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A New Approach for Euler-Lagrange Orbits on Compact Manifolds with Boundary

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Abstract: Consider a compact manifold with boundary, homeomorphic to the N-dimensional disk, and a Tonelli Lagrangian function defined on the tangent bundle. In this paper, we study the multiplicity problem for Euler-Lagrange orbits that satisfy the conormal boundary conditions and that lay on the boundary only in their extreme points. In particular, for suitable values of the energy function and under mild hypotheses, if the Tonelli Lagrangian is reversible then the minimal number of Euler-Lagrange orbits with prescribed energy that satisfies the conormal boundary conditions is N. If L is not reversible, then this number is two.

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1. Introduction

Let $\overline{\Omega}$ be a compact and connected N-manifold of class C^3 with boundary $\partial\Omega \in C^2$, homeomorphic to an N-dimensional disk $\mathbb{D}^N \subset \mathbb{R}^N$. For the sake of presentation, let $\overline{\Omega}$ be embedded into a larger N-manifold $\mathcal{M}$, which is the closure of an open set containing $\overline{\Omega}$. Let $L\colon T\mathcal{M} \to \mathbb{R}$ be a Tonelli Lagrangian, namely a fiberwise strictly convex and superlinear function of class C^2. The convexity assumption ensures that the Euler-Lagrange equation associated with L, which in local coordinates reads

$$d_q L(\gamma(t), \dot\gamma(t)) - \frac{d}{dt}\Big(d_v L(\gamma(t), \dot\gamma(t))\Big) = 0, \tag{1}$$

defines a locally well-posed Cauchy problem.

Definition 1. *An Euler-Lagrange chord is a curve $\gamma\colon [0,T] \to \overline{\Omega}$ such that*

- *γ satisfies the Euler-Lagrange Equation (1);*
- *$\gamma(]0,T[) \subset \Omega$ and $\gamma(0), \gamma(T) \in \partial\Omega$.*

If γ also satisfies the conormal boundary conditions, namely

$$\gamma(i) \in \partial\Omega, \quad d_v L\left(\gamma(i), \dot\gamma(i)\right)|_{T_{\gamma(i)}\partial\Omega} = 0, \quad i = 0, T, \tag{2}$$

then it is called Euler-Lagrange conormal chord (ELCC).

This work provides some existence and multiplicity results for ELCCs with suitable values of the energy function

$$E\colon T\mathcal{M} \to \mathbb{R}, \quad E(q,v) = d_v L(q,v)[v] - L(q,v).$$

Indeed, along the solutions of the Euler-Lagrange equation, the energy function is constant, namely if $\gamma\colon [0,T] \to \mathcal{M}$ satisfies (1), then there exists a constant κ such that $E(\gamma,\dot\gamma) = \kappa$. In our theorem, the existence and the multiplicity of ELCCs depend on the non existence of certain orbits, defined as follows.

Definition 2. *An Euler-Lagrange conormal-tangent chord (ELCTC) is an Euler-Lagrange chord $\gamma\colon [0,T] \to \overline{\Omega}$ such that*

$$d_vL(\gamma(0),\dot\gamma(0))|_{T_{\gamma(0)}\partial\Omega} = 0 \quad \textit{and} \quad \dot\gamma(T) \in T_{\gamma(T)}\partial\Omega.$$

In other words, an ELCTC is an Euler-Lagrange chord that satisfies the conormal boundary condition in its initial point and arrives tangentially on the boundary of Ω.

Let us define

$$m(L) = -\min_{q\in\overline{\Omega}}\min_{v\in T_q\mathcal{M}} L(q,v), \tag{3}$$

which is well defined since $\overline{\Omega}$ is compact and L is fiberwise convex. Finally, we say that two curves $\gamma_1\colon [0,T_1] \to \mathcal{M}$ and $\gamma_2\colon [0,T_2] \to \mathcal{M}$ are *geometrically distinct* if $\gamma([0,T_1]) \neq \gamma([0,T_2])$. Now we are ready to state our main theorem.

Theorem 1 (Main Theorem)**.** *Let $\overline{\Omega} \subset \mathcal{M}$ be an N-disk and $L\colon T\mathcal{M} \to \mathbb{R}$ a Tonelli Lagrangian. Then, for every fixed $\kappa > m(L)$, either:*

- *there exists an Euler-Lagrange conormal-tangent chord with energy κ*

or

- *if L is reversible, namely $L(q,v) = L(q,-v)$ for all $(q,v) \in T\mathcal{M}$, then there are at least N geometrically distinct Euler-Lagrange conormal chords with energy κ; if L is not reversible, then there are at least two Euler-Lagrange conormal chords with energy κ but with different values of the Lagrangian action.*

This work generalizes the ones on orthogonal Riemannian and Finsler geodesic chords. When the Lagrangian is the energy function of a Riemannian or Finsler metric, a solution of the Euler-Lagrange equations is a geodesic and the conormal boundary conditions are nothing but the orthogonality condition of the geodesic with the boundary. The Riemannian and Finsler geodesic chords on a manifold with boundary are strictly related with the brake orbits in a potential well for a Hamiltonian system of classical type, namely when the hamiltonian function is fiberwise even and convex (cf. [1]). Indeed, using a Legendre transform and the Maupertuis-Jacobi principle, every brake orbit of a Hamiltonian system of classical type corresponds to a geodesic in a disk with endpoints on the boundary, where the disk is endowed with a Jacobi-Finsler metric. Seifert conjectured in [2] that there are at least N brake orbits in an N-dimensional potential well of a natural Hamiltonian system, hence where the brake-orbits correspond to the geodesics of a Riemannian metric. This conjecture has been recently proved in [3], exploiting also some partial results achieved by the same authors in different previous works (cf. [4–9]), while a preliminary result for the Finsler case is presented in [10].

The proof of the main theorem is based on a variational approach, seeing ELCCs as critical points of the *free-time Lagrangian action functional*

$$\mathcal{L}_\kappa(\gamma) = \int_0^T \left(L(\gamma(t),\dot\gamma(t)) + \kappa\right) dt,$$

defined on the set of paths in $\overline{\Omega}$ with endpoints in $\partial\Omega$ and of class $H^{1,2}$, namely absolutely continuous with derivative in L^2. The existence and multiplicity results are then obtained through a minimax approach, exploiting a particular version of the Ljusternik and Schnirelman category.

This work directly extends the results achieved in [11], where the main differences are as follows. Firstly, the main theorem is stated in [11] with the additional hypothesis that if $\nu\colon \partial\Omega \to T\mathcal{M}$ is a unit normal vector field with respect along $\partial\Omega$, then

$$d_v L(q,\nu)[\xi] = 0, \qquad \forall q \in \partial\Omega, \forall \xi \in T_q\partial\Omega. \tag{4}$$

This condition, which is trivially satisfied when L is the energy of a Riemannian metric (cf. [9]), is a key ingredient to exploit the approach presented in [12] to prove that every critical curve of $\mathcal{L}_\kappa$ has $H^{2,\infty}$ regularity. Instead, in this work we prove the desired regularity following a *penalization method* and the hypothesis (4) is not required. Secondly, the geometric distinction of the ELCCs has not been proved in [11]. Finally, the minimax method applied in this paper is more simple than the one in [11]. Indeed, we reduce our study on a fixed-time problem, so we can avoid to take care of the possible sources of non-compactness of the time variable in the free-time Lagrangian action functional.

This work also extends [10], since Theorem 1 holds even when $L = F^2$, where $F\colon T\mathcal{M} \to \mathbb{R}$ is a Finsler metric on $\mathcal{M}$. In this case, the ELCCs are actually *orthogonal Finsler geodesic chords*, namely geodesics with respect the Finsler structure such that $\gamma(0), \gamma(T) \in \partial\Omega$, $\gamma(]0,T[) \subset \Omega$ and

$$d_v F^2\left(\gamma(i), \dot\gamma(i)\right)\Big|_{T_{\gamma(i)}\partial\Omega} = 0, \quad i = 0, T.$$

Theorem 1 cannot be directly applied with $L = F^2$, since in this case L is C^∞ on $T\mathcal{M}\backslash 0$ and only C^1 on the entire tangent bundle. However, for every fixed energy level $\kappa > 0 = m(F^2)$, we can construct (cf. ([13] Corollary 2.3)) a C^2 Tonelli Lagrangian $\widetilde{L}\colon T\overline{\Omega} \to \mathbb{R}$ such that $\widetilde{L}(q,v) = F^2(q,v)$ if $F^2(q,v) \geq \kappa/2$, with $\widetilde{L}$ reversible if F is reversible. As a consequence, every Euler-Lagrange chord for $\widetilde{L}$ with energy κ is actually a Finsler geodesic.

2. Framework Setup and Notation

For the sake of presentation, we suppose that $\overline{\Omega}$ is embedded into a N-manifold $\mathcal{M}$ including $\overline{\Omega}$. Using the Whitney embedding theorem, we can see $\mathcal{M}$ as a smooth (C^3) submanifold of $\mathbb{R}^{2N}$, endowed with the Riemannian structure of the euclidean scalar product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^{2N}$. A coordinate system $(q^i) = (q^1, \dots, q^N)$ on $\mathcal{M}$ naturally induces a coordinate system $(q^i, v^i), i = 1, \dots, N$ on the tangent bundle $T\mathcal{M}$. If f is a real-valued function defined on $T\mathcal{M}$, then $d_q f$ and $d_v f$ will denote the derivatives of f with respect to q and v respectively. In a local chart, the derivatives with respect to q^i and v^i will be denoted by ∂_{q^i} and ∂_{v^i}. We will use the Einstein notation, implying summation over a set of indexed terms in a formula. The norm $\|\cdot\|\colon T\mathcal{M} \to \mathbb{R}$ is that one induced by the euclidean product in $\mathbb{R}^{2N}$, while we denote by $\|\cdot\|_{L^p}$ the norm in a L^p space, for any $1 \leq p \leq \infty$.

2.1. Geometry of $\overline{\Omega}$

There exists a function $\Phi\colon \mathcal{M} \to \mathbb{R}$ of class C^2 such that $\Omega = \Phi^{-1}(]-\infty, 0[)$, $\partial\Omega = \Phi^{-1}(0)$ and $d\Phi(q) \neq 0$ for every $q \in \partial\Omega$. For all $\delta > 0$, we set

$$\Omega_\delta = \Phi^{-1}(]-\infty, \delta[).$$

By the C^2 regularity of Φ, there exists a $\delta_0 > 0$ such that

$$d\Phi(q) \neq 0, \quad \forall q \in \Phi^{-1}([-\delta_0, \delta_0]), \tag{5}$$

and such that $\overline{\Omega}_\delta$ is compact for any $\delta \in [0, \delta_0]$. We also set

$$K_0 = \max_{q \in \overline{\Omega}_{\delta_0}} \|\nabla\Phi(q)\|. \tag{6}$$

2.2. Sobolev and Functional Spaces

For any $[a, b] \subset \mathbb{R}$, we consider the Sobolev spaces

$$H^{1,2}([a,b],\mathbb{R}^{2N}) = \left\{ x\colon [a,b] \to \mathbb{R}^{2N} : x \text{ is absolutely continuous and } \dot{x} \in L^2([a,b],\mathbb{R}^{2N}) \right\}.$$

and

$$H_0^{1,2}([a,b],\mathbb{R}^{2N}) = \left\{ x \in H^{1,2}\left([a,b],\mathbb{R}^{2N}\right) : x(0) = x(1) = 0 \right\}.$$

For $S \subset \mathcal{M}$, set

$$H^{1,2}([a,b],S) = \left\{ x \in H^{1,2}([a,b],\mathbb{R}^{2N}) : x(s) \in S \text{ for all } s \in [0,1] \right\}.$$

It is well known that $H^{1,2}([a,b],\mathcal{M})$ is a manifold of class C^2 and its tangent space at x is

$$T_x\mathcal{M} = \left\{ \xi \in H^{1,2}([a,b],\mathbb{R}^{2N}) : \xi(s) \in T_{x(s)}\mathcal{M} \text{ for all } s \in [a,b] \right\}.$$

Due to the presence of the boundary $\partial\Omega$, not all the elements of $T_x\mathcal{M}$ are always admissible variations. So we give the following definition.

Definition 3. *Let $\mathcal{Q}$ be a non-empty subset of $H^{1,2}([a,b],\mathcal{M})$. Then $\xi \in T_x\mathcal{M}$ is an admissible infinitesimal variation of x in $\mathcal{Q}$ if there exists an $\epsilon > 0$ and a differentiable function $h\colon (-\epsilon,\epsilon) \times [a,b] \to \mathcal{M}$ such that*

- $h(0,s) = x(s)$;
- $h(\tau,\cdot) \in \mathcal{Q}$ *for all* $\tau \in (-\epsilon,\epsilon)$;
- $\frac{\partial h}{\partial \tau}(\tau,s)\Big|_{\tau=0} = \xi(s)$.

The set of all admissible infinitesimal variation of x in $\mathcal{Q}$ is denoted by $\mathcal{V}^-(x,\mathcal{Q})$.

We identify a curve $\gamma\colon [0,T] \to \mathcal{M}$ with the pair $(x,T) \in H^{1,2}([0,1],\mathcal{M}) \times (0,+\infty)$, where $x(s) = \gamma(Ts)$. Thus, the main functional space of our variational problem is

$$\mathfrak{M} = \{x \in H^{1,2}([0,1],\overline{\Omega}) : x(0), x(1) \in \partial\Omega\}.$$

If $x \in \mathfrak{M}$, then

$$\mathcal{V}^-(x,\mathfrak{M}) = \left\{ \begin{array}{c} \xi \in T_x\mathcal{M} : \langle \nabla\Phi(x(0)), \xi(0)\rangle = \langle \nabla\Phi(x(1)), \xi(1)\rangle = 0, \\ \langle \nabla\Phi(x(s)), \xi(s)\rangle \le 0 \text{ for any } s \in (0,1) \text{ s.t. } x(s) \in \partial\Omega \end{array} \right\}.$$

In other words, a vector field $\xi \in T_x\mathcal{M}$ is in $\mathcal{V}^-(x,\mathfrak{M})$ if $\xi(0)$ and $\xi(1)$ are tangent to $\partial\Omega$ and $\xi(s)$ points inside $\overline{\Omega}$ whenever $x(s) \in \partial\Omega$.

2.3. The Free-Time Action Functional

The main functional of our variational problem is

$$\mathcal{L}_\kappa\colon \mathfrak{M}\times]0,+\infty[\to \mathbb{R}, \quad \mathcal{L}_\kappa(x,T) = T\int_0^1 \left(L\left(x(s), \frac{\dot{x}(s)}{T} \right) + \kappa \right) ds.$$

Remark 1. *The functional $\mathcal{L}_\kappa$ is well-defined only if $L(q,v)$ is quadratic at infinity. Since we are considering the fixed energy problem for a Tonelli Lagrangian, the energy level $E^{-1}(\kappa)$ is a compact submanifold of $T\mathcal{M}$ and* *we can modify the Lagrangian outside a compact set $K \supseteq E^{-1}(\kappa)$ to achieve quadratic growth. In particular, we assume that $L(q, v)$ is quadratic at infinity, namely there is a constant $R > 0$ such that*

$$L(q,v) = \frac{1}{2}\|v\|^2 + \vartheta(q)[v] - V(q), \quad \forall \|v\| \geq R,$$

where ϑ is a smooth one-form and V is a smooth function on $\mathcal{M}$.

Through all this work, we need the following lemma (cf. ([14], Lemma 3.1)), which provides lower and upper bounds for the lagrangian function, its derivatives and the energy functions. Its proof is based on the quadratic construction given in Remark 1 and the compactness of $\overline{\Omega}_{\delta_0}$.

Lemma 1. *There exist four constants $a_0, a_1, b_1, A_1, B_1 > 0$ such that for all (q, v) with $q \in \overline{\Omega}_{\delta_0}$ and $v \in T_q\mathcal{M}$ we have*

$$\|d_q L(q,v)\| \leq a_0 \left(1 + \|v\|^2\right) \quad \textit{and} \quad \|d_v L(q,v)\| \leq a_0 \left(1 + \|v\|\right); \tag{7}$$

$$\|d_{qq} L(q,v)\| \leq a_0 \left(1 + \|v\|^2\right) \quad \textit{and} \quad \|d_{qv} L(q,v)\| \leq a_0 \left(1 + \|v\|\right); \tag{8}$$

$$a_1\|v\|^2 - b_1 \leq L(q,v) \leq A_1\|v\|^2 + B_1; \tag{9}$$

$$a_1\|v\|^2 - L(q,0) \leq E(q,v) \leq A_1\|v\|^2 - L(q,0); \tag{10}$$

$$a_1\|\xi\|^2 \leq d^2_{vv} L(q,v)[\xi,\xi] \leq A_1\|\xi\|^2, \qquad \forall \xi \in T_q\mathcal{M}. \tag{11}$$

Proposition 1. *The action functional $\mathcal{L}_\kappa$ is of class $C^{1,1}$, namely it is continuously differentiable and its differential $d\mathcal{L}_\kappa$ is locally Lipschitz continuous.*

Proof. See e.g., ([15] Theorem 2.3.2). □

The derivative of $\mathcal{L}_\kappa$ in the x-direction is given by

$$d_x\mathcal{L}_\kappa(x,T)[\xi] = T \int_0^1 \left(d_q L\left(x, \frac{\dot{x}}{T}\right)[\xi] + d_v L\left(x, \frac{\dot{x}}{T}\right)\left[\frac{\dot{\xi}}{T}\right] \right) ds$$

for $\xi \in T_x H^{1,2}([0,1], \mathcal{M})$. In the T-direction we have

$$\frac{\partial \mathcal{L}_\kappa}{\partial T}(x,T) = \int_0^1 \left(\kappa + L\left(x, \frac{\dot{x}}{T}\right) - d_v L\left(x, \frac{\dot{x}}{T}\right)\left[\frac{\dot{x}}{T}\right] \right) ds = \int_0^1 \left(\kappa - E\left(x, \frac{\dot{x}}{T}\right) \right) ds.$$

Hence, the differential of $\mathcal{L}_\kappa$ is

$$d\mathcal{L}_k(x,T)[\xi,H] = d_x\mathcal{L}_\kappa(x,T)[\xi] + \frac{\partial \mathcal{L}_\kappa}{\partial T}(x,T)H.$$

Definition 4. *Set $(x,T) \in \mathfrak{M} \times]0,+\infty[$. We say that (x,T) is a $\mathcal{V}^-$-critical curve for $\mathcal{L}_\kappa$ on $\mathfrak{M} \times]0,+\infty[$ if*

$$d_x\mathcal{L}_\kappa(x,T)[\xi] \geq 0, \qquad \forall \xi \in \mathcal{V}^-(x,\mathfrak{M}),$$

and

$$\frac{\partial \mathcal{L}_\kappa}{\partial T}(x,T) = 0.$$

The following lemma will be useful in different parts of this paper, so we state it here for the convenience of the readers.

Lemma 2. *Let $[a,b] \subset [0,1]$ and let $(x_n, T_n) \in, H^{1,2}([a,b], \overline{\Omega}_{\delta_0}) \times]0,+\infty[$ be a sequence such that T_n is bounded from above, namely there exists $T^* > 0$ such that $T_n < T^*$ for all $n \in \mathbb{N}$. If there exists $c \in \mathbb{R}$ such that*

$$T_n \int_a^b \left(L\left(x_n, \frac{\dot{x}_n}{T_n} \right) + \kappa \right) ds < c, \qquad \forall n \in \mathbb{N},$$

then $\dot{x}_n$ is uniformly bounded in $L^2([a,b], \mathbb{R}^{2N})$.

Proof. From (9) we have

$$c \geq T_n \int_a^b \left(a_1 \frac{\|\dot{x}_n\|^2}{T_n} - b_1 + \kappa \right) ds \geq a_1 \frac{\|\dot{x}_n\|^2_{L^2([a,b],\mathbb{R}^{2N})}}{T^*} - T^* \left| b_1 - \kappa \right|.$$

Hence

$$\|\dot{x}_n\|_{L^2([a,b],\mathbb{R}^{2N})} \leq \left(\frac{T^*}{a_1} \left(c + T^* \left| b_1 - \kappa \right| \right) \right)^{1/2}.$$

□

2.4. The Energy Critical Values

The behaviour of the free-time Lagrangian functional, hence of the Euler-Lagrangian flow induced by the Tonelli-Lagrangian L, changes when k is greater then some specific energy levels, called *critical values*. Here we only describe the critical values that affect our study; for more details about the different critical values of a Tonelli Lagragian function we refer, for instance, to [14,16,17].

We denote by $e_0(L)$ be the maximal critical value of the energy function E. Since L is a fiberwise convex and $\overline{\Omega}$ is compact, we have

$$e_0(L) = \max_{q \in \overline{\Omega}} E(q, 0). \tag{12}$$

The importance of this critical value is quite clear, since the projection of $E^{-1}(\kappa)$ on $\overline{\Omega}$ is surjective if and only if $\kappa \geq e_0(L)$. We will also prove that whenever $\kappa > e_0(L)$, for every path γ there exists a unique minimum of $\mathcal{L}_\kappa$ among all the linear orientation preserving reparametrizations of γ (see Section 4 for more details). As a consequence, we can reduce our analysis to a fixed time problem and this simplifies the minimax approach that we will exploit to find the critical points.

Another important value which affects the behaviour of $\mathcal{L}_\kappa$ is the *Mañé critical value* $c(L)$. In our setting, $c(L)$ can be defined as minus infimum of the mean Lagrangian action over all the closed curves γ, hence

$$c(L) = \inf \left\{ \kappa \in \mathbb{R} : \mathcal{L}_\kappa(x, T) \geq 0, \forall (x, T) \in \mathfrak{M} \times]0, +\infty[\text{ s.t. } x(0) = x(1) \right\}.$$

This Mañé critical value $c(L)$ marks an important changes in behaviour of the free-time action functional because, whenever $\kappa > c(L)$, $\mathcal{L}_\kappa$ is bounded from below and satisfies the Palais-Smale condition. Moreover, if $\kappa > c(L)$, then the Euler-Lagrange flow on $E^{-1}(\kappa)$ is conjugated up to a time-reparametrization to the geodesic flow which is induced by a Finsler metric on $\mathcal{M}$ (see ([16] Theorem 4.1)). However, our study cannot take advantage of this construction because the conormal boundary conditions (2) may not be preserved by the time-reparametrization, as shown in the following example.

Example 1. *Let Ω be $\mathbb{D}^2 = \{q \in \mathbb{R}^2 : \|q\| \leq 1\}$ and*

$$L(q, v) = \frac{1}{4}(v_1^4 + v_2^4) + \frac{1}{2}(v_1^2 + v_2^2), \qquad \forall q \in \overline{\Omega}, v \in \mathbb{R}^2,$$

where v_i indicates the component of v. Then for all $q \in \overline{\Omega}$ and $v \in \mathbb{R}^2$ we have

$$E(q,v) = \frac{3}{4}(v_1^4 + v_2^4) + \frac{1}{2}(v_1^2 + v_2^2),$$

and

$$d_v L(q,v) = (v_1^3 + v_1, v_2^3 + v_2).$$

Set an energy level κ and let F be the Finsler metric on $\overline{\Omega}$ such that the Euler-Lagrange flow on $E^{-1}(\kappa)$ is conjugated up to a time-reparametrization to the geodesic flow which is induced by F. Then $(F^2)^{-1}(1) = E^{-1}(\kappa)$ and

$$d_v F^2(q,v)[\xi] = 0 \iff d_v E(q,v)[\xi] = 0, \qquad \forall v \in E^{-1}(\kappa).$$

Hence the orthogonality condition for F on the boundary reads as

$$(3v_1^3 + v_1)\xi_1 + (3v_2^3 + v_2)\xi_2 = 0, \qquad \forall \xi \in T_q\partial\Omega,$$

that is different from the conormal boundary conditions for L

$$d_v L(q,v)[\xi] = (v_1^3 + v_1)\xi_1 + (v_2^3 + v_2)\xi_2 = 0, \qquad \forall \xi \in T_q\partial\Omega.$$

As a consequence, if a curve is an orthogonal Finsler geodesic chord for F, it is an Euler-Lagrange chord for L, but it may not satisfies the conormal boundary conditions (2).

We remark that the previous critical values satisfy the following chain of inequalities

$$e_0(L) \leq c(L) \leq m(L).$$

Thus, all the results we are going to prove will be available when $\kappa > m(L)$, as assumed in Theorem 1. Moreover, all the previous critical values coincide when the Lagrangian is reversible. However, if the reversibility assumption does not hold, all these values may be different.

3. Regularity of the $\mathcal{V}^-$-Critical Curves

Proposition 2. *Let (x,T) be $\mathcal{V}^-$-critical for $\mathcal{L}_\kappa$ on $\mathfrak{M}\times]0,+\infty[$. Then x has $H^{2,2}$ regularity, namely $\dot{x}$ is absolutely continuous and $\ddot{x} \in L^2([0,1],\mathbb{R}^{2N})$.*

Proposition 2 is the key ingredient of our variational approach to prove Theorem 1. Indeed, if there are no ELCTC in $\overline{\Omega}$ and $\kappa > m(L)$, then the regularity of the $\mathcal{V}^-$-critical curves for $\mathcal{L}_\kappa$ on $\mathfrak{M}\times]0,+\infty[$ implies that they are ELCCs.

While in [18] the regularity is proved exploiting directly the definition of critical curve in a manifold with boundary, we base our proof on a *penalization method*. We allow the curves to lay on an open set which contains $\overline{\Omega}$, adding a penalization term that is different from zero only when the curve does not lay on $\overline{\Omega}$. Since we are on an open set, the regularity of the critical curves can be obtained with standard techniques. Then, we prove the regularity of the $\mathcal{V}^-$-critical curves of the functional taking the limit to remove the penalization term. The penalization method in a manifold with boundary has been exploited, for instance, in [19] for the Riemannian, in [20] for the Lorentzian and in [21] for the Finsler case.

Let $p,q \in \overline{\Omega}$ be such that $p \neq q$ and $[a,b] \subset [0,1]$. We set

$$\mathcal{C}([a,b],p,q,\Omega_\delta) = \left\{ \gamma \in H^{1,2}([a,b],\Omega_\delta) : \gamma(a) = p,\ \gamma(b) = q \right\},$$

for any $\delta \leq \delta_0$, where δ_0 has been defined in (5). For the sake of presentation, we denote $\mathcal{C}([a,b],p,q,\Omega_\delta)$ by $\mathcal{C}_\delta$ when we have fixed $[a,b]$ and $p,q \in \overline{\Omega}$ and no confusion may arise. Then, fixing $[a,b] \subset [0,1]$ and $T > 0$, we define the functional

$$\mathcal{J} \colon H^{1,2}([a,b],\mathcal{M}) \to \mathbb{R} \quad \text{by} \quad \mathcal{J}(x) = T\int_a^b L\left(x, \frac{\dot{x}}{T}\right) ds.$$

We remark that the energy level κ does not appear in the definition of $\mathcal{J}$. Indeed, since $[a,b] \subset [0,1]$ and $T > 0$ are fixed, $T\int_a^b \kappa ds$ is a constant and does not affect the behaviour of $\mathcal{J}$. We consider on $\mathcal{C}([a,b],p,q,\Omega_\delta)$ the penalized functional

$$\mathcal{J}_\delta(x) = \mathcal{J}(x) + \int_a^b \chi_\delta(\Phi(x))\, ds,$$

where the function $\chi_\delta \colon]-\infty,\delta[\to \mathbb{R}$ is defined by

$$\chi_\delta(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \frac{t^2}{(\delta-t)^2}, & \text{if } 0 \leq t < \delta. \end{cases}$$

By definition of χ_δ we have

$$\chi_\delta'(t) = \frac{2\delta}{t(\delta-t)}\chi_\delta(t). \tag{13}$$

The regularity of critical points of $\mathcal{J}_\delta$ in $\mathcal{C}_\delta$ can be proved by a standard argument (see, for example, ([22] Theorem 4.1)) involving the global inversion theorem (cf. ([23] Theorem 1.8)), which is available since $d_{vv}L(q,v)$ is positive definite. Thus, we have the following lemma.

Lemma 3. *For any $\delta \in (0,\delta_0)$, let x_δ be a critical curve for $\mathcal{J}_\delta$ in $\mathcal{C}_\delta$. Then x_δ is C^2 and satisfies the equation*

$$T\left(d_q L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right) - \frac{1}{T}d_v L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)\right) = -\chi_\delta'(\Phi(x_\delta))d\Phi(x_\delta). \tag{14}$$

Remark 2. *If x_δ is a critical curve for $\mathcal{J}_\delta$ in $\mathcal{C}_\delta$, (14) implies the existence of a constant $E_{x_\delta} \in \mathbb{R}$ such that*

$$E_{x_\delta} = TE\left(x_\delta, \frac{\dot{x}_\delta}{T}\right) - \chi_\delta(\Phi(x_\delta)) \quad \text{on } [a,b]. \tag{15}$$

Moreover, in local coordinates (14) reads as

$$\frac{\ddot{x}^i_\delta}{T^2} = \ell^{ij}\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)\left(\partial_{q^j}L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right) - \partial^2_{q^k v^j}L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)\frac{\dot{x}^k_\delta}{T} + \frac{1}{T}\chi'(\Phi(x_\delta))\partial_{q^j}\Phi(x_\delta)\right). \tag{16}$$

The following result, known as Gordon's lemma (cf. [24]), is a key ingredient to prove the existence of a minimizer for $\mathcal{J}_\delta$ in the open set $\mathcal{C}_\delta$. Indeed, it allows proving that $\mathcal{C}_\delta$ contains at least a minimizing sequence which converges in $\mathcal{C}_\delta$.

Lemma 4 (Gordon's Lemma)**.** *Let $(x_n)_n \subset \mathcal{C}_\delta$ such that*

$$\mathcal{J}(x_n) \leq c < +\infty, \tag{17}$$

for some $c > 0$. Then if there exists a sequence $(s_n)_n \subset [a,b]$ such that

$$\lim_{n\to\infty} \Phi(x_n(s_n)) = \delta,$$

then

$$\lim_{n\to\infty}\int_a^b \chi_\delta(\Phi(x_n))ds = +\infty.$$

Proof. By (17), the sequence

$$((x_n, T_n))_n \subset H^{1,2}([a,b], \overline{\Omega}_{\delta_0}) \times]0, +\infty[$$

with $T_n = T$ for all $n \in \mathbb{N}$, satisfies the hypothesis of Lemma 2, hence $\int_a^b \|\dot{x}_n\|^2 ds$ is uniformly bounded. As a consequence, recalling the definition of K_0 in (6), for any $s \in [a, s_n]$ we have

$$\begin{aligned}\Phi(x_n(s_n)) - \Phi(x_n(s)) &= \int_s^{s_n} \langle \nabla\Phi(x(\sigma)), \dot{x}_n(\sigma)\rangle d\sigma \le \int_s^{s_n} K_0 \|\dot{x}_n(\sigma)\| d\sigma \\ &\le K_0 (s_n - s)^{\frac{1}{2}} \left(\int_a^b \|\dot{x}_n\|^2 ds\right)^{\frac{1}{2}} \le C(s_n - s)^{\frac{1}{2}},\end{aligned} \tag{18}$$

for some strictly positive constant C that does not depend on n. Then

$$0 < \delta - \Phi(x_n(s)) \le C(s_n - s)^{\frac{1}{2}} + (\delta - \Phi(x_n(s_n))),$$

and

$$\frac{1}{(\delta - \Phi(x_n(s)))^2} \ge \frac{1}{2\left(C^2(s_n - s) + (\delta - \Phi(x_n(s_n))^2\right)}. \tag{19}$$

Since $\Phi(x_n(s_n)) \to \delta > 0$, for n sufficiently large $\Phi(x_n(s_n)) > \frac{2}{3}\delta$ and there exists a sequence $\bar{s}_n < s_n$ such that

$$\Phi(x_n(\bar{s}_n)) = \frac{1}{3}\delta.$$

From (18) we get that

$$(s_n - \bar{s}_n)^{\frac{1}{2}} \ge \frac{1}{3C}\delta > 0.$$

Clearly we can choose $\bar{s}_n$ such that

$$\Phi(x_n(s)) > \frac{1}{3}\delta, \qquad \forall s \in (\bar{s}_n, s_n).$$

Thus, integrating both hands sides of (19) we obtain

$$\int_0^1 \chi_\delta(x_n(s))\, ds \ge \frac{1}{9}\int_{\bar{s}_n}^{s_n} \frac{\delta^2}{2\left(C^2(s_n - s) + (\delta - \Phi(x_n(s_n))^2\right)} ds$$

and passing to the limit we get the thesis. □

Lemma 5. *For every $\delta \in (0, \delta_0)$, the following statements hold:*

(i) for all $c \in \mathbb{R}$, the sublevels

$$\mathcal{J}_\delta^c = \{x \in \mathcal{C}_\delta : \mathcal{L}_\delta(x) \le c\}$$

are complete metric spaces;

(ii) $\mathcal{J}_\delta$ satisfies the Palais-Smale condition, namely if a sequence $(x_n)_n \subset \mathcal{C}_\delta$ is such that $\mathcal{J}_\delta(x_n)$ is bounded and $d\mathcal{J}_\delta(x_n) \to 0$, then $(x_n)_n$ admits a convergent subsequence.

Proof. (i) Fix $\delta \in (0, \delta_0)$. If (x_n) is a Cauchy sequence in $\mathcal{J}_\delta^c$, then it uniformly converges to a curve $\bar{x}$ with support in $\overline{\Omega}_\delta$. Arguing by contradiction, if there exists $\bar{s} \in [a, b]$ such that $\bar{x}(\bar{s})$ lies on the boundary of $\overline{\Omega}_\delta$, then there exists a sequence $(s_n)_n \in [a, b]$ such that $\lim_{n\to\infty}\Phi(x_n(s_n)) = \delta$.

By Lemma 4, $\mathcal{J}_\delta(x_n) \to +\infty$, which is absurd. As a consequence, $\bar{x}([a,b]) \in \mathcal{C}_\delta$ and, by the continuity of $\mathcal{J}_\delta$, $\bar{x} \in \mathcal{J}_\delta^c$.

(ii) Let $(x_n)_n \subset \mathcal{C}_\delta$ be a sequence such that $\mathcal{J}_\delta(x_n)$ is bounded and $d\mathcal{J}_\delta(x_n) \to 0$. By Lemma 2, there exists a constant $c_1 > 0$ such that $\int_a^b \|\dot{x}_n\|^2 ds \le c_1$. Hence, for all $[s_1, s_2] \in [a,b]$ we have

$$\mathrm{dist}(x_n(s_1), x_n(s_2)) \le \int_{s_1}^{s_2} \|\dot{x}_n\|^2 ds \le c_1\,(s_2 - s_1)^{1/2}.$$

By the Ascoli-Arzelá theorem, there exists a subsequence (x_n) that uniformly converges to a curve y, and such that $\dot{x}_n$ converges weakly to $\dot{y}$. By the completeness of the sublevels of $\mathcal{J}_\delta$, $y \in \mathcal{C}_\delta$. It remains to prove that $x_n \to y$ in $H^{1,2}([a,b], \Omega_\delta)$, hence that $\dot{x}_n \to \dot{y}$ strongly in L^2. Set

$$\xi_n(s) = \left(\exp_{x_n(s)}\right)^{-1} y(s), \qquad \forall s \in [a,b],$$

where exp is the exponential map of the Riemannian structure of $\mathcal{M}$. Since $x_n \to y$ uniformly, (ξ_n) is well defined for n sufficiently large. Moreover, ξ_n converges uniformly to zero, hence

$$\int_a^b \chi'_\delta(\Phi(x_n)) d\Phi(x_n)[\xi_n] \to 0.$$

As a consequence, $d\mathcal{J}_\delta(x_n) \to 0$ implies that

$$\lim_{n\to\infty} d\mathcal{J}(x_n)[\xi_n] = \lim_{n\to\infty} T \int_a^b \left(d_q L\left(x_n, \frac{\dot{x}_n}{T}\right)[\xi_n] + d_v L\left(x_n, \frac{\dot{x}_n}{T}\right)\left[\frac{\dot{\xi}_n}{T}\right]\right) = 0. \tag{20}$$

Since $d_q L(x_n, \dot{x}_n/T)$ is bounded in L^1 by (7) and ξ_n converges uniformly to zero, we have

$$\lim_{n\to\infty} \int_a^b d_q L\left(x_n, \frac{\dot{x}_n}{T}\right)[\xi_n] ds = 0.$$

Thus, from (20) we obtain

$$\lim_{n\to\infty} \int_a^b d_v L\left(x_n, \frac{\dot{x}_n}{T}\right)[\dot{\xi}_n] ds = 0.$$

Let $[s_0, s_1] \subset [a,b]$ such that $y([s_0, s_1])$ is in a single chart. If n is sufficiently large, also $x_n([s_0, s_1])$ is in this chart and

$$\dot{\xi}_n = \dot{y} - \dot{x}_n + \omega_n,$$

where w_n converges to zero in L^2. Then

$$\lim_{n\to\infty} \int_{s_0}^{s_1} d_v L\left(x_n, \frac{\dot{x}_n}{T}\right)[\dot{y} - \dot{x}_n] ds = 0. \tag{21}$$

Since x_n converges uniformly to y and $\dot{x}_n$ converges weakly to $\dot{y}$, we have

$$\lim_{n\to\infty} \int_{s_0}^{s_1} d_v L\left(x_n, \frac{\dot{y}}{T}\right)[\dot{y} - \dot{x}_n] ds = 0. \tag{22}$$

By (21) and (22) we obtain

$$\lim_{n\to\infty} \int_{s_0}^{s_1} \left(d_v L\left(x_n, \frac{\dot{y}}{T}\right) - d_v L\left(x_n, \frac{\dot{x}_n}{T}\right)\right)[\dot{y} - \dot{x}_n] ds = 0.$$

Since L is C^2, we can apply the mean value theorem and, by (11), there exists a constant $c_2 > 0$ such that

$$0 = \lim_{n\to\infty} \int_{s_0}^{s_1} \left(d_v L\left(x_n, \frac{\dot{y}}{T}\right) - d_v L\left(x_n, \frac{\dot{x}_n}{T}\right)\right)[\dot{y} - \dot{x}_n]ds$$
$$= \lim_{n\to\infty} \int_{s_0}^{s_1} ds \int_0^1 d_{vv} L\left(x_n, \frac{\dot{x}_n + \sigma(\dot{y} - \dot{x}_n)}{T}\right)[\dot{y} - \dot{x}_n][\dot{y} - \dot{x}_n]d\sigma$$
$$\geq \lim_{n\to\infty} c_2 \int_{s_0}^{s_1} \|\dot{y} - \dot{x}_n\|^2 ds.$$

Since the above inequality holds for in every local chart, $\dot{x}_n$ converges to $\dot{y}$ in L^2 and this ends the proof. □

Remark 3. *Since $\mathcal{J}_\delta$ satisfies the Palais-Smale condition and it is bounded from below, $\mathcal{J}_\delta$ has a minimum point $x_\delta \in \mathcal{C}_\delta$.*

Lemma 6. *For all $\delta \in (0, \delta_0)$, let x_δ be a minimum of $\mathcal{J}_\delta$ on $\mathcal{C}_\delta$. Then there exist two constants $k_1, k_2 \in \mathbb{R}$ such that for all $\delta \in (0, \delta_0)$*

$$\mathcal{J}_\delta(x_\delta) \leq k_1 < +\infty \tag{23}$$

and

$$E_{x_\delta} \leq k_2 < +\infty, \tag{24}$$

where E_{x_δ} are the constants defined in (15).

Proof. Let y be a curve in $\mathfrak{M}$ such that $y(a) = p$ and $y(b) = q$. Then $y \in \mathcal{C}_\delta$ for all $\delta \in (0, \delta_0)$. Hence

$$\mathcal{J}_\delta(x_\delta) \leq \mathcal{J}_\delta(y) = \mathcal{J}(y) = k_1 < +\infty, \qquad \forall \delta \in (0, \delta_0).$$

By (10) and (9), for all $\delta \in (0, \delta_0)$ we have the following chain of inequalities

$$\begin{aligned}(b-a)E_{x_\delta} &= T\int_a^b E\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)ds - \int_a^b \chi_\delta(\Phi(x_\delta))ds \leq T\int_a^b \left(A_1\left\|\frac{\dot{x}_\delta}{T}\right\|^2 - L(x_\delta, 0)\right)ds \\ &\leq T\int_a^b \left(\frac{A_1}{a_1}\left(L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right) + b_1\right) - L(x_\delta, 0)\right)ds \leq \frac{A_1}{a_1}k_1 + 2\frac{A_1}{a_1}b_1 T(b-a).\end{aligned} \tag{25}$$

As a consequence, from (25) we infer there exists a constant k_2 such that (24) holds. □

By Lemma 6, if $(x_{\delta_n})_n$ is a sequence such that x_{δ_n} is a minimum for $\mathcal{J}_{\delta_n}$ $\forall n$, then there exists a subsequence that is uniformly convergent to a curve y with support in $\overline{\Omega}$. However, we need the following two intermediate results to prove that y is a minimum for $\mathcal{J}$, which is a key ingredient to prove Proposition 2.

Lemma 7. *Let $(x_\delta)_{\delta\in(0,\delta_0)}$ be a family in $H^{1,2}([a,b], \mathcal{M})$ such that for any $\delta \in (0, \delta_0)$, x_δ is a minimum of $\mathcal{J}_\delta$ on $\mathcal{C}([a,b], p, q, \Omega_\delta)$. For any $\delta \in (0, \delta_0)$, set*

$$\lambda_\delta(s) = -\chi'_\delta(\Phi(x_\delta(s))), \quad s \in [a,b]. \tag{26}$$

Then there exists $\delta_1 \in (0, \delta_0)$ such that

$$\sup_{\delta\in(0,\delta_1)} \|\lambda_\delta\|_\infty = \sup_{\delta\in(0,\delta_1)} \max_{s\in[a,b]} |\lambda_\delta(s)| < +\infty. \tag{27}$$

Proof. For any $\delta \in (0, \delta_0)$, set $\rho_\delta(s) = \Phi(x_\delta(s))$ and let s_δ be a maximum point for ρ_δ. Since the derivative of χ_δ is non-decreasing and $\chi'_\delta(t) = 0$ for any $t \leq 0$, then

$$0 \leq \chi'_\delta(\Phi(\gamma_\delta(s))) \leq \chi'_\delta(\Phi(\gamma_\delta(s_\delta))), \quad \forall s \in [a, b].$$

Thus, it suffices to prove (27) assuming that

$$\Phi(\gamma_\delta(s_\delta)) \in]0, \delta[.$$

We will prove the existence of a constant $C > 0$ such that

$$\chi'_\delta(\Phi(\gamma_\delta(s_\delta))) \leq C\Big(1 + \chi_\delta(\Phi(\gamma_\delta(s_\delta)))\Big), \quad \text{for any } \delta \in (0, \delta_0), \tag{28}$$

from which we infer the thesis. Indeed, by (13) we obtain

$$\left(\frac{2\delta}{\Phi(\gamma_\delta(s_\delta))(\delta - \Phi(\gamma_\delta(s_\delta)))} - C\right)\chi_\delta(\Phi(\gamma(s_\delta))) \leq C.$$

Since

$$\inf_{t \in (0,\delta)} \frac{2\delta}{(t(\delta - t))} = \frac{8}{\delta},$$

then setting $\delta_1 = 8/C$ we obtain (27).

By Lemma 3, x_δ is twice differentiable. Since s_δ is a maximum for ρ_δ we have

$$\ddot{\rho}_\delta(s_\delta) = \partial^2{}_{q^i q^j}\Phi(x_\delta(s_\delta))\dot{x}^i_\delta \dot{x}^j_\delta + \partial_{q^i}\Phi(x_\delta(s_\delta))\ddot{x}^i_\delta(s_\delta) \leq 0.$$

By (16) we have

$$\begin{aligned}\partial^2{}_{q^i q^j}\Phi(x_\delta)\dot{x}^i_\delta \dot{x}^j_\delta + T\partial_{q^i}\Phi(x_\delta)\ell^{ij}\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)\Bigg[T\partial_{q^j}L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right) \\ - \partial^2_{q^k v^j}L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)\dot{x}^k_\delta + \chi'(\Phi(x_\delta))\partial_{q^j}\Phi(x_\delta)\Bigg] \leq 0,\end{aligned} \tag{29}$$

where we omitted the dependency on s_δ for the sake of presentation. Since we are on a compact subset of $\mathcal{M}$, there exists a constant $c_1 > 0$ such that

$$\partial^2{}_{q^i q^j}\Phi(x_\delta)\dot{x}^i_\delta \dot{x}^j_\delta \geq -c_1\|\dot{x}_\delta\|^2. \tag{30}$$

By (11), we have

$$\ell^{ij}(q, v)\xi_i\xi_j \geq \frac{1}{A_1}\|\xi\|^2, \qquad \forall q \in \mathcal{M}, \forall \xi \in T^*_q\mathcal{M}. \tag{31}$$

As a consequence, there exists a constant $c_2 > 0$ such that

$$\chi'_\delta(\Phi(x_\delta))\ell^{ij}\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)\partial_{q^i}\Phi(x_\delta)\partial_{q^j}\Phi(x_\delta) \geq \chi'_\delta(\Phi(x_\delta))c_2. \tag{32}$$

By (7) and (8), there exists a constant $c_3 > 0$ such that

$$\partial_{q^j}L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right) - \partial^2_{q^k v^j}L\left(x_\delta, \frac{\dot{x}_\delta}{T}\right)\frac{\dot{x}^k_\delta}{T} \geq -c_3\left(1 + \left\|\frac{\dot{x}_\delta}{T}\right\| + \left\|\frac{\dot{x}_\delta}{T}\right\|^2\right).$$

Then, using (31) and the compactness of $\overline{\Omega}_{\delta_0}$, there exists a constant $c_4 > 0$ such that

$$\partial_{q^i}\Phi(x_\delta)\ell^{ij}\left[\partial_{q^j}L\left(x_\delta,\frac{\dot{x}_\delta}{T}\right)-\partial^2_{q^k v^j}L\left(x_\delta,\frac{\dot{x}_\delta}{T}\right)\frac{\dot{x}^k_\delta}{T}\right]\geq -c_4\left(1+\left\|\frac{\dot{x}_\delta}{T}\right\|+\left\|\frac{\dot{x}_\delta}{T}\right\|^2\right). \tag{33}$$

Now, using (30), (32) and (33), from (29) we have

$$-c_1\|\dot{x}_\delta\|^2+Tc_2\chi'_\delta(\Phi(x_\delta))-T^2c_4\left(1+\left\|\frac{\dot{x}_\delta}{T}\right\|+\left\|\frac{\dot{x}_\delta}{T}\right\|^2\right)\geq 0.$$

Hence, there exists a constant $c > 0$ such that

$$\chi'_\delta(\Phi(x_\delta))\leq Tc\left(1+\left\|\frac{\dot{x}_\delta}{T}\right\|+\left\|\frac{\dot{x}_\delta}{T}\right\|^2\right). \tag{34}$$

By (10) and (24), we have

$$\begin{aligned}
&T\left(a_1\left\|\frac{\dot{x}_\delta}{T}\right\|^2-L(x_\delta,0)\right)-\chi_\delta(\Phi(x_\delta))\\
&\quad\leq T\left(d_vL\left(x_\delta,\frac{\dot{x}_\delta}{T}\right)\left[\frac{\dot{x}_\delta}{T}\right]-L\left(x_\delta,\frac{\dot{x}_\delta}{T}\right)\right)-\chi_\delta(\Phi(x_\delta))=E_\delta\leq k_2
\end{aligned}$$

and, consequently,

$$a_1T\left\|\frac{\dot{x}_\delta}{T}\right\|^2\leq k_2+\chi_\delta(\Phi(x_\delta))+TL(x_\delta,0).$$

Since $\overline{\Omega}_{\delta_0}$ is compact, $L(q,0)$ is a bounded function, so there exists a constant $c_5 > 0$ such that

$$T\left\|\frac{\dot{x}_\delta}{T}\right\|^2\leq c_5\Big(1+\chi_\delta(\Phi(x_\delta))\Big). \tag{35}$$

Since $\chi_\delta(\phi(q)) \geq 0$ for all $\delta \in (0,\delta_0)$ and $q \in \overline{\Omega}_{\delta_0}$, from (35) we also deduce that there exists a constant $c_6 > 0$ such that

$$T\left\|\frac{\dot{x}_\delta}{T}\right\|\leq c_6\Big(1+\chi_\delta(\Phi(x_\delta))\Big). \tag{36}$$

Finally, using (35) and (36), from (34) we infer that there exists a constant $C > 0$ such that (28) holds, and this ends the proof. □

Lemma 8. *Let $(x_n)_n$ be a sequence in $H^{1,2}([a,b],\mathcal{M})$ such that $x_n \to y \in H^{1,2}([a,b],\mathcal{M})$. Let $\xi \in T_y\mathcal{M}$ and set $\xi_n(s) = P_{x_n(s)}(\xi(s))$, where $P_q(\cdot)\colon \mathbb{R}^{2N} \to T_q\mathcal{M}$ is the orthogonal projection on $T_q\mathcal{M}$. Then ξ_n converges to ξ in $H^{1,2}([a,b],\mathbb{R}^{2N})$.*

Proof. For any $s \in [a,b]$, let (U,φ) be a local chart such that U is a neighbourhood of $y(s)$. If we denote by $(e^i)_{i=1,\ldots,N}$ the canonical basis of $\mathbb{R}^N$, then

$$e^i_n(s)=d(\varphi^{-1})(\varphi(x_n(s)))[e^i],\quad i=1,\ldots,N$$

is a basis for $T_{x_n(s)}\mathcal{M}$, if n is sufficiently large. Applying the Gram-Schmidt process to $(e^i_n(s))_{i\in 1,\ldots,N}$, we obtain an orthonormal basis $\hat{e}^i_n(s)$ for $T_{x_n(s)}\mathcal{M}$. Similarly, let us denote by $\hat{e}^i(s)$ the orthonormal basis of $T_{y(s)}\mathcal{M}$ obtained from $d(\varphi^{-1})(\varphi(y(s)))[e^i]$. With this notation, we can write

$$\xi_n(s)=\sum_{i=1}^N\langle\xi(s),\hat{e}^i_n(s)\rangle\hat{e}^i_n(s)\qquad\forall n. \tag{37}$$

Since x_n and y has $H^{1,2}$ regularity, $\hat{e}^i$ and $\hat{e}^i_n$ are in $H^{1,2}([a,b],\mathbb{R}^{2N})$ for all n. Moreover, since $x_n \to y$ in $H^{1,2}([a,b],\mathcal{M})$, we have that $\hat{e}^i_n$ converges to $\hat{e}^i$ in $H^{1,2}([a,b],\mathbb{R}^{2N})$. As a consequence, by (37) we obtain the thesis. □

Lemma 9. *Let $(x_\delta)_{\delta\in(0,\delta_0)}$ be a family in $H^{1,2}([a,b],\mathcal{M})$ such that for any $\delta \in (0,\delta_0)$, x_δ is a minimum of $\mathcal{J}_\delta$ on $\mathcal{C}([a,b],p,q,\Omega_\delta)$. Then there exists a subsequence $(\delta_n)_n$ in $(0,\delta_0)$ such that*

1. *$(x_{\delta_n})_n$ strongly converges to a curve $y \in \mathcal{C}([a,b],p,q,\overline{\Omega})$;*
2. *the sequence of functions $(\lambda_{\delta_n})_n$ weakly converges to a function $\lambda \in L^2([a,b],\mathbb{R})$;*
3. *the limit curve y satisfies*

$$T\left(d_qL\left(y,\frac{\dot y}{T}\right) - \frac{1}{T}\frac{d}{ds}d_vL\left(y,\frac{\dot y}{T}\right)\right) = \lambda\,\nabla\Phi(y) \quad a.e.; \tag{38}$$

and $y \in H^{2,2}([a,b],\mathcal{M})$;

4. *the limit curve y is a minimum of $\mathcal{J}$ on $\mathcal{C}([a,b],p,q,\overline{\Omega})$.*

Proof. (1) Since x_δ is a minimum of $\mathcal{L}_\delta$ for all $\delta \in (0,\delta_0)$, then there exists $k_1 > 0$ such that (23) holds. By Lemma 2, $\|\dot x_\delta\|_{L^2}$ is bounded and by the Ascoli-Arzelá theorem we obtain a decreasing sequence $(\delta_n)_n \subset (0,\delta_0)$ that converges to 0 such that x_{δ_n} uniformly converges to a curve y and $\dot x_{\delta_n}$ weakly converges to $\dot y$. By an argument analogous to that used in Lemma 5, x_{δ_n} strongly converges to y in $H^{1,2}([a,b],\mathcal{M})$. Since $x_{\delta_n}(s) \subset \Omega_{\delta_n}$ for all $n \in \mathbb{N}$ and $\delta_n \to 0$, the support of y is in $\overline{\Omega}$.

(2) Since $\delta_n \to 0$, we can assume that $\delta_n \in (0,\delta_1)$ and by Lemma 7, λ_{δ_n} is bounded in $L^\infty([a,b],\mathbb{R}) \subset L^2([a,b],\mathbb{R})$. Then, going if necessary to a subsequence, λ_{δ_n} weakly converges to a function $\lambda \in L^2([a,b],\mathbb{R})$.

(3) For any $\xi \in T_y\mathcal{M}$ such that $\xi(a) = \xi(b) = 0$, set $\xi_n = P_{x_{\delta_n}}(\xi) \in T_{x_{\delta_n}}\mathcal{M}$. By Lemma 8, ξ_n converges to ξ in $H^{1,2}([a,b],\mathbb{R}^{2N})$. Since x_{δ_n} is a minimum for $\mathcal{J}_{\delta_n}$ we obtain

$$\begin{aligned} d\mathcal{J}_{\delta_n}(x_{\delta_n})[\xi_n] = T\int_a^b \left(d_qL\left(x_{\delta_n},\frac{\dot x_{\delta_n}}{T}\right)[\xi_n] + d_vL\left(x_{\delta_n},\frac{\dot x_{\delta_n}}{T}\right)\left[\frac{\dot\xi_n}{T}\right]\right)ds \\ + \int_a^b \chi'_{\delta_n}(\Phi(x_{\delta_n}))d\Phi(x_{\delta_n})[\xi_n]ds = 0. \end{aligned}$$

Since $\lambda_{\delta_n} = -\chi'_{\delta_n}(x_{\delta_n})$ weakly converges to λ in $L^2([a,b],\mathbb{R})$, taking the limit in the above equation gives

$$T\int_a^b \left(d_qL\left(y,\frac{\dot y}{T}\right)[\xi] + d_vL\left(y,\frac{\dot y}{T}\right)\left[\frac{\dot\xi}{T}\right]\right)ds - \int_a^b \lambda\, d\Phi(y)[\xi]ds = 0. \tag{39}$$

Since $\xi \in H^{1,2}_0([a,b],\mathbb{R}^{2N})$, we obtain (38) by a partial integration. From (39), by a standard argument involving the implicit function theorem we obtain that $\ddot y$ has the same regularity of λ, so it is in $L^2([a,b],\mathbb{R}^{2N})$ and $y \in H^{2,2}([a,b],\mathcal{M})$.

(4) Recalling (13), by Lemma 7 there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \sup_{s\in[a,b]} |\chi_{\delta_n}(\Phi(\gamma_{\delta_n}(s)))| = \sup_{s\in[a,b]}\left|\chi'_{\delta_n}(\Phi(\gamma_{\delta_n}(s)))\frac{\Phi(\gamma_{\delta_n}(s))\,(\delta_n - \Phi(\gamma_{\delta_n}(s)))}{2\delta_n}\right| \\ \le k \sup_{t\in(0,\delta_n)}\left|\frac{t(\delta_n - t)}{2\delta_n}\right| = \frac{k}{8}\delta_n \to 0. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty}\int_a^b \chi_{\delta_n}(\Phi(x_{\delta_n}(s)))\,ds = 0,$$

and since $\mathcal{C}([a,b],p,q,\overline{\Omega}) \subset \mathcal{C}([a,b],p,q,\Omega_{\delta_n})$ for all δ_n, then

$$\mathcal{J}(y) = \lim_{n\to\infty} \mathcal{J}_{\delta_n}(x_{\delta_n}) \leq \lim_{n\to\infty} \mathcal{J}_{\delta_n}(x) = \mathcal{J}(x), \quad \text{for all } x \in \mathcal{C}([a,b],p,q,\overline{\Omega}).$$

□

Proof of Proposition 2. It suffices to prove the regularity of x when it touches the boundary $\partial\Omega$. Indeed, when x lies on Ω, it satisfies the Euler-Lagrange equations and it is C^2. Since the regularity is a local property, we can restrict our analysis on a single chart (U,φ) in a neighbourhood of a point $x(\bar{t}) \in \partial\Omega$. Let (a,b) be a neighbourhood of $\bar{t}$ such that $x([a,b]) \subset U$ and $x(a) \neq x(b)$. If $\bar{t} = 0$, then set $a = 0$ and, similarly, if $\bar{t} = 1$, then set $b = 1$. Please note that, for our purpose,s we can choose a and b as close as we desire.

Choosing T as in (x,T), for any $\delta \in (0,\delta_0)$, consider the functional $\mathcal{J}_\delta$ defined on $\mathcal{C}([a,b],x(a),x(b),\Omega_\delta)$. By Lemma 9, there exists a curve $y \in H^{2,2}([a,b],\mathcal{M})$ that is a minimum of $\mathcal{J}$ on $\mathcal{C}([a,b],x(a),x(b),\overline{\Omega})$. We shall prove that x has $H^{2,2}$ regularity by showing that $x = y$.

As a first step, let us show that if a and b are sufficiently close, then $y([a,b]) \subset U$. Looking for a contradiction, we assume that this is not true. Then, for every $\epsilon > 0$ there exists $\bar{s} \in (\bar{t}-\epsilon, \bar{t}+\epsilon)$ such that $y_\epsilon(\bar{s}) \notin U$, where

$$y_\epsilon \in \mathcal{C}([\bar{t}-\epsilon,\bar{t}+\epsilon],x(\bar{t}-\epsilon),x(\bar{t}+\epsilon),\overline{\Omega}) = \mathcal{C}_\epsilon$$

is the curve that minimizes $\mathcal{J}$ on $\mathcal{C}_\epsilon$. By the Cauchy-Schwarz inequality and (9) we have

$$\begin{aligned}\operatorname{dist}(x(a),\partial U) &\leq \int_{\bar{t}-\epsilon}^{\bar{s}} \|\dot{y}_\epsilon\| ds \leq \sqrt{2\epsilon}\left(\int_{\bar{t}-\epsilon}^{\bar{t}+\epsilon} \|\dot{y}_\epsilon\|^2 ds\right)^{1/2} \\ &\leq T\sqrt{2\epsilon}\left(\int_{\bar{t}-\epsilon}^{\bar{t}+\epsilon} \frac{1}{a_1}\left(L\left(y_\epsilon, \frac{\dot{y}_\epsilon}{T}\right) + b_1\right) ds\right)^{\frac{1}{2}} \leq 2T\sqrt{\frac{\epsilon}{a_1}\mathcal{J}(x)} + 2T\epsilon\sqrt{|b_1|},\end{aligned}$$

for every $\epsilon > 0$. As a consequence, $\operatorname{dist}(x(a),\partial U) = 0$, and this is absurd.

Now choose the map φ such that

$$d\varphi\left(\frac{\nabla\Phi(q)}{\|\nabla\Phi(q)\|}\right) \in \mathbb{R}^N$$

is constant on the chart. Then $\xi = y - x$ is an admissible variation of x in $\mathcal{C}([a,b],x(a),x(b),\overline{\Omega})$, since $\langle \xi(s), \nabla\Phi(x(s))\rangle \leq 0$ if $x(s) \in \partial\Omega$. Now define $f\colon [0,1] \to \mathbb{R}$ by

$$f(t) = \mathcal{J}(x + t\xi).$$

Since y is a minimum for $\mathcal{J}$ on $\mathcal{C}([a,b],x(a),x(b),\overline{\Omega})$, we have that

$$f(1) - f(0) = \mathcal{J}(y) - \mathcal{J}(x) \leq 0.$$

Setting $\xi(s) = 0$ for any $s \in [0,1]\backslash[a,b]$, we have that $\xi \in \mathcal{V}^-(x,\mathfrak{M})$. Since x is a $\mathcal{V}^-$-critical curve for $\mathcal{L}$ on $\mathfrak{M}$, we obtain

$$f'(0) = d\mathcal{J}(x)[\xi] = d_x\mathcal{L}_\kappa(x)[\xi] \geq 0.$$

Looking for a contradiction, we set $y \neq x$ and show that if a and b are sufficiently close, then

$$\int_0^1 (f'(t) - f'(0))dt > 0. \tag{40}$$

As a consequence,

$$0 \geq f(1) - f(0) = f'(0) + \int_0^1 (f'(t) - f'(0))dt \geq \int_0^1 (f'(t) - f'(0))dt > 0,$$

which is an absurd. By definition of f, we have

$$\begin{aligned} f'(t) - f'(0) &= d\mathcal{J}(x + t\xi)[\xi] - d\mathcal{J}(x)[\xi] \\ &= T \int_a^b \left[\left(d_q L \left(x + t\xi, \frac{\dot{x} + t\dot{\xi}}{T} \right) - d_q L \left(x, \frac{\dot{x}}{T} \right) \right) [\xi] + \right. \\ &\qquad \left. \left(d_v L \left(x + t\xi, \frac{\dot{x} + t\dot{\xi}}{T} \right) - d_v L \left(x, \frac{\dot{x}}{T} \right) \right) \left[\frac{\dot{\xi}}{T} \right] \right] ds. \end{aligned} \tag{41}$$

By the mean value theorem and (8), we have

$$\begin{aligned} &\left(d_q L \left(x + t\xi, \frac{\dot{x} + t\dot{\xi}}{T} \right) - d_q L \left(x, \frac{\dot{x}}{T} \right) \right) [\xi] \\ &= t \int_0^1 \left(d_{qq} L \left(x + t\xi, \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right) [\xi, \xi] + d_{qq} L \left(x + t\xi, \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right) \left[\xi, \frac{\dot{\xi}}{T} \right] \right) d\sigma \\ &\geq -a_0 t \int_0^1 \left[\left(1 + \left\| \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right\|^2 \right) \|\xi\|^2 + \frac{1}{T} \left(1 + \left\| \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right\| \right) \|\xi\| \|\dot{\xi}\| \right] d\sigma. \end{aligned} \tag{42}$$

Similarly, using also (11), we have

$$\begin{aligned} &\left(d_v L \left(x + t\xi, \frac{\dot{x} + t\dot{\xi}}{T} \right) - d_v L \left(x, \frac{\dot{x}}{T} \right) \right) \left[\frac{\dot{\xi}}{T} \right] \\ &\geq t \int_0^1 \left[a_1 \frac{\|\dot{\xi}\|^2}{T^2} - \frac{a_0}{T} \left(1 + \left\| \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right\| \right) \|\xi\| \|\dot{\xi}\| \right] d\sigma. \end{aligned} \tag{43}$$

Hence, by (42) and (43), from (41) we obtain

$$\begin{aligned} f'(t) - f'(0) \geq Tt \int_a^b ds \int_0^1 &\left[a_1 \frac{\|\dot{\xi}\|^2}{T^2} - \frac{2a_0}{T} \left(1 + \left\| \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right\| \right) \|\xi\| \|\dot{\xi}\| \right. \\ &\qquad \left. - a_0 \left(1 + \left\| \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right\|^2 \right) \|\xi\|^2 \right] d\sigma. \end{aligned} \tag{44}$$

Let us show that there exists a constant $c_1 > 0$, which depends only on x, such that

$$\int_a^b \left\| \frac{\dot{x} + \sigma t\dot{\xi}}{T} \right\|^2 ds \leq c_1.$$

Indeed, by (9) and since y is a minimum for $\mathcal{L}_T^{a,b}$, we have

$$\begin{aligned}
\int_a^b \left\| \frac{\dot{x}+\sigma t\dot{\xi}}{T} \right\|^2 &\leq 2\int_a^b \left(\left\| \frac{\dot{x}}{T} \right\|^2 + \left\| \frac{\dot{y}-\dot{x}}{T} \right\|^2 \right) ds \\
&\leq \frac{2}{T^2} \int_a^b \left(\|\dot{x}\|^2 + 2(\|\dot{y}\|^2 + \|\dot{x}\|^2) \right) ds \\
&\leq \frac{6}{T^2} \int_a^b \|\dot{x}\|^2 + \frac{4}{a_1} \int_a^b \left(L\left(y, \frac{\dot{y}}{T}\right) + b_1 \right) ds \\
&\leq \frac{6}{T^2} \int_a^b \|\dot{x}\|^2 + \frac{4}{a_1} \int_a^b \left(L\left(x, \frac{\dot{x}}{T}\right) + b_1 \right) ds = c_1 < +\infty.
\end{aligned}$$

As a consequence, there exists a strictly positive constant c_2 such that

$$\begin{aligned}
I_1 &= \frac{2a_0}{T} \int_a^b \left(\int_0^1 \left(1 + \left\| \frac{\dot{x}+\sigma t\dot{\xi}}{T} \right\| \right) \|\xi\| \|\dot{\xi}\| d\sigma \right) ds \\
&= \frac{2a_0}{T} \int_a^b \|\xi\| \|\dot{\xi}\| ds + \frac{2a_0}{T} \int_0^1 \left(\int_a^b \left\| \frac{\dot{x}+\sigma t\dot{\xi}}{T} \right\| \|\xi\| \|\dot{\xi}\| ds \right) d\sigma \\
&\leq \frac{2a_0}{T} \|\xi\|_{L^\infty} \|\dot{\xi}\|_{L^2} + \frac{2a_0}{T} \|\xi\|_{L^\infty} \|\dot{\xi}\|_{L^2} \int_0^1 \left(\int_a^b \left\| \frac{\dot{x}+\sigma t\dot{\xi}}{T} \right\|^2 ds \right)^{1/2} d\sigma \leq c_2 \|\xi\|_{L^\infty} \|\dot{\xi}\|_{L^2},
\end{aligned}$$

where we applied the Tonelli's theorem and the Hölder inequality. Similarly, there exists a constant $c_3 > 0$ such that

$$\begin{aligned}
I_2 &= a_0 \int_a^b \left(\int_0^1 \left(1 + \left\| \frac{\dot{x}+\sigma t\dot{\xi}}{T} \right\|^2 \right) \|\xi\|^2 d\sigma \right) ds \\
&\leq a_0 \|\xi\|_{L^\infty}^2 \int_0^1 \left(\int_a^b \left(1 + \left\| \frac{\dot{x}+\sigma t\dot{\xi}}{T} \right\|^2 \right) ds \right) d\sigma \leq c_3 \|\xi\|_{L^\infty}^2.
\end{aligned}$$

Then, by (44) we obtain

$$f'(t) - f'(0) \geq T\,t \left(\frac{a_1}{T^2} \|\dot{\xi}\|_{L^2}^2 - c_2 \|\xi\|_{L^\infty} \|\dot{\xi}\|_{L^2} - c_3 \|\xi\|_{L^\infty}^2 \right).$$

Since $\xi(a) = 0$, we have

$$\|\xi(s)\| = \left\| \xi(a) + \int_a^s \dot{\xi}(\sigma) d\sigma \right\| \leq \int_a^s \|\dot{\xi}(\sigma)\| d\sigma \leq \sqrt{s-a}\, \|\dot{\xi}\|_{L^2},$$

therefore

$$\|\xi\|_{L^\infty} \leq \sqrt{b-a}\, \|\dot{\xi}\|_{L^2}.$$

We have

$$f'(t) - f'(0) \geq T\,t \left(\frac{a_1}{T^2} - c_2\sqrt{b-a} - c_3(b-a) \right) \|\dot{\xi}\|_{L^2}^2.$$

As a consequence, if $b - a$ is sufficiently small, there exists a constant $c_4 > 0$ such that

$$f'(t) - f'(0) \geq c_4 t \|\dot{\xi}\|_{L^2}^2.$$

If $y \neq x$, then $\|\dot{\xi}\|_{L^2}^2 > 0$ and, consequently, (40) holds and this leads to a contradiction. □

To state our next result we need the following definition.

Definition 5. *We define the Hessian of Φ with respect to L in $(q,v) \in T\mathcal{M}$ by*

$$H^L_\Phi(q,v)[v,v] = \frac{d^2}{ds^2}\Big|_{s=0} \Phi(\gamma(s)),$$

where $\gamma\colon]-\epsilon,\epsilon[\to \mathcal{M}$ is the unique solution of the Euler-Lagrange equation associated with L such that $\gamma(0) = q$ and $\dot\gamma(0) = v$. In local coordinates the Hessian of Φ with respect to L in $(q,v) \in T\mathcal{M}$ is given by

$$H^L_\Phi(q,v)[v,v] = \partial^2_{q^i q^j}\Phi(q)v^i v^j + \partial_{q^i}\Phi(q)\ell^{ij}(q,v)\left(\partial_{q^j}L(q,v) - \partial^2_{q^k v^j}L(q,v)v^k\right), \tag{45}$$

where $\ell^{ij}(q,v)$ denote the components of the matrix $\left(d^2_{vv}L(q,v)\right)^{-1}$.

Corollary 1. *Let (x,T) be $\mathcal{V}^-$-critical for $\mathcal{L}_\kappa$ on $\mathfrak{M}$. Then $x \in H^{2,\infty}([0,1],\overline{\Omega})$ regularity. Moreover, setting $C_x = \{s \in [0,1] : x(s) \in \partial\Omega\}$ there exists a function $\lambda \in L^\infty([0,1],\mathbb{R})$ such that*

i. $\lambda \le 0$ a.e. in $[0,1]$, $\lambda(s) = 0$ if $s \notin C_x$ and

$$\lambda = T\frac{H^L_\Phi\left(x,\frac{\dot x}{T}\right)\left[\frac{\dot x}{T},\frac{\dot x}{T}\right]}{\ell^{ij}\left(x,\frac{\dot x}{T}\right)\partial_{q^i}\Phi(x)\partial_{q^j}\Phi(x)} \qquad \text{a.e. in } C_x; \tag{46}$$

ii. x satisfies the following equation

$$T\left(d_q L\left(x,\frac{\dot x}{T}\right) - \frac{1}{T}\frac{d}{ds}d_v L\left(x,\frac{\dot x}{T}\right)\right) = \lambda\nabla\Phi(x) \qquad \text{a.e.}. \tag{47}$$

Moreover, (x,T) satisfies the conservation law

$$E\left(x(s),\frac{\dot x(s)}{T}\right) = \kappa, \qquad \forall s \in [0,1]. \tag{48}$$

Proof. By the proof of Proposition 2, in every chart the curve x coincides with the curve y obtain from Lemma 9. Then x satisfies (47) a.e., where $\lambda \in L^2([0,1],\mathbb{R})$ is the limit of functions defined in (26). Consequently, $\lambda(s) = 0$ for all $s \notin C_x$ and $\lambda \le 0$ a.e.. Set $\rho(s) = \Phi(x(s))$. Since $\rho(s) = 0$ on C_x and $\dot\rho$ is a $H^{1,2}$ function, by ([25] Lemma 7.7) we have

$$\ddot\rho(s) = \partial^2_{q^i q^j}\Phi(x)\dot x^i \dot x^j + \partial_{q^j}\Phi(x)\ddot x^i = 0, \quad \text{a.e. on } C_x. \tag{49}$$

Using (38) in local coordinates, we get that x satisfies the equations

$$\frac{\ddot x^i}{T^2} = \ell^{ij}\left(x,\frac{\dot x}{T}\right)\left(\partial_{q^j}L\left(x,\frac{\dot x}{T}\right) - \partial_{q^k v^j}L\left(x,\frac{\dot x}{T}\right)\frac{\dot x^k}{T} - \frac{1}{T}\lambda\,\partial_{q^j}\Phi(x)\right). \tag{50}$$

Substituting the expression of $\ddot x^i$ of (50) in (49) and using (45), we obtain (46). By (46) and since $\dot x$ is a continuous function, also λ is a continuous function in $[0,1]$. Then $\lambda \in L^\infty([0,1],\mathbb{R})$ and using (50) we obtain that $\ddot x \in L^\infty([0,1],\mathbb{R}^N)$.

In order to prove (48), we can contract both terms of (47) with $\dot x/T$. Since $\lambda = 0$ on $[0,1]\backslash C_x$ and $\langle\nabla\Phi(x),\dot x,=\rangle 0$ on C_x, we have

$$\frac{d}{ds}E\left(x,\frac{\dot x}{T}\right) = T\left(d_q L\left(x,\frac{\dot x}{T}\right) - \frac{1}{T}\frac{d}{ds}d_v L\left(x,\frac{\dot x}{T}\right)\right)\left[\frac{\dot x}{T}\right] = 0, \quad \text{on } [0,1],$$

thus $E(x,\dot x/T)$ is a constant. Since $\frac{\partial\mathcal{L}_\kappa}{\partial T}(x,T) = 0$, we obtain (48). □

Lemma 10. *If (x,T) is a $\mathcal{V}^-$-critical curve for $\mathcal{L}_\kappa$ on $\mathfrak{M}\times]0,+\infty[$, then it satisfies the conormal boundary conditions (2).*

Proof. Take any vector field $\xi \in \mathcal{V}^-(x,\mathfrak{M})$ such that

$$\langle \nabla\Phi(x(s)), \xi(s)\rangle = 0, \text{ for all } s \text{ such that } \Phi(x(s)) = 0.$$

In this case, also $-\xi$ is in $\mathcal{V}^-(x,\mathfrak{M})$ and, by the $\mathcal{V}^-$-critical assumption on x, we obtain $d_x\mathcal{L}_\kappa(x,T)[\xi] = 0$. Integrating by parts and using (47), we have

$$d_x\mathcal{L}_\kappa(x,T)[\xi] = \left[\frac{1}{T}d_vL\left(x,\frac{\dot{x}}{T}\right)[\xi]\right]_0^1 + \int_0^1 \left(d_qL\left(x,\frac{\dot{x}}{T}\right) - \frac{1}{T}\frac{d}{ds}d_vL\left(x,\frac{\dot{x}}{T}\right)\right)[\xi]ds = 0,$$

then

$$d_vL\left(x(1),\frac{\dot{x}(1)}{T}\right)[\xi(1)] - d_vL\left(x(0),\frac{\dot{x}(0)}{T}\right)[\xi(0)] = 0.$$

Since $\xi(0)$ and $\xi(1)$ are arbitrary tangent vectors to $\partial\Omega$, then (x,T) satisfies the conormal boundary conditions. $\square$

Proposition 3. *For every $\kappa > m(L)$, one and only one of the following statements holds:*

i. *there exists at least one ELCTC with energy κ*

or

ii. *every $\mathcal{V}^-$-critical curve for $\mathcal{L}_\kappa$ on $\mathfrak{M}\times]0,+\infty[$ is an ELCC.*

Proof. Let (x,T) be a $\mathcal{V}^-$-critical curve for $\mathcal{L}_\kappa$ on $\mathfrak{M}\times]0,+\infty[$. Since $\kappa > m(L) \geq e_0(L)$, by (48) we infer that $\dot{x}(s) \neq 0$ for all $s \in [0,1]$. Since $x([0,1]) \in \overline{\Omega}$, $\dot{x}(0)$ cannot point outside Ω. By Lemma 10, (x,T) satisfies the conormal boundary conditions (2). Hence, if $\dot{x}(0) \in T_{x(0)}\partial\Omega$, then by (2) we have

$$\kappa = E\left(x(0),\frac{\dot{x}(0)}{T}\right) = d_vL\left(x(0),\frac{\dot{x}(0)}{T}\right)\left[\frac{\dot{x}(0)}{T}\right] - L\left(x(0),\frac{\dot{x}(0)}{T}\right) = -L\left(x(0),\frac{\dot{x}(0)}{T}\right) < \kappa,$$

which is absurd. Thus, $\dot{x}(0)$ points inside Ω. Let us suppose that $C_x = \{s \in]0,1[: x(s) \in \partial\Omega\} \neq \emptyset$ and set $s_0 = \min(C_x)$, namely the first positive time at which $x(s) \in \partial\Omega$. By Proposition 2, x is of class C^1, then $\dot{x}(s_0)$ must be tangent to $\partial\Omega$. Then $x(T\cdot)|_{[0,s_0]}$ is an ELCTC. Otherwise, if $C_x = \emptyset$, (x,T) is an ELCC. $\square$

4. The Functional $\mathcal{F}_\kappa$

In this section, we prove that if $\kappa > e_0(L)$, then we can restrict our analysis on a fixed-time variational problem, since for every x which is not constant there exists one and only one $T(x) \in]0,+\infty[$ such that $\frac{\partial\mathcal{L}_\kappa}{\partial T}(x,T(x)) = 0$. This result will simplify the construction of a descent vector field that will allow using our minimax approach to prove the existence of $\mathcal{V}^-$-critical curves (cf. [11]).

Let us denote by $\mathfrak{C}_0$ the subset of $\mathfrak{M}$ that are constant curves in $\partial\Omega$, thus

$$\mathfrak{C}_0 = \left\{x \in \mathfrak{M} : \exists q \in \overline{\Omega}\,\text{s.t.}\, x(s) = q\; \forall s \in [0,1]\right\}.$$

We remark that if $x \in \mathfrak{M}\backslash\mathfrak{C}_0$, then $\|\dot{x}\|_{L^2} > 0$.

Proposition 4. *Set $\kappa > e_0(L)$, where $e_0(L)$ is defined in (12). Then for every $x \in \mathfrak{M}\backslash\mathfrak{C}_0$ there exists an unique $T(x) > 0$ such that*

$$\frac{\partial\mathcal{L}_\kappa}{\partial T}(x,T(x)) = 0. \tag{51}$$

Moreover,

$$\mathcal{L}_\kappa(x,(T(x)) \leq \mathcal{L}_\kappa(x,T), \qquad \forall T \in]0,+\infty[. \tag{52}$$

Proof. In order to prove the existence and uniqueness of a $T(x) > 0$ such that (51) holds, we are going to prove that for every $x \in \mathfrak{M}\backslash\mathfrak{C}_0$ the function

$$f\colon]0,+\infty[\to \mathbb{R}, \qquad f(T) = \frac{\partial \mathcal{L}_\kappa}{\partial T}(x,T) = \int_0^1 \left(\kappa - E\left(x, \frac{\dot{x}}{T}\right)\right) ds$$

is strictly increasing,

$$\lim_{T\to 0^+} f(T) = \lim_{T\to 0^+} \frac{\partial \mathcal{L}_\kappa}{\partial T}(x,T) = -\infty, \tag{53}$$

and

$$\lim_{T\to +\infty} f(T) = \lim_{T\to +\infty} \frac{\partial \mathcal{L}_\kappa}{\partial T}(x,T) > 0.$$

The above properties prove also (52).

The following inequality shows that f is strictly increasing

$$f'(T) = \frac{\partial^2 \mathcal{L}_\kappa}{\partial T^2}(x,T) = \frac{1}{T}\int_0^1 d^2_{vv}L\left(x, \frac{\dot{x}}{T}\right)\left[\frac{\dot{x}}{T}, \frac{\dot{x}}{T}\right] ds \geq \frac{\alpha}{T^3}\int_0^1 \|\dot{x}\|^2 ds > 0.$$

By (10) we have

$$f(T) \leq \int_0^1 \left(L(x,0) + \kappa - \frac{a_1}{2T^2}\|\dot{x}^2\|\right) ds = \int_0^1 (L(x,0)+\kappa)\, ds - \frac{a_1}{2T^2}\|\dot{x}\|^2_{L^2}.$$

Consequently, (53) holds. Now let $(T_n)_n$ be a sequence such that $T_n \to \infty$. As a consequence,

$$\frac{\dot{x}}{T_n} \to 0 \quad a.e. \quad \text{and} \quad \left\|\frac{\dot{x}}{T_n}\right\| \leq \|\dot{x}\| \in L^2([0,1],\mathbb{R}).$$

By definition of $e_0(L)$, we have that if $\kappa > e_0(L)$, then $L(q,0) + \kappa > 0, \forall q \in \overline{\Omega}$.

By (7), we can apply the dominated convergence theorem and we obtain

$$\lim_{n\to\infty} f(T_n) = \int_0^1 (L(q,0)+\kappa)\, ds > 0,$$

and this ends the proof. □

Lemma 11. *There exist two constant $c_1, c_2 > 0$ such that for all $x \in \mathfrak{M}\backslash\mathfrak{C}_0$ we have*

$$\frac{c_2}{A_1} \leq \frac{\|\dot{x}\|^2_{L^2}}{T^2(x)} \leq \frac{c_1}{a_1}. \tag{54}$$

As a consequence, for every sequence $(x_n)_n \subset \mathfrak{M}\backslash\mathfrak{C}_0$, we have that $T(x_n) \to 0$ if and only if $\|\dot{x}_n\|^2_{L^2} \to 0$.

Proof. By definition of $T(x)$ and by (10) we have

$$0 = \kappa - \int_0^1 E\left(x, \frac{\dot{x}}{T}\right) ds \leq \kappa - \int_0^1 \left(a_1 \frac{\|\dot{x}\|^2}{T^2(x)} + L(x,0)\right) ds \leq -a_1 \frac{\|\dot{x}\|^2_{L^2}}{T^2(x)} + c_1,$$

from which the right-hand side inequality of (54) follows at once. Similarly we have

$$0 = \kappa - \int_0^1 E\left(x, \frac{\dot{x}}{T}\right) ds \geq \kappa - \int_0^1 \left(A_1 \frac{\|\dot{x}\|^2}{T^2(x)} + L(x,0)\right) ds \geq -A_1 \frac{\|\dot{x}\|^2_{L^2}}{T^2(x)} + c_2,$$

and we get the left-hand side inequality of (54). We remark that, since $\kappa > e_0(L) = -\min_{q\in\overline{\Omega}} L(q,0)$, c_2 can be chosen strictly positive. □

Lemma 12. *Let $(x_n) \in \mathfrak{M}\backslash\mathfrak{C}_0$ such that $x_n \to x_0 \in \mathfrak{C}_0$ with respect to the $H^{1,2}$-convergence. Then $\mathcal{F}_\kappa(x_n) \to 0$.*

Proof. By Lemma 11 we have $T(x_n) \to 0$. Moreover, by (9) and (54) we obtain

$$\mathcal{F}_\kappa(x_n) \geq T(x_n) \int_0^1 \left(a_1 \frac{\|\dot{x}_n\|^2}{T^2(x)} - b_1 + \kappa \right) ds \geq T(x_n) \left(\frac{a_1}{A_1} c_2 + \kappa - b_1 \right) \to 0.$$

Similarly,

$$\mathcal{F}_\kappa(x_n) \leq T(x_n) \int_0^1 \left(A_1 \frac{\|\dot{x}_n\|^2}{T^2(x)} + B_1 + \kappa \right) ds \leq T(x_n) \left(\frac{A_1}{a_1} c_1 + \kappa + B_1 \right) \to 0.$$

□

By Proposition 4 and Lemma 12, we can define the continuous functional $\mathcal{F}\colon \mathfrak{M} \to \mathbb{R}^+$ by

$$\mathcal{F}_\kappa(x) = \begin{cases} \mathcal{L}_\kappa(x, T(x)), & \text{if } x \notin \mathfrak{C}_0, \\ 0, & \text{if } x \in \mathfrak{C}_0. \end{cases}$$

Moreover, $\mathcal{F}_\kappa$ is a C^1 functional on $\mathfrak{M}\backslash\mathfrak{C}_0$, and its differential is

$$d\mathcal{F}_\kappa(x)[\xi] = d_x\mathcal{L}_\kappa(x, T(x))[\xi], \qquad \forall x \in \mathfrak{M}\backslash\mathfrak{C}_0,\ \forall \in \xi \in \mathcal{V}^-(x, \mathfrak{M}). \tag{55}$$

Accordingly to the definition of $\mathcal{V}^-$-critical curves for the functional $\mathcal{L}_\kappa$, we give the following definition.

Definition 6. *We say that a curve $x \in \mathfrak{M}\backslash\mathfrak{C}_0$ is a $\mathcal{V}^-$-critical curve for $\mathcal{F}_\kappa$ on $\mathfrak{M}\backslash\mathfrak{C}_0$ if*

$$d\mathcal{F}_\kappa(x)[\xi] \geq 0, \qquad \forall \xi \in \mathcal{V}^-(x, \mathfrak{M}).$$

A number $c > 0$ is a $\mathcal{V}^-$-critical value for $\mathcal{F}_\kappa$ if there exists $x \in \mathfrak{M}$ that is a $\mathcal{V}^-$-critical curve for $\mathcal{F}_\kappa$ on $\mathfrak{M}\backslash\mathfrak{C}_0$ such that $\mathcal{J}(x) = c$. Otherwise, c is said $\mathcal{V}^-$-regular value for $\mathcal{F}_\kappa$ on $\mathfrak{M}$.

From (55) and the definition of $T(x)$ we can infer the following result.

Proposition 5. *A curve x is $\mathcal{V}^-$-critical for $\mathcal{F}_\kappa$ on $\mathfrak{M}\backslash\mathfrak{C}_0$ if and only if $(x, T(x))$ is a $\mathcal{V}^-$-critical for $\mathcal{L}_\kappa$ on $\mathfrak{M}$.*

As a consequence, exploiting also Proposition 3, we can find ELCCs by looking for the $\mathcal{V}^-$-critical curves of $\mathcal{F}_\kappa$ on $\mathfrak{M}\backslash\mathfrak{C}_0$.

5. $\mathcal{V}^-$-Palais-Smale Condition

Let $\|\cdot\|_*\colon H^{1,2}([0,1], \mathbb{R}^{2N}) \to \mathbb{R}$ be the norm given by

$$\|\xi\|_* = \max\{\|\xi(0)\|, \|\xi(1)\|\} + \left(\int_0^1 \|\dot{\xi}(s)\|^2 ds \right)^{\frac{1}{2}}.$$

Definition 7. *A sequence $(x_n)_n \subset \mathfrak{M}\backslash\mathfrak{C}_0$ is said $\mathcal{V}^-$-Palais-Smale sequence for $\mathcal{F}_\kappa$ at level $c \in \mathbb{R}$ if*

i. $\mathcal{F}_\kappa(x_n) \to c$;

ii. *for all (sufficiently large) $n \in \mathbb{N}$ and for all $\xi_n \in \mathcal{V}^-(x_n, \mathfrak{M})$ such that $\|\xi_n\|_* = 1$,*

$$d\mathcal{F}_\kappa(x_n)[\xi_n] \geq -\epsilon_n,$$

where $\epsilon_n \to 0^+$.

In this section, we shall prove the following result.

Proposition 6. *If $\kappa > c(L)$, then any $\mathcal{V}^-$-Palais-Smale sequence x_n for $\mathcal{F}_\kappa$ at level $c \neq 0$ admits a strongly convergent subsequence.*

Lemma 13. *Let $(x_n)_n \subset \mathfrak{M} \backslash \mathfrak{C}_0$ be a $\mathcal{V}^-$-Palais-Smale sequence for $\mathcal{F}_\kappa$ at level $c \in \mathbb{R}$ with $0 < T_* \leq T(x_n) \leq T^* < +\infty$. Then $(x_n)_n$ admits a strongly convergent subsequence.*

Proof. See ([11] Proposition 4.3). □

Lemma 14. *Let $(x_n)_n \subset \mathfrak{M} \backslash \mathfrak{C}_0$ be a $\mathcal{V}^-$-Palais-Smale sequence for $\mathcal{F}_\kappa$ at level $c \in \mathbb{R}$. If $T(x_n) \to 0$, then $c = 0$.*

Proof. It is an immediate consequence of Lemma 11. Indeed, for all $(x_n) \in \mathfrak{M} \backslash \mathfrak{C}_0$ we have

$$\mathcal{F}_\kappa(x_n) \geq T(x_n) \left(\frac{a_1}{A_1} c_2 + \kappa - b_1 \right) \to 0$$

and

$$\mathcal{F}_\kappa(x_n) \leq T(x_n) \left(\frac{A_1}{a_1} c_1 + \kappa + B_1 \right) \to 0.$$

□

Lemma 15. *If $\kappa \geq c(L)$, then $\mathcal{F}_\kappa$ is bounded from below.*

Proof. By definition of $\mathcal{F}_\kappa$, it suffices to prove that $\mathcal{L}_\kappa$ is bounded from below.
For any $(p, q) \in \partial\Omega \times \partial\Omega$, we can choose a curve $x_{p,q} \in \mathfrak{M}$ such that $x(0) = p$ and $x(1) = q$. Then we define $f : \partial\Omega \times \partial\Omega \to \mathbb{R}$ by

$$f(p, q) = \mathcal{L}_\kappa(x_{p,q}, 1).$$

Since $\partial\Omega \times \partial\Omega$ is compact, there exists a constant C such that $f(p, q) \leq C$, for all $p, q \in \partial\Omega$. Now let (x, T) be an element in $\mathfrak{M} \times]0, +\infty[$. Then the curve

$$y = \begin{cases} x\left(\frac{T+1}{T} s\right), & \text{if } s \in [0, T/(T+1)], \\ x_{x(1),x(0)}((T+1)s - T), & \text{if } s \in [T/(T+1), 1], \end{cases}$$

is a closed curve. By definition of the Mañé critical value $c(L)$, since $\kappa \geq c(L)$ we have $\mathcal{L}_\kappa(y, T+1) \geq 0$. As a consequence

$$0 \leq \mathcal{L}_\kappa(y, T+1) = \mathcal{L}_\kappa(x, T) + \mathcal{L}_\kappa(x_{x(1),x(0)}, 1) \leq \mathcal{L}_\kappa(x, T) + C,$$

so $\mathcal{L}_\kappa(x, T) \geq -C$. By the arbitrariness of (x, T), we have the thesis. □

Lemma 16. *If $\kappa > c(L)$, then for any $\mathcal{V}^-$-Palais Smale sequence $(x_n)_n \subset \mathfrak{M} \backslash \mathfrak{C}_0$ there exists $T^* > 0$ such that $T(x_n) < T^*$ for all $n \in \mathbb{N}$.*

Proof. Let $c \in \mathbb{R}$ such that $\mathcal{F}_\kappa(x_n) \to c$. Then we have

$$c + 1 \geq \mathcal{F}_\kappa(x_n) = \mathcal{F}_{c(L)}(x_n) + (\kappa - c(L))T(x_n).$$

Hence

$$T(x_n) \leq \frac{1}{\kappa - c(L)} \left(c + 1 - \mathcal{F}_{c(L)}(x_n) \right).$$

By Lemma 15, $\mathcal{F}_{c(L)}$ is bounded from below, and the thesis follows at once. □

Proof of Proposition 6. By Lemma 16, there exists T^* such that $T(x_n) < T^*$ for all $n \in \mathbb{N}$. On the other hand, since $c \neq 0$, by Lemma 14 there must exists T_* such that $T(x_n) > T_*$. Hence, Lemma 13 applies and there exists a subsequence of (x_n) that strongly converges. □

6. Proof of the Main Theorem

In the following we assume that $\kappa > m(L) \geq c(L) \geq e_0(L)$. Hence, all the previous results are available and

$$\mathcal{F}_\kappa(x) \geq 0,\ \forall x \in \mathfrak{M}, \qquad \mathcal{F}_\kappa(x) = 0 \iff x \in \mathfrak{C}_0.$$

Moreover, for every $c > 0$, we denote the sublevel of $\mathcal{F}_\kappa$ at c by

$$\mathcal{F}_\kappa^c = \{x \in \mathfrak{M} : \mathcal{F}_\kappa(x) \leq c\}.$$

Definition 8. *Let $x \in \mathfrak{M} \backslash \mathfrak{C}_0$ and $\mu > 0$ be fixed. We say that $\mathcal{F}_\kappa$ has $\mathcal{V}^-$-steepness greater than or equal to μ if there exists $\xi \in \mathcal{V}^-(x, \mathfrak{M})$ such that $\|\xi\|_* = 1$ and $d\mathcal{F}_\kappa(x)[\xi] \leq -\mu$.*

Lemma 17. *Let $M > m > 0$ be such that $C = \mathcal{F}_\kappa^{-1}([m, M])$ does not contain any $\mathcal{V}^-$-critical curves for $\mathcal{F}_\kappa$. Then there exists μ_C such that every $x \in C$ has $\mathcal{V}^-$-steepness greater than or equal to μ_C.*

Proof. This is a consequence of Proposition 6. Indeed, if such μ_C does not exist, we can choose a $\epsilon_n \to 0^+$ and a sequence $(x_n) \in C$ such that

$$d\mathcal{F}_\kappa(x_n)[\xi_n] \geq -\epsilon_n, \qquad \forall \xi_n \in \mathcal{V}^-(x_n, \mathfrak{M}),\ \|\xi_n\|_* = 1.$$

Since $\mathcal{F}_\kappa(x_n) \subset [m, M]$, $\mathcal{F}_\kappa(x_n) \to c \neq 0$, going if necessary to a subsequence. Hence, $(x_n) \subset \mathfrak{M} \backslash \mathfrak{C}_0$ is a $\mathcal{V}^-$-Palais-Smale sequence and, by Proposition 6, there is a $\mathcal{V}^-$-critical curve in C, that is absurd. □

Thanks to the previous lemma, we can construct a pseudogradient vector field for $\mathcal{F}_\kappa$ (cf. [10,11]), that is the key ingredient to prove two deformation lemmas on the sublevels of $\mathcal{F}_\kappa$ which are necessary for our minimax approach. We need some preliminary definitions to state these lemmas.

We define the backward parametrization map $\mathcal{R} \colon \mathfrak{M} \to \mathfrak{M}$ by

$$(\mathcal{R}x)(s) = x(1 - s), \qquad \forall s \in [0, 1].$$

We say that $\mathcal{N} \subset \mathfrak{M}$ is $\mathcal{R}$-invariant if $\mathcal{R}(\mathcal{N}) = \mathcal{N}$. On any $\mathcal{R}$-invariant set $\mathcal{N}$, the backward parametrization map $\mathcal{R}$ induces an equivalence relation and we denote by $\widetilde{\mathcal{N}}$ the quotient space. Through this equivalence relation, we identify any element x of $\mathfrak{M}$ with its backward parametrization. The map $\mathcal{R}$ induces a map $\overline{\mathcal{R}} \colon \mathcal{V}^-(x, \mathfrak{M}) \to \mathcal{V}^-(\mathcal{R}x, \mathfrak{M})$ defined by

$$(\overline{\mathcal{R}}\xi)(s) = \xi(1 - s).$$

Remark 4. *If L is reversible, then x is a $\mathcal{V}^-$-critical curve for $\mathcal{F}_\kappa$ if and only if $\mathcal{R}x$ is a $\mathcal{V}^-$-critical curve for $\mathcal{F}_\kappa$.*

Definition 9. *Let $\mathcal{N}$ be a subset of $\mathfrak{M}$. Then a continuous function $h\colon [0,1]\times\mathcal{N}\to\mathcal{N}$ is said admissible homotopy if*

(i) $h(0,x)=x$ *for all* $x\in\mathcal{N}$;
(ii) $h(\tau,x)\in\mathfrak{C}_0$ *for all* $x\in\mathcal{N}\cap\mathfrak{C}_0$ *and* $\tau\in[0,1]$;
(iii) *if* $x\notin\mathcal{N}\cap\mathfrak{C}_0$, *then* $h(\tau,x)\notin\mathcal{N}\cap\mathfrak{C}_0$, *for all* $\tau\in[0,1]$;
(iv) *if L is a reversible, $\mathcal{N}$ must be $\mathcal{R}$-invariant and $h(\tau,\mathcal{R}x)=\mathcal{R}h(\tau,x)$ for every $\tau\in[0,1]$.*

We are ready to state the first following lemma.

Lemma 18. *Let $c>0$ be a $\mathcal{V}^-$-regular value for $\mathcal{F}_\kappa$ on $\mathfrak{M}\backslash\mathfrak{C}_0$. Then there exists an $\epsilon>0$ and an admissible homotopy $h\colon[0,1]\times\mathcal{F}_\kappa^{c+\epsilon}\to\mathcal{F}_\kappa^{c+\epsilon}$ such that $h(1,\mathcal{F}_\kappa^{c+\epsilon})\subset\mathcal{F}_\kappa^{c-\epsilon}$.*

Proof. See ([10] Lemma 6.3). □

We need some other definitions in order to state the other deformation lemma.
For every $x,y\in\mathfrak{M}$, set

$$\mathrm{dist}_*(x,y)=\max\left\{\|x(0)-u(0)\|,\|x(1)-y(1)\|\right\}+\left(\int_0^1\|x(s)-y(s)\|^2\,ds\right)^{\frac{1}{2}},$$

and for every $x\in\mathfrak{M}$ and $r>0$ set

$$B_r(x)=\{y\in\mathfrak{M}:\mathrm{dist}_*(y,x)\le r\}.$$

Now assume that the number of $\mathcal{V}^-$-critical curves of $\mathcal{F}_\kappa$ on $\mathfrak{M}\backslash\mathfrak{C}_0$ is finite, say $(y_i)_{i\in I}$.
Then we can fix an $r^*>0$ such that

- $\overline{B_{r_*}(y_i)}\cap\overline{B_{r_*}(y_j)}=\emptyset$ for every $i\neq j$;
- any $\overline{B_{r_*}(y_i)}$ is contractible in itself;
- any $\overline{B_{r_*}(y_i)}$ does not include constant curves.

Thus, we define
$$\mathcal{O}^*=\bigcup_{i\in I}B_{r^*}(x_i).$$

We remark that, if L is reversible, then $\mathcal{O}^*$ is $\mathcal{R}$-invariant by Remark 4.

Lemma 19. *Assume that the number of non-constant $\mathcal{V}^-$-critical curves for $\mathcal{F}_\kappa$ on $\mathfrak{M}\backslash\mathfrak{C}_0$ is finite and let $c>0$ be a $\mathcal{V}^-$-critical value for $\mathcal{F}_\kappa$. Then there exists an $\epsilon>0$ and an admissible homotopy $h\colon[0,1]\times\mathcal{F}_\kappa^{c+\epsilon}\to\mathcal{F}_\kappa^{c+\epsilon}$ such that $h(1,\mathcal{F}_\kappa^{c+\epsilon}\backslash\mathcal{O}^*)\subset\mathcal{F}_\kappa^{c-\epsilon}$.*

Proof. See ([10] Lemma 6.4). □

Lemma 20. *If $k>m(L)$, then there exists δ_1 such that, if $x\in\mathcal{F}_\kappa^{\delta_1}$, then*

$$|\Phi(x(s))|\le\delta_0,\qquad\forall s\in[0,1],\tag{56}$$

that is, every curve of the sublevel $\mathcal{F}_\kappa^{\delta_1}$ lies on a tubular neighbourhood of $\partial\Omega$.

Proof. Since $\overline{\Omega}$ is compact and L is quadratic at infinity, for every $\kappa>m(L)$ there exist two constants $\alpha_\kappa,\delta_\kappa>0$ such that

$$L(q,v)+\kappa\ge\alpha_\kappa\|v\|^2+\delta_\kappa>0,\qquad\forall q\in\overline{\Omega}_{\delta_0},\forall v\in T_q\mathcal{M}.$$

Set $\delta > 0$. As a consequence, for every $(x, T) \in \mathcal{L}^{\delta}_{\kappa} \subset \mathfrak{M} \times]0, +\infty[$ we have

$$\delta \geq \mathcal{F}_\kappa(x) = T(x) \int_0^1 \left(L\left(x, \frac{\dot{x}}{T(x)}\right) + \kappa \right) ds \geq T(x)\delta_\kappa,$$

and

$$\delta \geq \mathcal{F}_\kappa(x) \geq \frac{\alpha_\kappa}{T(x)} \int_0^1 \|\dot{x}\|^2 ds,$$

from which we infer

$$\|\dot{x}\|_{L^2} \leq \frac{\delta}{\sqrt{\alpha_\kappa \delta_\kappa}}.$$

Since $x(0) \in \partial\Omega$, then $\Phi(x(0)) = 0$ and we obtain

$$\begin{aligned} |\Phi(x(s))| = |\Phi(x(s)) - \Phi(x(0))| &\leq \int_0^s |\langle \nabla\Phi(\sigma), \dot{x}(\sigma)\rangle| \, d\sigma \\ &\leq K_0 \int_0^s \|\dot{x}\| d\sigma \leq K_0 \|\dot{x}\|_{L^2} \leq K_0 \frac{\delta}{\sqrt{\alpha_\kappa \delta_\kappa}}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality twice. By the arbitrariness of δ, there exists $\delta_1 > 0$ such that (56) holds for every $x \in \mathcal{F}^{\delta_1}_\kappa$. □

Lemma 21. *There exists an admissible homotopy* $h\colon [0,1] \times \mathcal{F}^{\delta_1}_\kappa \to \mathfrak{M}$ *such that* $h(1, \mathcal{F}^{\delta_1}_\kappa) \subset \mathfrak{C}_0$.

Proof. By Lemma 20, $x([0,1]) \subset \Phi^{-1}([0, \delta_0])$ for every $x \in \mathcal{F}^{\delta_1}_\kappa$. By (5), there exists a retraction $r\colon \Phi^{-1}([0, \delta_0]) \to \partial\Omega$ of class C^1 defined in terms of flow of $\nabla\Phi$. Hence, there exists a homotopy g such that $g(1, x)(s) \in \partial\Omega$, for all $x \in \mathcal{F}^{\delta_1}_\kappa$. Now define the homotopy $k(\tau, x)(s) = x((1-\tau)s + \tau/2)$, so that $k(1, x)(s) = x(1/2) \in \partial\Omega$ for every curve that lies in $\partial\Omega$. Combining the two homotopies g and k we define

$$h(\tau, x) = \begin{cases} g(2\tau, x), & \text{if } \tau \in [0, 1/2], \\ k(2\tau - 1, h(1, x)), & \text{if } \tau \in [1/2, 1], \end{cases}$$

which is an $\mathcal{R}$-invariant homotopy such that $h(1, \mathcal{F}^{\delta_1}_\kappa) \subset \mathfrak{C}_0$. □

We require one last definition to complete our proof, which is actually a relative Ljusternik and Schnirelmann category.

Definition 10. *Let X be a topological space and Y a closed subset of X. A closed subset Z of X has relative category equal to* $k \in \mathbb{N}$,

$$cat_{X,Y}(Z) = k,$$

if k is the minimal positive integer such that $Z \subset \bigcup_{i=0}^k A_i$, *where* $\{A_i\}_{i=0}^k$ *is a family of open subset of X satisfying:*

- $Z \cap Y \subset A_0$;
- *if* $i \neq 0$, A_i *is contractible in* $X \setminus Y$;
- *if* $i = 0$, *there exists* $h_0 \in C^0([0,1] \times A_0, X)$ *such that* $h_0(1, A_0) \subset Y$ *and* $h_0([0,1], A_0 \cap Y) \subset Y$.

Proof of Theorem 1. Let L be a reversible Tonelli Lagrangian. By Lemma 4, we can identify each curve with its backward reparametrization and we can study our variational problem in $\widetilde{\mathfrak{M}}$. To prove our existence and multiplicity result, we exploit the relative category

$$\mathrm{cat}_{\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{C}}_0} \widetilde{\mathfrak{M}} \geq N,$$

that has been proved in [6]. Let $\mathcal{D}$ be the set of all closed $\mathcal{R}$-invariant subsets of $\mathfrak{M}$ and define

$$\Gamma_i = \left\{ D \in \mathcal{D} : \mathrm{cat}_{\widetilde{\mathfrak{M}},\widetilde{\mathfrak{C}}_0} \widetilde{D} \geq i \right\}, \qquad i = 1, \ldots, N.$$

Since each Γ_i is non-empty, the following quantities are well defined

$$c_i = \inf_{D \in \Gamma_i} \sup_{x \in D} \mathcal{J}(x), \quad \text{for any } i = 1, \ldots, N.$$

With similar arguments applied in [11], by Lemma 21 we have that $c_1 > \delta_1$, by Lemma 18 we have that each c_i is a critical value for $\mathcal{F}_\kappa$ in $\mathfrak{M} \backslash \mathfrak{C}_0$ and by Lemma 19 we have that $c_i < c_i + 1$ for all $i = 1, \ldots, N-1$. Hence there are at least N $\mathcal{V}^-$-critical curves for $\mathcal{F}_\kappa$ on $\mathfrak{M} \backslash \mathfrak{C}_0$. By Propositions 3 and 5, either there exists an ELCTC or they are all ELCCs with energy κ. It remains to prove the geometric distinction of the ELCCs. Let us denote by x_i the $\mathcal{V}^-$-critical curve such that $\mathcal{F}_\kappa(x_i) = c_i$. Seeking a contradiction, let us assume that $x_i([0,1]) = x_j([0,1])$, with $i \neq j$. Then either $x_i(0) = x_j(0)$ or $x_i(0) = x_j(1)$. If $x_i(0) = x_j(0) = q$, then $\dot{x}_i(0)$ and $\dot{x}_j(0)$ have the same direction and since the two curves $(x_i, T(x_i))$ and $(x_j, T(x_j))$ have the same energy κ, it must be $\dot{x}_i / T(x_i) = \dot{x}_j / T(x_j)$. As a consequence, the two curves $\gamma_i \colon [0, T(x_i)] \to \mathcal{M}$ and $\gamma_j \colon [0, T(x_j)] \to \mathcal{M}$ defined by

$$\gamma_i(t) = x_i(t/T(x_i)) \quad \text{and} \quad \gamma_j(t) = x_j(t/T(x_j))$$

have the same initial velocity $\dot{\gamma}_i(0) = \dot{\gamma}_j(0) = v \in T_q\mathcal{M}$. By the uniqueness of the solution of the Cauchy problem

$$\begin{cases} d_q L(\gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\Big(d_v L(\gamma(t), \dot{\gamma}(t)) \Big) = 0, \\ \gamma(0) = q, \\ \dot{\gamma}(0) = v \in T_q\mathcal{M}, \end{cases}$$

we infer that $\gamma_i = \gamma_j$. Since γ_i and γ_j satisfy the conormal boundary conditions and since $k > m(L)$, the same argument applied in Proposition 3 shows that $\dot{\gamma}_i(T(x_i))$ and $\dot{\gamma}_j(T(x_j))$ point outside $\overline{\Omega}$. As a consequence, $T(x_i) = T(x_j)$ and $x_i = x_j$. If $x_i(0) = x_j(1)$, the same argument shows that $x_i = \mathcal{R}x_j$. As a consequence, it must be $\mathcal{F}_\kappa(x_i) = \mathcal{F}_\kappa(x_j)$, so $c_i = c_j$, which is absurd.

If L is not reversible, we get the thesis by applying the same minimax argument and the relative category

$$\mathrm{cat}_{\mathfrak{M},\mathfrak{C}_0} \mathfrak{M} \geq 2,$$

that has been proved in [8]. However, in this case the geometric distinction of two ELCCs with different values of the energy functional cannot be ensured. □

7. Conclusions

In this paper, we proved the existence and multiplicity of ELCCs with fixed energy $\kappa > m(L)$ in a compact manifold with boundary $\overline{\Omega}$, where $m(L)$ is defined in (3). As previously stated, this work generalizes [11], since it does not require (4) to hold, and [10], where only the energy functional of a Finsler metric is considered.

Moreover, we proved that if $\kappa > e_0(L)$, where $e_0(L)$ is given by (12), then the non-constant critical curves of the free-time action functional $\mathcal{L}_\kappa$ can be searched among the critical curves of a fixed-time action functional, simplifying the problem by avoiding the compactness issues that arise from the time variable $T \in]0, +\infty[$. This result could be applied in similar contexts to simplify the proofs of some known results (cf. [15–17]).

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