$a_{m_k} \leq a_{m_k+1}, \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

3. Algorithm and Convergence Analysis

In this section, we describe our algorithm and prove its convergence under suitable conditions. Let H be a real Hilbert space and C be a nonempty, closed, and convex subset of H. We suppose that the following assumptions hold.

Assumption 1.

(A1) *For $i = 1, 2, \ldots, N$, $A_i : H \to H$ are pseudomonotone, uniformly continuous and weakly sequentially continuous operators;*
(A2) *For $j = 1, 2, \ldots, M$, $T_j : H \to H$ are κ_j-demicontractive mappings with $\kappa = \max\{\kappa_j : 1 \le j \le M\}$ such that $I - T_j$ are demiclosed at zero;*
(A3) *$f : H \to H$ is an α-contraction mapping with $\alpha \in (0, 1)$;*
(A4) *For $k = 1, 2, \ldots, \overline{N}$, $B_k : H \to H$ are strongly positive bounded linear operators with coefficients $\gamma_k > 0$, where $\overline{\gamma} = \min\{\gamma_k : 1 \le k \le \overline{N}\}$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$;*
(A5) *The solution set*

$$Sol = \bigcap_{i=1}^{N} VI(C, A_i) \cap \bigcap_{j=1}^{M} Fix(T_j)$$

is nonempty.

We now present our method as follows.

Remark 1. *Observe that we are at a solution of Problem (2) if $x_n = y_n = u_n$. We will implicitly assume that this does not occur after finitely many iterations so that our Algorithm 3 generates an infinitely sequence for our analysis.*

Algorithm 3: EFEM

Initialization: Choose $\sigma \in (0, 1)$, $\rho \in (0, 1)$, $\{\alpha_n\}$, $\{\delta_{n,j}\}_{j=0}^{M} \subset (0, 1)$. Let $x_1 \in C$ be given arbitrarily and set $n = 1$.

Iterative step:

Step 1: For $i = 1, 2, \ldots, N$, compute in parallel

$$y_n^i = P_C(x_n - A_i x_n).$$

If $\theta^i(x_n) = x_n - y_n^i = 0$: set $x_n = w_n$ and do Step 3. Otherwise: do Step 2.

Step 2. Compute $z_n^i = x_n - \rho^{l_n} \theta^i(x_n)$, where l_n is the smallest non-negative integer satisfying

$$\langle A_i z_n^i, \theta^i(x_n) \rangle \ge \frac{\sigma}{2} \|\theta^i(x_n)\|^2. \tag{12}$$

Set $w_n = P_{D_n}(x_n)$, where

$$D_n = \left\{ x \in H : \sum_{i=1}^{N} \beta_n^i h_n^i(x_n) \le 0 \right\},$$

$\{\beta_n^i\}_{i=1}^{N} \subset (0, 1)$ such that $\sum_{i=1}^{N} \beta_n^i = 1$, and $h_n^i(x) = \langle A_i z_n^i, x - z_n^i \rangle$.

Step 3. Compute

$$u_n = \delta_{n,0} w_n + \sum_{n=1}^{M} \delta_{n,j} T_j w_n,$$

and

$$x_{n+1} = \alpha_n \gamma f(x_n) + \left(I - \alpha_n \sum_{k=1}^{\overline{N}} c_k B_k \right) u_n, \tag{13}$$

where $\{c_k\}_{k=1}^{\overline{N}} \subset (0, 1)$ such that $\sum_{k=1}^{\overline{N}} c_k = 1$.

Stopping criterion: If $x_n = y_n = u_n$, then stop; otherwise, set $n := n + 1$ and go back to **Step 1.**

In order to prove the convergence of our algorithm, we assume that the control parameters satisfy the following conditions.

Assumption 2.

(B1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
(B2) $\liminf_{n\to\infty}(\delta_{n,0} - \kappa) > 0$.

We begin the convergence analysis of Algorithm 3 by proving some useful Lemmas.

Lemma 6. *Let $u^* \in Sol$ and h_n^i be as defined in Algorithm 3. Then $h_n^i(x_n) \geq \rho^{l_n}\frac{\sigma}{2}\|x_n - y_n^i\|^2$ and $h_n^i(u^*) \leq 0$. In particular, if $\theta^i(x_n) \neq 0$, then $h_n^i(x_n) > 0$ for all $n \in \mathbb{N}$.*

Proof. As $z_n^i = x_n - \rho^{l_n}(x_n - y_n^i)$ for $i = 1, 2, \ldots, N$, then

$$\begin{aligned} h_n^i(x_n) &= \langle Az_n^i, x_n - z_n^i\rangle \\ &= \rho^{l_n}\langle Az_n^i, x_n - y_n^i\rangle \\ &\geq \rho^{l_n}\frac{\sigma}{2}\|x_n - y_n^i\|^2. \end{aligned} \tag{14}$$

Furthermore, if $x_n \neq y_n^i$ for $i = 1, 2, \ldots, N$, then $h_n^i(x_n) > 0$. As $u^* \in Sol$ and A_i are pseudomonotone, then

$$\langle Ay, y - u^*\rangle \geq 0 \quad \forall y \in C.$$

Therefore,

$$\langle Az_n^i, z_n^i - u^*\rangle \geq 0. \tag{15}$$

Therefore,

$$h_n^i(u^*) = \langle Az_n^i, u^* - z_n^i\rangle \leq 0.$$

□

Remark 2. *Lemma 6 shows that $D_n \neq \emptyset$ and so P_{D_n} is well defined. Consequently, Algorithm 3 is well defined.*

Now we show that the sequence $\{x_n\}$ generated by Algorithm 3 is bounded.

Lemma 7. *Let $\{x_n\}$ be the sequence generated by Algorithm 3. Then $\{x_n\}$ is bounded.*

Proof. Let $u^* \in Sol$, then from (11), we have

$$\begin{aligned} \|w_n - u^*\|^2 &= \|P_{D_n}x_n - u^*\|^2 \\ &\leq \|x_n - u^*\|^2 - \|P_{D_n}x_n - x_n\|^2 \\ &= \|x_n - u^*\|^2 - dist(x_n, D_n)^2 \\ &\leq \|x_n - u^*\|^2. \end{aligned} \tag{16}$$

Moreover, from Lemma 2 (iii), we get

$$\begin{aligned}
\|u_n-u^*\|^2 &= \left\|\delta_{n,0}(w_n-u^*)-\sum_{j=1}^{M}\delta_{n,j}(T_jw_n-u^*)\right\|^2 \\
&= \delta_{n,0}\|w_n-u^*\|^2+\sum_{j=1}^{M}\delta_{n,j}\|T_jw_n-u^*\|^2-\delta_{n,0}\delta_{n,j}\|T_jw_n-w_n\|^2 \\
&\le \delta_{n,0}\|w_n-u^*\|^2+\sum_{j=1}^{M}\delta_{n,j}\Big(\|w_n-u^*\|^2+\kappa_j\|w_n-T_jw_n\|^2\Big)-\delta_{n,0}\delta_{n,j}\|T_jw_n-w_n\|^2 \qquad (17)\\
&\le \|w_n-u^*\|^2-\sum_{j=1}^{M}(\delta_{n,0}-\kappa)\delta_{n,j}\|w_n-T_jw_n\|^2 \\
&\le \|x_n-u^*\|^2-\sum_{j=1}^{M}(\delta_{n,0}-\kappa)\delta_{n,j}\|w_n-T_jw_n\|^2.
\end{aligned}$$

This implies that

$$\|u_n-u^*\|\le\|x_n-u^*\|.$$

Then from (13), we obtain

$$\begin{aligned}
\|x_{n+1}-u^*\| &= \left\|\alpha_n\gamma f(x_n)+\left(1-\alpha_n\sum_{k=0}^{N}c_kB_k\right)u_n-u^*\right\| \\
&= \left\|\alpha_n\left(\gamma f(x_n)-\sum_{k=0}^{N}c_kB_ku^*\right)+\left(I-\alpha_n\sum_{k=0}^{N}c_kB_k\right)u_k-\left(I-\alpha_n\sum_{k=0}^{N}c_kB_k\right)u^*\right\| \\
&\le \left\|\left(I-\alpha_n\sum_{k=0}^{N}c_kB_k\right)(u_n-u^*)\right\|+\alpha_n\gamma\|f(x_n)-f(u^*)\|+\alpha_n\left\|\gamma f(u^*)-\sum_{k=0}^{N}c_kB_ku^*\right\| \\
&\le \left(1-\alpha_n\sum_{k=0}^{N}c_k\gamma_k\right)\|u_n-u^*\|+\alpha_n\alpha\gamma\|x_n-u^*\|+\alpha_n\left\|\gamma f(u^*)-\sum_{k=0}^{N}c_kB_ku^*\right\| \\
&\le \left(1-\alpha_n\sum_{k=0}^{N}c_k\overline{\gamma}\right)\|x_n-u^*\|+\alpha_n\alpha\gamma\|x_n-u^*\|+\alpha_n\left\|\gamma f(u^*)-\sum_{k=0}^{N}c_kB_ku^*\right\| \\
&= (1-\alpha_n\overline{\gamma})\|x_n-u^*\|+\alpha_n\alpha\gamma\|x_n-u^*\|+\alpha_n\left\|\gamma f(u^*)-\sum_{k=0}^{N}c_kB_ku^*\right\| \\
&= (1-\alpha_n(\overline{\gamma}-\alpha\gamma))\|x_n-u^*\|+\alpha_n(\overline{\gamma}-\alpha\gamma)\frac{\|\gamma f(u^*)-\sum_{k=0}^{N}c_kB_ku^*\|}{(\overline{\gamma}-\alpha\gamma)} \\
&\le \max\left\{\|x_n-u^*\|,\frac{\|\gamma f(u^*)-\sum_{k=0}^{N}c_kB_ku^*\|}{(\overline{\gamma}-\alpha\gamma)}\right\}.
\end{aligned}$$

By induction, we have

$$\|x_n-u^*\|\le\max\left\{\|x_1-u^*\|,\frac{\|\gamma f(u^*)-\sum_{k=0}^{N}c_kB_ku^*\|}{(\overline{\gamma}-\alpha\gamma)}\right\}.$$

This implies that $\{x_n\}$ is bounded. □

Lemma 8. *Let $u^*\in Sol$ and $\{x_n\}$ be the sequence generated by Algorithm 3, then $\{x_n\}$ satisfies the following estimates.*

(i)

$$\|w_n-u^*\|^2\le\|x_n-u^*\|^2-\left(\frac{\sigma\rho^{l_n}}{L}\sum_{i=1}^{N}\beta_n^i\|x_n-y_n^i\|^2\right)$$

for some $L \geq 0$;

(ii)

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n$$

where

$$s_n = \|x_n - u^*\|^2, \quad a_n = \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n \alpha\gamma},$$
$$b_n = \alpha_n M + \frac{\langle \gamma f(u^*) - \sum_{k=0}^{N} c_k B_k u^*, x_{n+1} - u^* \rangle}{\overline{\gamma} - \alpha_n \gamma},$$

for some $M > 0$.

Proof. As $\{x_n\}$ is bounded and A_i are continuous on bounded subsets of H, then $\{A_i x_n\}$ are bounded, and thus there exists some constants $Li > 0$ such that

$$\|A_i x_n\| \leq \frac{L_i}{2} \quad \forall n \in \mathbb{N},\ i = 1, 2, \ldots, N.$$

Consequently,

$$\|A_i x_n\| \leq \frac{L}{2} \quad \text{where} \quad L = \max\{L_i,\ i = 1, 2, \ldots, N\}.$$

Therefore from Lemma 3, we have

$$dist(x_n, D_n) \geq \frac{2}{L} \sum_{i=1}^{N} \beta_n^i h_n^i(x_n), \quad \forall n \geq 0. \tag{18}$$

Thus from (16) and (18), we get

$$\begin{aligned} \|w_n - u^*\|^2 &= \|x_n - u^*\|^2 - dist(x_n, D_n)^2 \\ &\leq \|w_n - u^*\|^2 - \left(\frac{2}{L} \sum_{i=1}^{N} \beta_n^i h_n^i(x_n) \right)^2. \end{aligned}$$

Hence from Lemma 6, we obtain

$$\|w_n - u^*\|^2 \leq \|x_n - u^*\|^2 - \left(\frac{\sigma \rho^{l_n}}{L} \sum_{i=1}^{N} \beta_n^i \|x_n - y_n^i\|^2 \right).$$

This established (i).

Moreover, we have from Algorithm 3 that

$$\begin{aligned}
\|x_{n+1}-u^*\|^2 &= \|\alpha_n\gamma f(x_n)+(I-\alpha_n\sum_{k=0}^{\overline{N}}c_kB_k)u_n-u^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n)-\sum_{k=0}^{\overline{N}}c_kB_ku^*)+(1-\alpha_n\sum_{k=0}^{\overline{N}}c_kB_k)u_n-(1-\alpha_n\sum_{k=0}^{\overline{N}}c_kB_k)u^*\|^2 \\
&\leq \|(1-\alpha_n\sum_{k=0}^{\overline{N}}c_kB_k)u_n-(1-\alpha_n\sum_{k=0}^{\overline{N}}c_kB_k)u^*\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-\sum_{k=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle \\
&\leq (1-\alpha_n\sum_{k=0}^{\overline{N}}c_k\gamma_k)^2\|u_n-u^*\|^2+2\alpha_n\gamma\langle f(x_n)-f(u^*),x_{n+1}-u^*\rangle \\
&\quad +2\alpha_n\langle\gamma f(u^*)-\sum_{k=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle \\
&\leq (1-\alpha_n\sum_{k=0}^{\overline{N}}c_k\overline{\gamma})^2\|x_n-u^*\|^2+2\alpha_n\alpha\gamma\|x_n-u^*\|\|x_{n+1}-u^*\| \\
&\quad +2\alpha_n\langle\gamma f(u^*)-\sum_{k=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle \\
&\leq (1-\alpha_n\overline{\gamma})^2\|x_n-u^*\|^2+\alpha_n\gamma(\|x_n-u^*\|^2+\|x_{n+1}-u^*\|^2) \\
&\quad +2\alpha_n\langle\gamma f(u^*)-\sum_{k=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\|x_{n+1}-u^*\|^2 &\leq \frac{(1-\alpha_n\overline{\gamma})^2+\alpha_n\alpha\gamma}{1-\alpha_n\alpha\gamma}\|x_n-u^*\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha\gamma}\langle\gamma f(u^*)-\sum_{k=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle \\
&= \left(1-\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\alpha\gamma}\right)\|x_n-u^*\|^2+\frac{\alpha_n^2\overline{\gamma}^2}{2-\alpha_n\alpha\gamma}\|x_n-u^*\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\alpha\gamma}\langle\gamma f(u^*)-\sum_{k=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle \\
&= \left(1-\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\alpha\gamma}\right)\|x_n-u^*\|^2 \\
&\quad +\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\alpha\gamma}\left(\alpha_nM+\frac{\langle\gamma f(u^*)-\sum_{k=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle}{\overline{\gamma}-\alpha_n\gamma}\right) \\
&= (1-a_n)s_n+a_nb_n,
\end{aligned}$$

where the existence of M follows from the boundedness of $\{x_n\}$. This completes the proof. □

Lemma 9. *Let $\{x_{n_l}\}$ be a subsequence of $\{x_n\}$ generated by Algorithm 3 such that $\{x_{n_l}\}$ converges weakly to $p\in H$ and $\lim_{l\to\infty}\|x_{n_l}-y^i_{n_l}\|=0$, for all $i=1,2,\dots,N$. Then*

(i) $0\leq\liminf_{l\to\infty}\langle A_ix_{n_l},x-x_{n_l}\rangle$, *for all $x\in C$, $i=1,2,\dots,N$;*
(ii) $p\in\bigcap_{i=1}^{N}VI(C,A_i)$.

Proof. (i) From the Definition of y^i_n and (10), we have

$$\langle x_{n_l}-A_ix_{n_l}-y^i_{n_l},x-y^i_{n_l}\rangle\leq 0,\quad\forall x\in C,\ i=1,2,\dots,N.$$

Thus,
$$\langle x_{n_l} - y^i_{n_l}, x - y^i_{n_l}\rangle \le \langle A_i x_{n_l}, x - y^i_{n_l}\rangle, \quad \forall x \in C,\ i = 1, 2 \ldots, N.$$

This implies that

$$\langle x_{n_l} - y^i_{n_l}, x - y_{n_l}\rangle + \langle A_i x_{n_l}, y^i_{n_l} - x_{n_l}\rangle \le \langle A x_{n_l}, x - x_{n_l}\rangle, \quad \forall x \in C, i = 1, 2, \ldots, N. \tag{19}$$

Fix $x \in C$ and let $l \to \infty$ in (19), since $\|y^i_{n_l} - x_{n_l}\| \to 0$, then

$$0 \le \liminf_{l\to\infty} \langle A_i x_{n_l}, x - x_{n_l}\rangle, \quad \forall x \in C, i = 1, 2, \ldots, N.$$

(ii) Suppose $\{\xi_l\}$ is a decreasing sequence of non-negative numbers such that $\xi_l \to 0$ as $l \to \infty$. For each ξ_l, we denote by N_l the smallest positive integer such that

$$\langle A_i x_{n_l}, x - x_{n_l}\rangle + \xi_l \ge 0, \quad \forall\, l \ge N_l,\ i = 1, 2, \ldots, N,$$

where the existence of N_l follows from (i). This means that

$$\langle A_i x_{n_l}, x + \xi_l t^i_{n_l} - x_{n_l}\rangle \ge 0, \quad \forall\, l \ge N_l, i = 1, 2, \ldots, N,$$

for some $t^i_{n_l} \in H$ satisfying $1 = \langle A_i x_{n_l}, t^i_{n_l}\rangle$ (since $A_i x_{n_l} \ne 0$ for $i = 1, 2, \ldots, N$). As A_i are pseudomonotone, it follows from (i) that

$$\langle A_i(x + \xi_l t^i_{n_l}), x + \xi_l t^i_{n_l} - x_{n_l}\rangle \ge 0 \quad \forall\, l \ge N_l,\ i = 1, 2, \ldots, N. \tag{20}$$

Furthermore, $x_{n_l} \rightharpoonup p$ as $l \to \infty$ and A_i are weakly sequentially continuous, then $A_i x_{n_l} \rightharpoonup A_i p$ for $i = 1, 2, \ldots, N$. Therefore,

$$0 < \|A_i p\| \le \liminf_{l\to\infty} \|A_i x_{n_l}\|, \quad \forall i = 1, 2, \ldots, N.$$

Moreover, $\{x_{N_l}\} \subset \{x_{n_l}\}$ and $\xi_l \to 0$ as $l \to \infty$. Thus, we obtain

$$\begin{aligned} 0 &\le \limsup_{l\to\infty} \|\xi_l t^i_{n_l}\| = \limsup_{l\to\infty} \left(\frac{\xi_l}{\|A_i x_{n_l}\|}\right) \\ &\le \frac{\limsup_{l\to\infty} \xi_l}{\liminf_{l\to\infty} \|A_i x_{n_l}\|} \le \frac{0}{\|A_i p\|} = 0, \end{aligned}$$

which implies that $\lim_{l\to\infty} \|\xi_l t^i_{n_l}\| = 0$. Thus, taking limit of (20) as $l \to \infty$, we obtain

$$\langle A_i x, x - p\rangle \ge 0, \quad \forall\, i = 1, 2, \ldots, N.$$

Using Lemma 1, we have $p \in VI(C, A_i)$ for all $i = 1, 2, \ldots, N$. Therefore $p \in \bigcap_{i=1}^N VI(C, A_i)$. This completes the proof. □

We now present our main result.

Theorem 1. *Suppose $\{x_n\}$ is generated by Algorithm 3. Then $\{x_n\}$ converges strongly to a point $z \in Sol$.*

Proof. Let $u^* \in Sol$ and put $\Gamma_n = \|x_n - x^*\|^2$. We consider the following possible cases.
Case A: Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}$ is monotonically decreasing for $n \ge n_0$. Since $\{\Gamma_n\}$ is bounded, then

$$\Gamma_n - \Gamma_{n+1} \to 0 \quad \text{as } n \to \infty. \tag{21}$$

Moreover, from Lemma 8 (i), we have

$$\begin{aligned}
\|x_{n+1}-u^*\|^2 &= \|\alpha_n\gamma f(x_n)-(1-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_k)u_n-u^*\|^2\\
&= \|\alpha_n(\gamma f(x_n)-\sum_{n=0}^{\overline{N}}c_kB_ku^*)+(1-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_k)(u_n-u^*)\|^2\\
&\leq \|(I-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_k)(u_n-u^*)\|^2+2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle\\
&\leq (1-\alpha_n\sum_{n=0}^{\overline{N}}c_k\gamma_k)\|u_n-u^*\|^2+2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle \qquad (22)\\
&\leq (1-\alpha_n\sum_{n=0}^{\overline{N}}c_k\overline{\gamma})\|w_n-u^*\|^2+2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle\\
&\leq (1-\alpha_n\overline{\gamma})\left[\|x_n-u^*\|^2-\frac{\sigma\rho^{l_n}}{L}\sum_{i=1}^{N}\beta_n^i\|x_n-y_n^i\|^2\right]\\
&\quad +2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
(1-\alpha_n\overline{\gamma})\frac{\sigma\rho^{l_n}}{L}\sum_{i=1}^{N}\beta_n^i\|x_n-y_n^i\|^2 &\leq (1-\alpha_n\overline{\gamma})\|x_n-u^*\|^2-\|x_{n+1}-u^*\|^2\\
&\quad +2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle.
\end{aligned}$$

As $\alpha_n\to 0$ as $n\to\infty$ and from (21), we have

$$\lim_{n\to\infty}\frac{\sigma\rho^{l_n}}{L}\sum_{i=1}^{N}\beta_n^i\|x_n-y_n^i\|^2=0. \qquad (23)$$

Furthermore, from (18), we obtain

$$\begin{aligned}
\|x_{n+1}-u^*\|^2 &\leq (1-\alpha_n\overline{\gamma})\|u_n-u^*\|^2+2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle\\
&\leq (1-\alpha_n\overline{\gamma})[\|x_n-u^*\|^2-\sum_{j=1}^{M}(\delta_{n,0}-\kappa)\delta_{n,j}\|w_n-T_jw_n\|^2]\\
&\quad +2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1-\alpha_n\overline{\gamma})\sum_{j=1}^{M}(\delta_{n,0}-\kappa)\delta_{n,j}\|w_n-T_jw_n\|^2 &\leq (1-\alpha_n\overline{\gamma})\|x_n-u^*\|^2-\|x_{n+1}-u^*\|^2\\
&\quad +2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\sum_{j=1}^{M}(\delta_{n,0}-\kappa_j)\delta_{n,j}\|w_n-T_jw_n\|=0.$$

Using condition (C2), we obtain

$$\lim_{n\to\infty}\|w_n-T_jw_n\|=0. \tag{24}$$

Consequently,

$$\|u_n-w_n\|\leq\delta_{n,0}\|w_n-w_n\|+\sum_{j=1}^{M}\delta_{n.j}\|w_n-T_jw_n\|\to 0 \quad \text{as } n\to\infty. \tag{25}$$

Furthermore, from (11), we have

$$\begin{aligned}\|w_n-u^*\|^2 &= \|P_{D_n}(x_n)-u^*\|^2\\ &\leq \|x_n-u^*\|^2-\|w_n-x_n\|^2.\end{aligned}$$

Then, from (23), we have

$$\begin{aligned}\|w_n-x_n\|^2 &\leq \|x_n-u^*\|^2-\|w_n-u^*\|^2\\ &= \|x_n-u^*\|^2-\|x_{n+1}-u^*\|^2+\|x_{n+1}-u^*\|^2-\|w_n-u^*\|^2\\ &\leq \|x_n-u^*\|^2-\|x_{n+1}-u^*\|^2+\|w_n-u^*\|^2+2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle\\ &\quad -\|w_n-u^*\|^2\\ &= \|x_n-u^*\|^2-\|x_{n+1}-u^*\|^2+2\alpha_n\langle\gamma f(x_n)-\alpha_n\sum_{n=0}^{\overline{N}}c_kB_ku^*,x_{n+1}-u^*\rangle.\end{aligned}$$

Moreover, as $\alpha_n\to 0$ as $n\to\infty$, then

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{26}$$

From (25) and (26), we get

$$\|u_n-x_n\|\leq\|u_n-w_n\|+\|w_n-x_n\|\to 0 \quad \text{as } n\to\infty.$$

Therefore,

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n(\gamma f(x_n)-\sum_{n=0}^{\overline{N}}c_kB_kx_n)+(I-\alpha_n\sum_{n=0}^{\overline{N}})(u_n-x_n)\|\\ &\leq \alpha_n\|\gamma f(x_n)-\sum_{n=0}^{\overline{N}}c_kB_kx_n\|+(1-\alpha_n\overline{\gamma})\|u_n-x_n\|\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{27}$$

Now, we show that $\Omega_w(x_n)\subset Sol$, where $\Omega_w(x_n)$ is the set of weak subsequential limits of $\{x_n\}$. Let $p\in\Omega_w(x_n)$, then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l}\rightharpoonup p$ as $l\to\infty$. Let $\{y^i_{n_l}\}$ be subsequences of $\{y^i_n\}$ for $i=1,2,\ldots,N$. From (23), we have

$$\lim_{l\to\infty}\gamma^{m_{n_l}}\sum_{i=1}^{N}\beta_{n_l}\|x_{n_l}-y^i_{n_l}\|=0, \quad \forall i=1,2,\ldots,N.$$

Now we claim that

$$\lim_{l\to\infty} \|x_{n_l} - y^i_{n_l}\| = 0, \quad \forall i = 1,2,\ldots,N.$$

Indeed, we consider two distinct cases depending on the behavior of subsequence $\{\gamma^{m_{n_l}}\}$.
(i) If $\liminf_{l\to\infty} \gamma^{m_{n_l}} > 0$, then

$$0 \leq \|x_{n_l} - y^i_{n_l}\|^2 = \frac{\gamma^{m_{n_l}} \|x_{n_l} - y_{n_l}\|^2}{\gamma^{m_{n_l}}}.$$

This implies that

$$\begin{aligned}
\limsup_{l\to\infty} \|x_{n_l} - y_{n_l}\|^2 &\leq \limsup_{l\to\infty} \left(\gamma^{m_{n_l}} \|x_{n_l} - y_{n_l}\|^2\right) \left(\limsup_{l\to\infty} \frac{1}{\gamma^{m_{n_l}}}\right) \\
&= \limsup_{l\to\infty} \left(\gamma^{m_{n_l}} \|x_{n_l} - y_{n_l}\|^2\right) \left(\frac{1}{\liminf_{l\to\infty} \gamma^{m_{n_l}}}\right) \\
&= 0.
\end{aligned}$$

Therefore,

$$\lim_{l\to\infty} \|x_{n_l} - y^i_{n_l}\| = 0.$$

(ii) Suppose $\liminf_{l\to\infty} \gamma^{m_{n_l}} = 0$. Then we may assume without loss of generality that $\lim_{l\to\infty} \gamma^{m_{n_l}} = 0$ and $\lim_{l\to\infty} \|x_{n_l} - y_{n_l}\| = a > 0$. Let us define $\bar{z}^i_{n_l} = \frac{1}{t}\gamma^{m_{n_l}} y^i_{n_l} + \left(1 - \frac{1}{t}\gamma^{m_{n_l}}\right) x_n$ for $i = 1,2,\ldots,N$. This implies that $\bar{z}^i_{n_l} - x_n = \frac{1}{t}\gamma^{m_{n_l}}(y^i_{n_l} - x_{n_l})$ for $i = 1,2,\ldots,N$. Since $\{y^i_{n_l} - x_{n_l}\}$ are bounded and $\lim_{l\to\infty} \gamma^{m_{n_l}} = 0$, then

$$\lim_{l\to\infty} \|\bar{z}^i_{n_l} - x_{n_l}\| = 0.$$

As A_i are uniformly continuous, then

$$\lim_{l\to\infty} \|A_i \bar{z}^i_{n_l} - A_i x_{n_l}\| = 0, \quad \forall i = 1,2,\ldots,N. \tag{28}$$

Using (12) and from the Definition of $\bar{z}^i_{n_l}$ for $i = 1,2,\ldots,N$, we know that

$$\langle A_i \bar{z}^i_{n_l}, x_{n_l} - y^i_{n_l}\rangle < \frac{\sigma}{2}\|x_{n_l} - y^i_{n_l}\|, \quad \forall i = 1,2,\ldots,N.$$

Therefore,

$$2\langle A_i x_{n_l}, x_{n_l} - y^i_{n_l}\rangle + 2\langle A_i \bar{z}^i_{n_l} - A_i x_{n_l}, x_{n_l} - y^i_{n_l}\rangle < \sigma\|x_{n_l} - y^i_{n_l}\|^2, \quad \forall i = 1,2,\ldots,N.$$

Putting $v^i_{n_l} = x_{n_l} - A_i x_{n_l}$, for all $i = 1,2,\ldots,N$, we obtain

$$2\langle x_{n_l} - v^i_{n_l}, x_{n_l} - y^i_{n_l}\rangle + 2\langle A_i \bar{z}^i_{n_l} - A_i x_{n_l}, x_{n_l} - y^i_{n_l}\rangle < \sigma\|x_{n_l} - y^i_{n_l}\|^2, \quad \forall i = 1,2,\ldots,N. \tag{29}$$

Moreover, from Lemma 2 (i), we have

$$2\langle x_{n_l} - y_{n_l}, x_{n_l} - y^i_{n_l}\rangle = \|x_{n_l} - v^i_{n_l}\|^2 + \|x_{n_l} - y^i_{n_l}\|^2 - \|y^i_{n_l} - v^i_{n_l}\|^2. \tag{30}$$

Substituting (30) into (29), we have

$$\|x_{n_l} - v^i_{n_l}\|^2 - \|y^i_{n_l} - v^i_{n_l}\|^2 < (\sigma - 1)\|x_{n_l} - y^i_{n_l}\|^2 - 2\langle A_i \bar{z}^i_{n_l} - A_i x_{n_l}, x_{n_l} - y^i_{n_l}\rangle.$$

Passing limit to the last inequality as $l \to \infty$ and using (28), we get

$$\lim_{l\to\infty}\left(\|x_{n_l}-v^i_{n_l}\|^2-\|y^i_{n_l}-v^i_{n_l}\|^2\right)\le(\sigma-1)a<0.$$

For $\epsilon=\frac{-(\sigma-1)a}{2}>0$, there exists $m\in\mathbb{N}$ such that

$$\|x_{n_l}-v^i_{n_l}\|^2-\|y^i_{n_l}-v^i_{n_l}\|^2\le(\sigma-1)a+\epsilon=\frac{(\sigma-1)a}{2}<0,$$

for all $l\in\mathbb{N}, n\ge m, i=1,2,\ldots,N$. Therefore

$$\|x_{n_l}-v^i_{n_l}\|^2<\|y^i_{n_l}-v^i_{n_l}\|^2,\quad\forall l\in\mathbb{N}, n\ge m, i=1,2,\ldots,N.$$

This contradicts the Definition of metric projection as $y^i_{n_l}=P_C(x_{n_l}-A_ix_{n_l})$. Thus $a=0$. Therefore, we obtain

$$\lim_{l\to\infty}\|x_{n_l}-y^i_{n_l}\|=0,\quad\forall i=1,2,\ldots,N. \tag{31}$$

Consequently from Lemma 9, we have $p\in\bigcap_{i=1}^N VI(C,A_i)$. Furthermore, as $w_{n,l}\rightharpoonup p$ and $\|v_{n_l,j}-w_{n_l}\|\to 0$, then by the demi-closedness of T_j, $j=1,2,\ldots,M$, we have that $p\in Fix(T_j)$, for each $j=1,2,\ldots,M$. This means that $p\in\bigcap_{j=1}^M Fix(T_j)$. Therefore, $p\in Sol$, which show that $\Omega_w(x_n)\subset Sol$. We now show that $\{x_n\}$ converges strongly to a point $u^*\in Sol$. As $x_{n_l}\rightharpoonup p$ and $\|x_{n_l+1}-x_{n_l}\|\to 0$ as $l\to\infty$, then$x_{n_l+1}\rightharpoonup p$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(u^*)-\sum_{n=0}^{N}c_kB_ku^*,x_{n+1}-u^*\rangle&=\lim_{l\to\infty}\langle\gamma f(u^*)-\sum_{n=0}^{N}c_kB_ku^*,x_{n_l+1}-u^*\rangle\\&=\langle\gamma f(u^*)-\sum_{n=0}^{N}c_kB_ku^*,p-u^*\rangle.\end{aligned}\tag{32}$$

As $p\in Sol$, then it follows from (10) and (32) that

$$\limsup_{n\to\infty}\langle\gamma f(u^*)-\sum_{n=0}^{N}c_kB_ku^*,x_{n+1}-u^*\rangle\le 0.$$

Therefore, using Lemma 4 and Lemma 8 (ii), we have that $\lim_{n\to\infty}\|x_n-u^*\|=0$. This implies that $\{x_n\}$ converges strongly to u^*.

Case B: Suppose $\{\Gamma_n\}$ is not monotonically decreasing. Let $\tau:\mathbb{N}\to\mathbb{N}$ for all $n\ge n_0$ (for some n_0 large enough) be defined by

$$\tau(n)=\max\{k\in\mathbb{N}:k\le n,\Gamma_k\le\Gamma_{k+1}\}.$$

Clearly τ is non-decreasing, $\tau(n)\to\infty$ as $n\to\infty$ and

$$0\le\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1},\quad\forall n\ge n_0.$$

As $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$ still denoted by $\{x_{\tau(n)}\}$ which converges weakly to $p\in C$. Following similar arguments as in Case A, we get

$$\begin{aligned}\lim_{n\to\infty}\|w_{\tau(n)}-x_{\tau(n)}\|=\lim_{n\to\infty}\|u_{\tau(n)}-x_{\tau(n)}\|=\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\|=0,\\ \lim_{n\to\infty}\|x_{\tau(n)}-y^i_{\tau(n)}\|=0,\quad\forall i=1,2\ldots,N,\ \lim_{n\to\infty}\|v_{\tau(n),j}-w_{\tau(n)}\|=0,\quad j=1,2,\ldots,M,\end{aligned}$$

and $\Omega_w(x_{\tau(n)}) \subset Sol$, where $\Omega_w(x_{\tau(n)})$ is the set of weak subsequential limits of $\{x_{\tau(n)}\}$. Furthermore,

$$\limsup_{n\to\infty}\langle \gamma f(u^*) - \sum_{n=0}^{N} c_k B_k u^*, x_{\tau(n)+1} - u^*\rangle \leq 0. \tag{33}$$

From Lemma 8 (ii), we have

$$\begin{aligned}\|x_{\tau(n)+1} - u^*\|^2 \leq &\left(1 - \frac{2\alpha_{\tau(n)}(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_{\tau(n)}\alpha\gamma}\right)\|x_{\tau(n)} - u^*\|^2 \\ &+ \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_{\tau(n)}\alpha\gamma}\left(\alpha_{\tau(n)}M + \frac{\langle \gamma f(u^*) - \sum_{k=0}^{N} c_k B_k u^*, x_{\tau(n)+1} - u^*\rangle}{\overline{\gamma} - \alpha_{\tau(n)}\gamma}\right),\end{aligned} \tag{34}$$

for some $M > 0$. As $0 \leq \|x_{\tau(n)} - u^*\|^2 \leq \|x_{\tau(n)+1} - u^*\|^2$, then we get

$$\|x_{\tau(n)} - u^*\|^2 \quad\leq\quad \alpha_{\tau(n)}M + \frac{\langle \gamma f(u^*) - \sum_{k=0}^{N} c_k B_k u^*, x_{\tau(n)+1} - u^*\rangle}{\overline{\gamma} - \alpha_{\tau(n)}\gamma}.$$

Then from (33) and the fact that $\alpha_{\tau(n)} \to 0$, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - u^*\| = 0. \tag{35}$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_n \leq \Gamma_{\tau(n)+1}$. As a consequence, we get for all sufficiently large n that $0 \leq \Gamma_n \leq \Gamma_{\tau(n)+1}$. Thus, $\lim_{n\to\infty}\|x_n - u^*\| = 0$. Therefore, $\{x_n\}$ converges strongly to u^*. Consequently, $\{y_n^i\}, \{z_n^i\}, \{w_n\}$ and $\{u_n\}$ converges strongly to u^*. This completes the proof. □

Remark 3.

(i) *Instead of finding the farthest element to the iterate x_n, we construct a sub-level set using the convex combination of the finite functions and perform a single projection onto the sub-level set. Note that this projection can be calculated explicitly irrespective of the feasible set C.*
(ii) *We emphasize that the convergence of our Algorithm 3 is proved without using a prior estimate of any Lipshitz constant. Moreover, the cost operators do not even need to satisfy the Lipschitz condition. Note that the previous results of [6,8,33] and references therein cannot be applied in this situation.*
(iii) *We give an example of a finite family of $A_i : H \to H$ which does not satisfy Lipschitz condition.*

Example 1. *Let $H = \Re^n$ defined by $\Re^n = \{\bar{x} = (x_1, x_2, \ldots, x_n),\ x_l \in \Re : \sum_{l=1}^{n}|x_l|^2 < \infty\}$ with norm $\|\cdot\| : \Re^n \to [0.\infty)$ defined by $\|\bar{x}\| := (\sum_{l=1}^{n}|x_l|^2)^{\frac{1}{2}}$ for arbitrary $\bar{x} = (x_1, x_2, \ldots, x_n) \in \Re^n$. Let $C_i = C = \{\bar{x} \in \Re^n : \|\bar{x}\| \leq 1\}$ and for $i = 1, \ldots, N$, let $A_i : C \to H$ be defined by*

$$A_i x = \left(\|x\| + \frac{i}{\|x\| + 1}\right)x, \quad i = 1, \ldots, N.$$

It is clear that $VI(C, A_i) \neq \emptyset$ as $0 \in VI(C, A_i)$ for each $i = 1, \ldots, N$. First, we show that A_i is pseudomonotone and not Lipschitz continuous for $i = 1, 2, \ldots, N$. Let $u, v \in C$ be such that $\langle A_i u, v - u\rangle \geq 0$. This means that $\langle u, v - u\rangle \geq 0$. Thus,

$$\begin{aligned}\langle A_i v, v-u\rangle &= \left(\|v\|+\frac{i}{\|v\|+1}\right)\langle v, v-u\rangle \\ &> \left(\|v\|+\frac{i}{\|v\|+1}\right)(\langle v, v-u\rangle - \langle u, v-u\rangle) \\ &= \left(\|v\|+\frac{i}{\|v\|+1}\right)\|v-u\|^2 > 0.\end{aligned}$$

Therefore, A_i is pseudomonotone for $i = 1, \ldots, N$. To see that A_i is not L-Lipschitz continuous for $i = 1, 2, \ldots, N$, let $u = (L_i, 0, \ldots, 0)$ and $v = (0, 0, \ldots, 0)$. Then,

$$\|Au - Av\| = \|Au\| = \left(\|u\| + \frac{i}{\|u\|+1}\right)\|u\| = \left(L_i + \frac{i}{L_i+1}\right)L_i.$$

Moreover, $\|A_i u - A_i v\| \le L_i \|u - v\|$ implies that

$$\left(L_i + \frac{i}{L_i+1}\right)L_i \le L_i^2.$$

Thus, $\frac{i}{L_i+1} \le 0$, which is a contradiction. Therefore, A_i is not Lipschitz continuous for $i = 1, \ldots, N$.

4. Numerical Experiments

In this section, we present some numerical experiments to illustrate the performance of the proposed algorithm. We compare our Algorithm 3 with Algorithm 1 of Anh and Phuong [8], Algorithm 2 of Hieu [33], Algorithm 1 of Suantai et al. [40], and other algorithms in the literature. The projections onto C_i are computed explicitly. All codes are written with a HP PC with the following specification: Intel(R)core i7-9700, CPU 3.00GHz, RAM 4.0GB, MATLAB version 9.9 (R2020b).

Example 2. *First, we consider the variational inequalities with operators $A_i : \Re^m \to \Re^m$ for $i = 1, 2, \ldots, N$, defined by $A_i(x) = G_i(x) + q_i$, where*

$$G_i = S_i S_i^T + Q_i + R_i, \quad i = 1, 2, \ldots, N,$$

such that for each i, S_i is a $m \times m$ matrix, Q_i is a $m \times m$ skew symmetric matrix, R_i is a $m \times m$ diagonal matrix, whose diagonal entries are non-negative (so G_i is positive definite) and q_i is a vector in $\Re^m$. The feasible set C is given by $C_i = C = \{x \in \Re^m : \langle x, a \rangle \le c\}$, where $a \in \Re^m$ is generated randomly and c is a positive real number randomly in $[1, m]$. It is clear that for each i, G_i is monotone (hence, pseudomonotone) and Lipschitz continuous with Lipschitz constant $L_i = \max\{\|G_i\| : i = 1, 2, \ldots, N\}$. The entries of matrices S_i, Q_i, R_i are generated randomly and uniformly in $[-m, m]$, diagonal entries of R_i are in $[1, m]$ and q_i is equal to the zero vector. In this case, it is easy to see that the $VI(C, A_i) = \{0\}$. For $j = 1, 2, \ldots, M$, let $T_j : \Re^m \to \Re^m$ be defined by $T_j x = \frac{x}{2j}$, for $j \in \mathbb{N}$. Then T_j is 0-demicontractive mapping, $Fix(T_j) = \{0\}$ and $(I - T_j)$ is demiclosed at 0. Also for $k = 1, 2, \ldots, \overline{N}$, let $B_k = \frac{1}{2k} I$, $f = I$ (I being the identity operator on H), we choose $\alpha = 1, \gamma = \frac{1}{8}, \sigma = 0.28, \rho = 0.36, \lambda = \frac{1}{4}$, $c_k = \frac{1}{\overline{N}}$, $\delta_{n,j} = \frac{1}{M+1}$, $\alpha_n = \frac{1}{n+1}$, $\beta_{n,i} = \frac{1}{N+1}$, for all $n \in \mathbb{N}$. We compare Algorithm 3 with Algorithm 1 of Anh and Phuong [8], Algorithm 2 of Hieu [33], and Algorithm 1 of Suantai et al. [40]. We test the algorithms using the following parameters.

- *Anh and Phuong alg.: $\lambda_{n,i} = \frac{0.99}{2L_i}$, $\alpha_{n,i} = \frac{1}{ni+1}$, $\gamma_{n,i} = \frac{1}{3}$,*
- *Hieu alg.: $\lambda = \frac{1}{1.5L}$,*
- *Suantai et al. alg.: $\rho = 0.34$, $\mu = 0.06$,*

and

Case I: $m = 5, \quad N = 5, \quad M = 2, \quad \overline{N} = 1,$
Case II: $m = 10, \quad N = 10, \quad M = 5, \quad \overline{N} = 5,$
Case III: $m = 20, \quad N = 15, \quad M = 10, \quad \overline{N} = 10,$

Case IV: $m = 50, \quad N = 20, \quad M = 5, \quad \overline{N} = 15.$

We also use $\|x_n - x^*\| < 10^{-5}$ *as stopping criterion for each algorithm and plot the graphs of* $D_n = \|x_n - x^*\|^2$ *against the number of iteration. The computational results are shown in Table 1 and Figure 1.*

Table 1. Computational result for Example 2.

		Algorithm 3	Anh-Phuong [8]	Hieu [33]	Suantai et al. [40]
Case I	No of Iter.	16	34	39	67
	Time (sec)	0.0038	0.0034	0.0032	0.0061
Case II	No of Iter.	15	63	107	98
	Time (sec)	0.0020	0.0054	0.0100	0.0097
Case III	No of Iter.	14	57	93	183
	Time (sec)	0.0020	0.0053	0.0093	0.0236
Case IV	No of Iter.	10	53	114	183
	Time (sec)	0.0019	0.0047	0.0136	0.0244

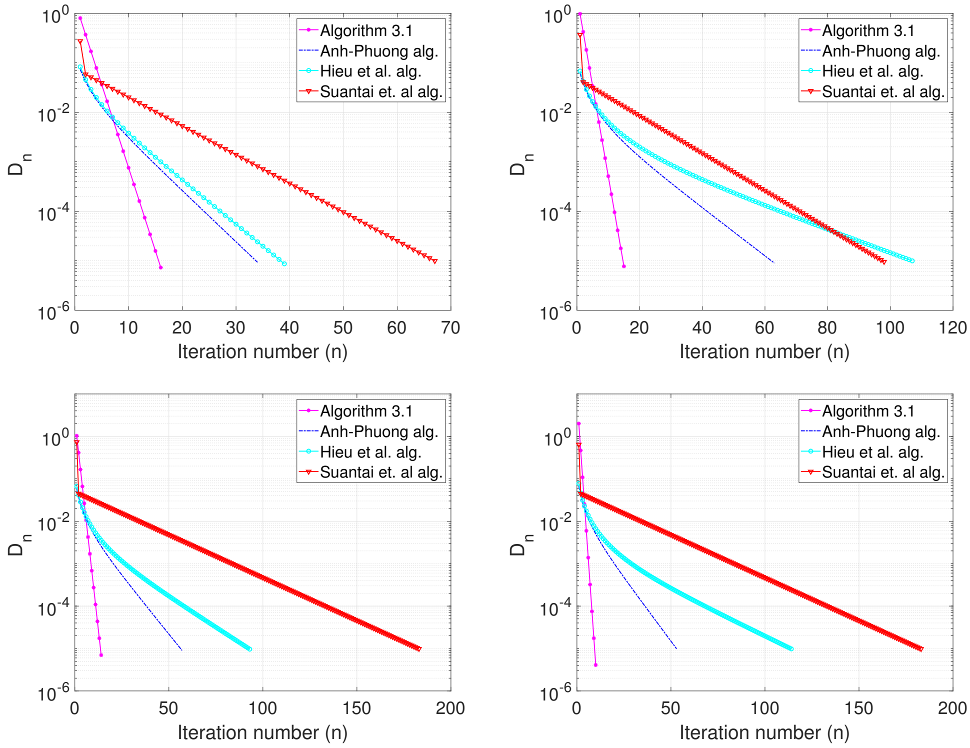

Figure 1. Example 2, From Top to Bottom: Case I, Case II, Case III, and Case IV.

Now, we consider the case when $N = 1$ with finite family of demicontractive mappings in infinite dimensional spaces. In this example, we compare our algorithm with Algorithm 1 of Anh et al. [41] and Algorithm 2.1 of Hieu [42].

Example 3. *Let* $H = \ell_2(\Re)$ *and define* $A : H \to H$ *by* $Ax = \frac{2}{2+\|x\|}$. *It is easy to see that* A *is easy to see that* A *is strongly pseudomonotone and Lipschitz continuous with* $L = \frac{1}{2}$. *We defined the feasible set* $C = \{x = (x_1, x_2, \dots) \in \ell_2 : \|x\| \le 1\}$ *and for* $j = 1, 2, \dots, M$, $T_j : H \to H$ *is defined by* $T_j x = \frac{-(1+j)}{j} x$, *for* $j \in \mathbb{N}$. *Then* T_j *is demicontractive mapping with* $\kappa_j = \frac{1}{1+2j}$, $Fix(T_j) = \{0\}$ *and* $(I - T_j)$ *is demiclosed at* 0. *We choose* $\overline{N} = 1$, $B_k = \frac{1}{2}I$, $f(x) = \frac{x}{4}$, $\sigma = 0.02, \rho = 0.036, \gamma = \frac{1}{16}$, $\alpha_n = \frac{1}{\sqrt{(n+1)}}$, $\delta_{n,j} = \frac{1}{M+1}$, $\beta_n^i = 1$, $c_k = 1$. *For Anh et al. alg., we take* $\lambda_n = \frac{1}{\sqrt{n+1}}$, $\alpha_n = \frac{1}{2n+4}$; *and for Hieu alg., we take* $\lambda_n = \frac{1}{n+1}$, $\beta_n^i = \frac{n}{2n+i}$, $\gamma_n^i = \frac{1}{M+1}$ *(where* $i = j$ *in this context). We test the algorithms for* $M = 5, 15, 20, 30$ *and study the* *behavior of the sequence generated by the algorithms using* $D_n = \|x_n - x^*\| < 10^{-5}$ *as stopping criterion. The numerical results are shown in Table 2 and Figure 2.*

Table 2. Computational result for Example 2.

		Algorithm 3	**Anh et al. [41]**	**Hieu [42]**
$M = 5$	No of Iter.	7	14	67
	Time (sec)	5.836e-04	0.0014	0.0021
$M = 15$	No of Iter.	8	13	25
	Time (sec)	6.300e-04	0.0014	0.0018
$M = 20$	No of Iter.	11	16	37
	Time (sec)	0.0010	0.0033	0.0088
$M = 30$	No of Iter.	10	17	43
	Time (sec)	7.583e-04	0.0018	0.0043

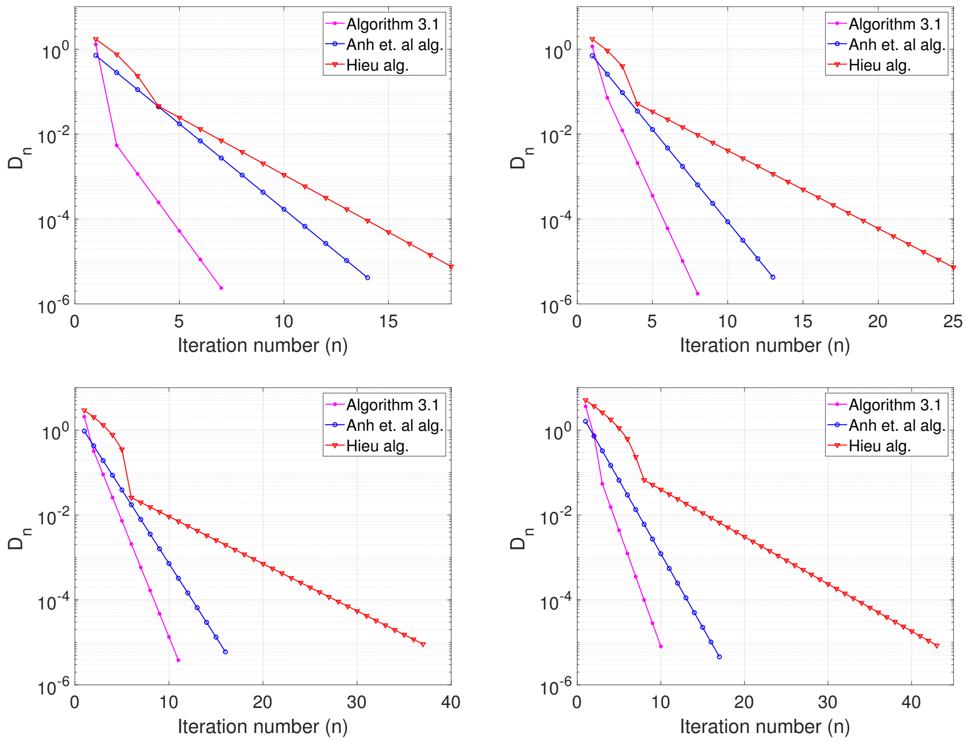

Figure 2. Example 3, From Top to Bottom: $M = 5, 15, 20, 30$.

5. Conclusions

In this paper, we introduced a new efficient parallel extragradient method for solving systems of variational inequalities involving common fixed point of demicontractive mappings in real Hilbert spaces. The algorithm is designed such that its step size is determined by an Armijo line search technique and a projection onto a sub-level set is computed for determining the next iterate. A strong convergence result is proved under suitable conditions on the control parameters. Finally, some numerical results were reported to show the performance of the proposed method with respect to some other methods in the literature.

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