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Best Approximations by Increasing Invariant Subspaces of Self-Adjoint Operators

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Abstract: The paper describes approximations properties of monotonically increasing sequences of invariant subspaces of a self-adjoint operator, as well as their symmetric generalizations in a complex Hilbert space, generated by its positive powers. It is established that the operator keeps its spectrum over the dense union of these subspaces, equipped with quasi-norms, and that it is contractive. The main result is an inequality that provides an accurate estimate of errors for the best approximations in Hilbert spaces by these invariant subspaces.

Keywords: spectral approximation; exact errors estimations; self-adjoint operator

MSC: 47A58; 41A17

1. Introduction

Our purpose is to study the approximation properties of monotonically increasing family of invariant subspaces $\left\{L_p^\tau(A)\colon \tau \geq 1\right\}$ relative to a given self-adjoint unbounded operator A in a complex Hilbert space H. The monotonicity property of $\left\{L_p^\tau(A)\colon \tau \geq 1\right\}$ is crucial to obtain an accurate error estimate for the best approximations in the space H using the A-invariant subspaces $L_p^\tau(A)$. In the paper, we propose the construction of the increasing family $L_p^\tau(A)$ on the basis of positive operator degrees A^s $(s \geq 0)$. Such subspaces have the following form:

$$L_p^\tau(A) = \left\{x \in H\colon \|x\|_{L_p^\tau(A)} < \infty\right\}, \quad \|x\|_{L_p^\tau(A)} = \left(\int_0^\infty \|(A/\tau)^s x\|_H^p \, e^{-s} ds\right)^{1/p}$$

in the case $1 \leq p < \infty$ (specified also for $p = \infty$), where the index $\tau \geq 1$ appears as a parameter for the monotonic ordering of these subspaces relative to the contractive inclusions

$$L_p^\tau(A) \looparrowright L_p^t(A), \quad \tau < t.$$

It is also examined the monotonically increasing symmetric family of interpolation subspaces in H with the parameter $1 \leq q < \infty$,

$$L_{p,q}^\tau(A) = \left(L_{p_0}^\tau(A), L_{p_1}^\tau(A)\right)_{\vartheta,q}, \quad 1 \leq p_0, p_1 \leq \infty, \quad p = \{p_0, p_1\},$$

generated by a quadratic modified real interpolation method with $0 < \vartheta < 1$, as well as the increasing family of Lorentz-type subspaces $L_{p,q}^\tau(A) = (L_r^\tau(A), L_\infty^\tau(A))_{\vartheta,q}$ in H with a scalar index $p = r/\vartheta$ $(r \geq 1)$.

Each of the subspaces $L_p^\tau(A)$ (also their symmetric versions $L_{q,p}^\tau(A)$) is complete and A-invariant. Moreover, the restrictions A to $L_p^\tau(A)$ are bounded, namely $\|A\,|_{L_p^\tau(A)}\|_{\mathcal{L}(L_p^\tau(A))} \le \tau$ for all $r \ge 1$ (see Theorem 1). On the quasi-normed union of these subspaces,

$$L_p(A) = \bigcup_{\tau\ge1} L_p^\tau(A) \quad \text{with the quasi-norm} \quad |x|_{L_p(A)} = \inf\left\{\tau \ge 1 \colon x \in L_p^\tau(A)\right\}$$

the operator A is contractive and keeps its spectrum (see, Theorem 2), i.e.,

$$\sigma(A) = \bigcup_{\tau\ge1} \sigma\left(A\,|_{L_p^\tau(A)}\right).$$

To estimate the best approximation errors in the Hilbert space H by monotonically increasing family of invariant subspaces $L_p^\tau(A)$ (also their symmetric versions $L_{q,p}^\tau(A)$), we apply the so-called best approximation E-functional (more details in [1,2])

$$E_p\left(t, x; L_p(A), H\right) = \inf\left\{\, \|x - x_0\|_H : x \in L_p(A),\ \|x_0\|_{L_p(A)} < t\right\}$$

and the corresponding quasi-normed approximative space

$$\mathcal{E}_{2\vartheta}\left(L_p(A), H\right) = \left\{x \in L_p(A) + H \colon \|x\|_{\mathcal{E}_{2\vartheta}} < \infty\right\},$$
$$\|x\|_{\mathcal{E}_{2\vartheta}} = \left(\int_0^\infty \left[t^{-1+1/\vartheta} E_p(t,x)\right]^{2\vartheta} \frac{dt}{t}\right)^{1/2\vartheta}.$$

The main result is the following isomorphism of quasi-normed spaces

$$\mathcal{E}_{2\vartheta}\left(L_p(A), H\right) = \left(L_p(A), H\right)_{\vartheta,2}^{1/\vartheta},$$

where on the right-hand side is the $1/\vartheta$-power of best approximation space generated by the quadratic modified real interpolation method with $0 < \vartheta < 1$. This isomorphism provides the validity of the following estimation for approximation errors:

$$E_p\left(t, x; L_p(A), H\right) \le t^{1-1/\vartheta}\left((1/\vartheta\pi)\sin(\vartheta\pi)\right)^{1/2\vartheta} \|x\|_{\mathcal{E}_{2\vartheta}}, \quad t > 0$$

for all elements $x \in \mathcal{E}_{2\vartheta}\left(L_p(A), H\right)$ (see Theorem 3 that is also true for symmetric spaces $L_{p,q}(A)$). This inequality fully characterizes the subspace of elements from H in relation to rapidity of approximations.

Finally, note that inverse and direct theorems on best approximation estimates are proven in [3] where, instead of the E-functional, the modulus of smoothness was used. Exact estimations for approximation errors of spectral approximations for unbounded operators in Banach spaces, using the Besov-type quasi-norms, as well as many examples of Besov-type spaces generated by various unbounded operators, in particular elliptical operators, are given in the papers [4,5].

In the present paper, notions about interpolation and approximations tools are used without additional mentions from well-known books [1,6].

2. Quadratic Real Interpolation of Invariant Subspaces

We assume everywhere that, on a Hilbert complex space H, endowed with the norm $\|\cdot\|_H := \langle\cdot\mid\cdot\rangle^{1/2}$, a self-adjoint unbounded linear operator A with the dense domain $\mathcal{D}(A)$ is given. By the spectral theorem, the operator A and its positive powers have the following spectral expansions:

$$A = \int_{\sigma(A)} \lambda\, d\mu^A(\lambda), \quad A^s = \int_{\sigma(A)} \lambda^s\, d\mu^A(\lambda), \quad s \in [0, \infty)$$

such that $A^0x = x$ for all $x \in H$ (see, e.g., [7]), where μ^A is a unique projection-valued measure determined on its spectrum $\sigma(A) \subset \mathbb{R}$ with values in the Banach space of bounded linear operators $\mathcal{L}(H)$ that can be extended on $\mathbb{R} \setminus \sigma(A)$ as zero. For any Borel set $\Omega \subset \mathbb{R}$, the spectral subspace is defined to be the range

$$H_\Omega := \mathcal{R}[\mu^A(\Omega)] \subset H.$$

Consider subspaces generated by domains of all powers A^s of the operator A

$$\mathcal{D}^\infty(A) := \bigcap_{s\geq 0} \mathcal{D}^s(A), \quad \mathcal{D}^s(A) := \{x \in H\colon A^s x \in H\}.$$

Let $1 \leq \tau < \infty$ and $1 \leq p \leq \infty$. The following mappings

$$\mathcal{D}^\infty(A) \ni x \longmapsto (A/\tau)^s x \in H, \quad s \geq 0$$

are well defined. Hence, for any $\tau \geq 1$, we can assign the linear subspace $L_p^\tau(A) \subset H$ which by definition contains all elements $x \in H$ such that the scalar function

$$x_A^\tau\colon [0,\infty) \ni s \longmapsto \|(A/\tau)^s x\|_H$$

has the finite norm

$$\|x\|_{L_p^\tau(A)} := \begin{cases} \left(\int_0^\infty \|(A/\tau)^s x\|_H^p \, e^{-s} ds\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{s\geq 0} \| \, e^{-s}(A/\tau)^s x\|_H & \text{if } p = \infty. \end{cases}$$

This definition is correct, since, for any $x, y \in H$ and $\alpha \in \mathbb{C}$, the following inequality holds:

$$\begin{aligned} \|x + \alpha y\|_{L_p^\tau(A)} &= \left(\int_0^\infty \|(A/\tau)^s (x + \alpha y)\|_H^p \, e^{-s} ds\right)^{1/p} \\ &= \left(\int_0^\infty \|(A/\tau)^s x + \alpha (A/\tau)^s y\|_H^p \, e^{-s} ds\right)^{1/p} \leq \|x\|_{L_p^\tau(A)} + |\alpha| \|y\|_{L_p^\tau(A)} \end{aligned}$$

for any $1 \leq p < \infty$. The case $p = \infty$ is similar, which proves the linearity of $L_p^\tau(A)$.

We will apply a quadratic modified real interpolation method. Given a couple of normed spaces $L_\imath^\tau(A)$ $(\imath = 0, 1)$,

$$\left(L_{p_0}^\tau(A), L_{p_1}^\tau(A)\right), \quad 1 \leq p_0, p_1 \leq \infty,$$

where elements $x = x_0 + x_1$ of the algebraic sum $L_{p_0}^\tau(A) + L_{p_1}^\tau(A)$ are such that $x_\imath \in L_\imath^\tau(A)$, we assign the quadratic K-functional with $t > 0$

$$K(t, x) = K\left(t, x; L_{p_0}^\tau(A), L_{p_1}^\tau(A)\right) = \inf_{x = x_0 + x_1} \left(\|x_0\|^2_{L_{p_0}^\tau(A)} + t^2 \|x_1\|^2_{L_{p_1}^\tau(A)}\right)^{1/2}$$

(see [2] (Definition 3.3), [8] (p. 318)). Note that this couple of normed spaces with a fixed $\tau \geq 1$ and an operator A can be considered as a subspace in $L_{p_0}^\tau(A) + L_{p_1}^\tau(A) \subset H$ endowed with the quasi-norm

$$\|f\|_{L_{p_0}^\tau(A) + L_{p_1}^\tau(A)} = \inf_{x = x_0 + x_1} \left(\|x_0\|^2_{L_{p_0}^\tau(A)} + \|x_1\|^2_{L_{p_1}^\tau(A)}\right)^{1/2},$$

which guarantees its compatibility ([1] (3.11)).

For any pair indexes $\{0 < \vartheta < 1,\ 1 \le q < \infty\}$ or $\{0 < \vartheta \le 1,\ q = \infty\}$ with the help of a quadratic K-functional, we define the interpolation space

$$\left(L^\tau_{p_0}(A), L^\tau_{p_1}(A)\right)_{\vartheta,q} = \left\{x \in L^\tau_{p_0}(A) + L^\tau_{p_1}(A) \colon \|x\|_{\left(L^\tau_{p_0}(A), L^\tau_{p_1}(A)\right)_{\vartheta,q}} < \infty\right\},$$

$$\|x\|_{\left(L^\tau_{p_0}(A), L^\tau_{p_1}(A)\right)_{\vartheta,q}} = \begin{cases} \left(\int_0^\infty \left[t^{-\vartheta} K(t,x)\right]^q \dfrac{dt}{t}\right)^{1/q} & \text{if } 1 \le q < \infty \\ \sup\limits_{0<t<\infty} t^{-\vartheta} K(t,x) & \text{if } q = \infty \end{cases}$$

endowed with the norm $\|\cdot\|_{\left(L^\tau_{p_0}(A), L^\tau_{p_1}(A)\right)_{\vartheta,q}}$ that is determined in the case $q < \infty$ using the Haar measure dt/t on the multiplicative group $(0,\infty)$. For this interpolation space, we will briefly denote

$$L^\tau_{p,q}(A) := \left(L^\tau_{p_0}(A), L^\tau_{p_1}(A)\right)_{\vartheta,q} \quad \text{with a vector index} \quad p = (p_0, p_1). \tag{1}$$

In particular, for any $1 \le r < \infty$ and $1 \le q \le \infty$, the Lorentz-type subspace in H can be defined with the help of the following linear isomorphism (see, e.g., [1] (p. 109), [9] (Proposition 2))

$$L^\tau_{p,q}(A) = (L^\tau_r(A), L^\tau_\infty(A))_{\vartheta,q} \quad \text{with a scalar index} \quad p = r/\vartheta. \tag{2}$$

In accordance with this definition, the Lorentz subspace $L^\tau_{p,q}(A) \subset H$ contains all scalar functions $x^\tau_A(s) = \|(A/\tau)^s x\|_H$ in the variable $s \in [0,\infty)$ belonging to $L_q[0,\infty)$, i.e., such that

$$\|x^\tau_A\|_{L_{p,q}(A)} := \left\|s^{1/p} \hat{x}^\tau_A(s)\right\|_{L_q[0,\infty)} < \infty,$$

where the non-increasing rearrangement $\hat{x}^\tau_A$ of the function x^τ_A,

$$\hat{x}^\tau_A(s) := \inf\left\{\varrho \colon \lambda\left[z \in [0,\infty) \colon |x^\tau_A(z)| > \varrho\right] \le s\right\},$$

is defined via the Lebesgue measure λ on $[0,\infty)$. In other words, the function $[0,\infty) \ni s \mapsto s^{1/p} \hat{x}^\tau_A(s)$ should be L_q-integrable.

Let us describe the basic properties of the normed spaces $L^\tau_p(A)$ and $L^\tau_{q,p}(A)$.

Theorem 1. *(a) The subspaces $L^\tau_p(A)$ and $L^\tau_{q,p}(A)$ are invariant with respect to the operator A and the following inclusions*

$$\begin{gathered} L^\tau_p(A) \hookrightarrow H, \quad L^\tau_{p,q}(A) \hookrightarrow H, \\ L^\tau_p(A) \hookrightarrow L^t_p(A), \quad L^\tau_{p,q}(A) \hookrightarrow L^t_{p,q}(A) \end{gathered} \tag{3}$$

for any $t > \tau \ge 1$ are contractive.

(b) The restrictions $A\mid_{L^\tau_p(A)}$ and $A\mid_{L^\tau_{p,q}(A)}$ of A on the subspaces $L^\tau_p(A)$ and $L^\tau_{q,p}(A)$, respectively, are bounded operators satisfying the inequalities

$$\left\|A\mid_{L^\tau_p(A)}\right\|_{\mathcal{L}(L^\tau_p(A))} \le \tau, \quad \left\|A\mid_{L^\tau_{p,q}(A)}\right\|_{\mathcal{L}(L^\tau_p(A))} \le \tau. \tag{4}$$

(c) The spaces $L^\tau_p(A)$ and $L^\tau_{p,q}(A)$ with $\tau \ge 1$ are complete.

(d) Every spectral subspace H_Ω with a Borel $\Omega \subset \sigma(A)$ is contained in some $L^\tau_p(A)$ and $L^\tau_{p,q}(A)$ with a large enough $\tau \ge 1$.

Proof. (a) The inequality $\|x\|_H \le \sup_{s\ge0} e^{-s}\|(A/\tau)^s x\|_H = \|x\|_{L^\tau_\infty(A)}$ immediately yields the contractive embedding $L^\tau_\infty(A) \hookrightarrow H$.

For $1 \leq p < \infty$, the following inequality with arbitrary $\varepsilon > 0$ holds:

$$\begin{aligned} \|x\|^p_{L^\tau_p(A)} &= \int_0^\infty \|(A/\tau)^s x\|^p_H e^{-s} ds \\ &= \left(\int_0^\varepsilon + \int_\varepsilon^\infty\right) \|(A/\tau)^s x\|^p_H e^{-s} ds \geq \int_0^\varepsilon \|(A/\tau)^s x\|^p_H e^{-s} ds. \end{aligned}$$

By Lagrange's mean value theorem, for any ε, there exists $0 < c < \varepsilon$ such that

$$\int_0^\varepsilon \|(A/\tau)^s x\|^p_H e^{-s} ds = \|x\|^p_H + \varepsilon e^{-c} \|(A/\tau)^c x\|^p_H \geq \|x\|^p_H. \tag{5}$$

This yields the contractive embedding $L^\tau_p(A) \hookrightarrow H$ for any $1 \leq p < \infty$.

Consider the case of a vector index $p = (p_0, p_1)$ with $1 \leq p_0, p_1 < \infty$. Let $1 \leq q \leq \infty$. By the known interpolation property of K-functionals (see [1] (p. 81) or [8] (Theorem B.2)), the contractive inclusions $L^\tau_{p_\imath}(A) \hookrightarrow H$ with both indexes $\imath = 0, 1$ imply that the inclusion

$$\left(L^\tau_{p_0}(A), L^\tau_{p_1}(A)\right)_{\vartheta,q} \hookrightarrow H \quad \text{with} \quad 0 < \vartheta < 1$$

is contractive. In particular, it holds for the Lorentz-type spaces. Thus, as a summary result, we obtain the inequality

$$\|x\|_H \leq \|x\|_{L^\tau_{p,q}(A)}, \quad x \in L^\tau_{p,q}(A). \tag{6}$$

For any $\tau > t \geq 1$ and $1 \leq p < \infty$, we have $\|(A/\tau)^s x\|^p_H \leq \|(A/t)^s x\|^p_H$. Hence,

$$\begin{aligned} \|x\|^p_{L^\tau_p(A)} &= \int_0^\infty \|(A/\tau)^s x\|^p_H e^{-s} ds \\ &\leq \int_0^\infty \|(A/t)^s x\|^p_H e^{-s} ds = \|x\|^p_{L^t_p(A)} \end{aligned}$$

yields the contractive embedding $L^t_p(A) \hookrightarrow L^\tau_p(A)$. Likewise,

$$\begin{aligned} \|x\|_{L^\tau_\infty(A)} &= \sup_{s \geq 0} e^{-s} \|(A/\tau)^s x\|_H \\ &\leq \sup_{s \geq 0} e^{-s} \|(A/t)^s x\|_H = \|x\|_{L^t_\infty(A)}. \end{aligned}$$

Thus, $L^t_p(A) \hookrightarrow L^\tau_p(A)$ for $p = \infty$ is also contractive.

Similarly, for the subspaces $L^\tau_{q,p}(A)$, we obtain the inequality

$$\|x\|_{L^\tau_{p,q}(A)} \leq \|x\|_{L^t_{p,q}(A)}, \quad x \in L^t_{p,q}(A), \quad t < \tau. \tag{7}$$

The inequality (6) and (7) together yield all inclusions (3).

(b) Since $A(A/\tau)^s x = \tau (A/\tau)^{s+1} x$ for all $\tau \geq 1$, we get

$$\begin{aligned} \|Ax\|^p_{L^\tau_p(A)} &= \int_0^\infty \|(A/\tau)^s Ax\|^p_H e^{-s} ds = \int_0^\infty \|A(A/\tau)^s x\|^p_H e^{-s} ds \\ &= \tau^p \int_0^\infty \|(A/\tau)^{s+1} x\|^p_H e^{-s} ds \\ &= \tau^p \int_1^\infty \|(A/\tau)^s x\|^p_H e^{-s} ds^{1/p} \leq \tau^p \|x\|^p_{L^\tau_p(A)}. \end{aligned}$$

Applying the already mentioned interpolation property of K-functionals (see, [1] (p. 81) or [8] (Theorem B.2)), we obtain the inequality

$$\|Ax\|_{L^\tau_{p,q}(A)} \le \tau \|x\|_{L^\tau_{p,q}(A)} \quad \text{for} \quad 1 \le q \le \infty$$

and any $p = (p_0, p_1)$ with $1 \le p_0, p_1 < \infty$. It at once follows (4).

(c) Let (x_n) be a fundamental sequence in $L^\tau_p(A)$. For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_n - x_m\|_{L^\tau_p(A)} < \varepsilon$ for all $n, m \ge n_\varepsilon$. By the inequality (5), for any $\varepsilon > 0$ and $t \ge 0$, we find

$$\begin{aligned}
\|(A/\tau)^t x\|^p_{L^\tau_p(A)} &= \int_0^\infty \|(A/\tau)^{t+s} x\|^p_H \, e^{-s} ds \\
&= \left(\int_0^\varepsilon + \int_\varepsilon^\infty \right) \|(A/\tau)^{t+s} x\|^p_H \, e^{-s} ds \\
&\ge \int_0^\varepsilon \|(A/\tau)^{t+s} x\|^p_H \, e^{-s} ds \ge \|(A/\tau)^t x\|^p_H .
\end{aligned}$$

In particular, this inequality hold for the indexes $p = 1, \infty$. Consequently, by the known interpolation property (see, e.g., [9] (Theorem 4)), we obtain

$$\|(A/\tau)^s x\|_{L^\tau_{p,q}(A)} \ge \|(A/\tau)^s x\|_H \quad \text{for} \quad 1 \le q \le \infty, \quad s \ge 0.$$

It follows that $\{x_n \colon n \in \mathbb{N}\}$ and $\{(A/\tau)^s x_n \colon n \in \mathbb{N}\}$, for every $s \ge 0$, are fundamental sequences in H. By the completeness of H, there exist $x, y \in H$ such that $x_n \to x$ and $(A/\tau)^s x_n \to y$ by the norm in H. The graph of the operator A^s is a closed subspace in $H \times H$; therefore, $y = (A/\tau)^s x$ and $x \in \mathcal{D}^s(A)$. Since, it holds for any $s \ge 0$, we have $x \in \mathcal{D}^\infty(A)$. Hence, $(A/\tau)^s x_n \to (A/\tau)^s x$ is convergent by the norm in H for any $s \ge 0$.

Furthermore, we may apply a standard reasoning. For every $s \ge 0$, there exists the following limits $(A/\tau)^s (x_n - x_m) \to 0$ and $(A/\tau)^s (x_m - x) \to 0$ for all $m_{\varepsilon,k} \ge n_\varepsilon$ such that

$$\|(A/\tau)^s (x_m - x_n)\|_H < \varepsilon \, e^{-s} \quad \text{and} \quad \|(A/\tau)^s (x_m - x)\|_H < \varepsilon \, e^{-s}$$

for all $m \ge m_{\varepsilon,k}$. From $\|(A/\tau)^s x\|_H \le \|(A/\tau)^s x_{n_\varepsilon}\|_H + \|(A/\tau)^s (x_m - x_{n_\varepsilon})\|_H + \|(A/\tau)^s (x_m - x)\|_H$, it follows that

$$\|(A/\tau)^s x\|_H < \|(A/\tau)^s x_{n_\varepsilon}\|_H + 2\varepsilon \, e^{-s} \quad \text{for all} \quad s \ge 0.$$

By integration with the weight e^{-s}, we find

$$\|x\|_{L^\tau_{p,q}(A)} \le \|x_{n_\varepsilon}\|_{L^\tau_{p,q}(A)} + 4\varepsilon \, e^{-s}.$$

Hence, $x \in L^\tau_{p,q}(A)$, which is the same in the case $L^\tau_p(A)$. Moreover, by integration with the weight e^{-s}, the inequality

$$\|(A/\tau)^s (x_n - x)\|_H \le \left\| (A/\tau)^s (x_{m_{\varepsilon,k}} - x) \right\|_H + \left\| (A/\tau)^s (x_n - x_{m_{\varepsilon,k}}) \right\|_H ,$$

and we find that $\|x_n - x\|_{L^\tau_{p,q}(A)} \le 4\varepsilon \, e^{-s}$ for all $n \ge n_\varepsilon$. Thus, $L^\tau_{p,q}(A)$ is complete. The case $L^\tau_p(A)$ is fully similar.

An alternative reasoning can also be used. Since both spaces $L^\tau_p(A)$ and $L^\tau_\infty(A)$ are complete, the interpolation space $L^\tau_{p,q}(A)$ is also complete in accordance with ([1] (Theorem 3.4.2 & Lemma 3.10.2)), [6].

(d) Let $1 \le \sup\{|\lambda| \colon \lambda \in \Omega\} \le \tau$ and $1 \le p < \infty$. In accordance with the spectral theorem, the restriction $A|_{H_\Omega}$ is a bounded operator on the spectral subspace H_Ω. Hence, using the inequality $\|Ax\|_H \le \|A|_{H_\Omega}\| \|x\|_H$, we get

$$\int_0^\infty \|(A/\tau)^s x\|^p_H \, e^{-s} ds \le \left(\|A|_{H_\Omega}\| \|x\|_H \right)^p \int_0^\infty \tau^{-sp} \, e^{-s} ds$$

for any $x \in H_\Omega$, where

$$\int_0^\infty \tau^{-sp}\, e^{-s} ds = \frac{1}{1 + p \ln \tau} \quad \text{and} \quad \|A|_{H_\Omega}\| < \infty.$$

Thus, every spectral subspace H_Ω is contained in some $L_p^\tau(A)$ with a large enough τ.

For $p = \infty$, we similarly obtain

$$\sup_{s \geq 0} \|(A/\tau)^s x\|_H \leq \|A|_{H_\Omega}\| \|x\|_H \sup_{s \geq 0} (1/\tau)^s\, e^{-s} \leq \|A|_{H_\Omega}\| \|x\|_H$$

for all $x \in H_\Omega$, i.e., $H_\Omega \subset L_\infty^\tau(A)$ for a large enough $\tau \geq 1$. □

3. Dense Quasi-Normed Invariant Subspaces

Now, we consider the union

$$L_p(A) := \bigcup_{\tau \geq 1} L_p^\tau(A) \text{ with the quasi-norm } |x|_{L_p(A)} := \inf\left\{\tau \geq 1 \colon x \in L_p^\tau(A)\right\}$$

which is a linear subspace in H, in view of the monotonicity of subspaces resulting from Theorem 1(a). Similarly, we define the linear subspace in H

$$L_{p,q}(A) := \bigcup_{\tau \geq 1} L_{p,q}^\tau(A) \text{ with the quasi-norm } |x|_{L_{p,q}(A)} := \inf\left\{\tau \geq 1 \colon x \in L_{p,q}^\tau(A)\right\}.$$

The correctness of these definitions follows from the elementary considerations that

$$\begin{aligned} |x + y|_{L_{p,q}(A)} &= \inf\left\{\tau \geq 1 \colon x + y \in L_{p,q}^\tau(A)\right\} \\ &= \inf\left\{t + s \geq 1 \colon x + y \in L_{p,q}^{t+s}(A)\right\} \\ &\leq \inf\left\{t + s \geq 1 \colon x \in L_{p,q}^t(A),\, y \in L_{p,q}^s(A)\right\} \leq |x|_{L_{p,q}(A)} + |y|_{L_{p,q}(A)} \end{aligned}$$

for all $x \in L_{p,q}^t(A)$, $y \in L_{p,q}^s(A)$. Obviously, $|x|_{L_{p,q}(A)} = |-x|_{L_{p,q}(A)}$. As a result, $|\cdot|_{L_{p,q}(A)}$ is a quasi-norm.

In the following statements, we describe the basic properties of quasi-normed spaces $L_p(A)$ and $L_{p,q}(A)$.

Theorem 2. *(a) The linear subspaces $L_p(A)$ and $L_{p,q}(A)$ are dense in H and the restrictions of A to these both subspaces are contractive.*

(b) The spectrum $\sigma(A)$ of the operator A allows the following decompositions:

$$\sigma(A) = \bigcup_{\tau \geq 1} \sigma\left(A\,|_{L_p^\tau(A)}\right) = \bigcup_{\tau \geq 1} \sigma\left(A\,|_{L_{p,q}^\tau(A)}\right).$$

Proof. (a) By the spectral theorem, the collection of spectral subspaces H_Ω with all Borel subsets $\Omega \subset \sigma(A)$ is total in H. Hence, from Theorem 1(c), it immediately follows that the union $L_{p,q}(A) = \bigcup_{\tau \geq 1} L_{p,q}^\tau(A)$ is dense in H.

Since $A[L_{p,q}^\tau(A)] \subset L_{p,q}^\tau(A)$ and $L_{p,q}^\tau(A) \subset L_{p,q}^t(A)$ for all $t \geq \tau$, we find that

$$\begin{aligned} |Ax|_{L_{p,q}(A)} &= \inf\left\{\tau \geq 1 \colon Ax \in L_{p,q}^\tau(A)\right\} \\ &\leq \inf\left\{\tau \geq 1 \colon x \in L_{p,q}^\tau(A)\right\} = |x|_{L_{p,q}(A)}. \end{aligned}$$

The case of spaces $L_{p,q}^\tau(A)$ is similar.

(b) For any $\lambda \in \mathbb{C} \setminus \sigma(A)$ and $x \in L_p^\tau(A)$, the equality

$$(\lambda - A)^{-1}(A/\tau)^s x = (A/\tau)^s(\lambda - A)^{-1}x, \quad s \geq 0$$

holds. Thus, in the case $1 \leq p < \infty$, we get

$$\begin{aligned}
\|(\lambda - A)^{-1}x\|^p_{L_p^\tau(A)} &= \int_0^\infty \|(\lambda - A)^{-1}(A/\tau)^s x\|_H^p e^{-s} ds \\
&\leq \|(\lambda - A)^{-1}\|^p_{\mathcal{L}(H)} \int_0^\infty \|(A/\tau)^s x\|_H^p e^{-s} ds \\
&= \|(\lambda - A)^{-1}\|^p_{\mathcal{L}(H)} \|x\|^p_{L_p^\tau(A)}.
\end{aligned}$$

Thus, $\lambda \in \mathbb{C} \setminus \sigma(A|_{L_p^\tau(A)})$. It follows that $\sigma\left(A\,|_{L_p^\tau(A)}\right) \subset \sigma(A)$ for any $\tau \geq 1$. For $p = \infty$, the reasoning is similar.

Let $\Omega \subset \sigma(A)$ be a Borel set. By Theorem 1(c) for the corresponding spectral subspace H_Ω, there exists $\tau \geq 1$ such that $\Omega \subset [-\tau, \tau]$. Then, for any $\lambda \in \mathbb{C} \setminus \sigma(A|_{L_p^\tau(A)})$, we have

$$\|(\lambda - A)^{-1}x\|^p_{L_p^\tau(A)} \leq \|(\lambda - A)^{-1}\|_{\mathcal{L}(L_p^\tau(A))} \|x\|_{L_p^\tau(A)}.$$

If (x_n) is a fundamental sequence in $L_p^\tau(A)$ with the limit $x \in L_p^\tau(A)$, then the following sequences $\big((\lambda - A)^{-1}(A/\tau)^s x_n\big)$ for every $s \geq 0$ are fundamental in H and

$$\lim_{n\to\infty} (\lambda - A)^{-1}(A/\tau)^s x_n = (\lambda - A)^{-1}(A/\tau)^s x$$

by the closeness of operators $(A/\tau)^s$ on H.

By Theorem 1(c) $H_\Omega \subset L_p^\tau(A)$, hence the resolvent $(\lambda - A)^{-1}$ is well defined and closed on H_Ω. By the closed graph theorem, the resolvent $(\lambda - A)^{-1}$ is bounded on H_Ω for any $\lambda \in \mathbb{C} \setminus \sigma(A|_{L_p^\tau(A)})$, i.e., $\lambda \in \mathbb{C} \setminus \sigma(A|_{H_\Omega})$. As a result,

$$\sigma(A|_{H_\Omega}) \subset \sigma(A|_{L_p^\tau(A)}) \quad \text{if} \quad \Omega \subset [-\tau, \tau].$$

The inclusions that are implied from the spectral theorem still need to be used. As a result,

$$\sigma(A) = \bigcup_{\Omega \subset \sigma(A)} \sigma(A|_{H_\Omega}) \subset \bigcup_{\tau \geq 1} \sigma\left(A\,|_{L_p^\tau(A)}\right) \subset \sigma(A).$$

The case of the space $L_{p,q}^\tau(A)$ with $1 \leq q < \infty$ is completely similar. □

Remark 1. *Since $A^{t-s}(A/\tau)^s x = \tau^{t-s}(A/t)^r x$ for all $t > s \geq 0$, we can rewrite the first inequality (4) as*

$$\begin{aligned}
\|A^{t-s}x\|_{L_p^\tau(A)} &= \left(\int_0^\infty \|A^{t-s}(A/\tau)^s x\|_H^p e^{-s} ds\right)^{1/p} \\
&= \tau^{t-s}\left(\int_0^\infty \|(A/\tau)^{s+t-s} x\|_H^p e^{-s} ds\right)^{1/p} \\
&= \tau^{t-s}\left(\int_{t-s}^\infty \|(A/\tau)^s x\|_H^p e^{-s} ds\right)^{1/p} \leq \tau^{t-s}\|x\|_{L_p^\tau(A)}.
\end{aligned}$$

4. Estimates of Best Approximation Errors

We study in this section the case of best approximation, where the compatible pairs are quasi-normed invariant subspaces $L_{p,q}(A)$ in the initial Hilbert space H, generated by a given self-adjoint operator

$$A\colon \mathcal{D}(A) \ni x \mapsto Ax \in H.$$

Let $0 < \vartheta < 1$. For the pair indexes $1 \leq \tau, q < \infty$, we assign the Banach spaces

$$L^{\tau}_{p,q}(A) = \begin{cases} \left(L^{\tau}_{p_0}(A), L^{\tau}_{p_1}(A)\right)_{\vartheta,q}, & p = (p_0, p_1), \quad 1 \leq p_0, p_1 \leq \infty, \\ (L^{\tau}_{r}(A), L^{\tau}_{\infty}(A))_{\vartheta,q}, & p = r/\vartheta, \qquad 1 \leq r < \infty. \end{cases}$$

We will investigate the compatible couple of spaces $\left(L_{p,q}(A), H\right)$ in which

$$L_{p,q}(A) = \bigcup_{\tau \geq 1} L^{\tau}_{p,q}(A), \text{ endowed with } |x|_{L_{p,q}(A)} := \inf\left\{\tau \geq 1 \colon x \in L^{\tau}_{p,q}(A)\right\},$$

is a quasi-normed subspace in the Hilbert space H. This couple is compatible, since the sum $L_{p,q}(A) + H$ possesses the well defined quasi-norm

$$\|x\|_{L_{p,q}(A)+H} = \inf_{x=x_0+x_1} \left(|x_0|^2_{L_{p,q}(A)} + \|x_1\|^2_H\right)^{1/2}.$$

Apply now to this compatible couple $\left(L_{p,q}(A), H\right)$ the quadratic modified real interpolation method. Let us define the suitable quadratic K-functional with $t > 0$

$$K_{p,q}(t,x) = K_{p,q}\left(t, x; L_{p,q}(A), H\right) = \inf_{x=x_0+x_1} \left(|x_0|^2_{L_{p,q}(A)} + t^2\|x_1\|^2_H\right)^{1/2}.$$

Using this functional, we define the corresponding real interpolation space

$$\begin{gathered} \left(L_{p,q}(A), H\right)_{\vartheta,2} = \left\{x \in L_{p,q}(A) + H \colon \|x\|_{\left(L_{p,q}(A),H\right)_{\vartheta,2}} < \infty\right\}, \\ \|x\|_{\left(L_{p,q}(A),H\right)_{\vartheta,2}} := \left(\int_0^\infty \left[t^{-\vartheta} K_{p,q}(t,x)\right]^2 \frac{dt}{t}\right)^{1/2}, \end{gathered}$$

endowed with the quasi-norm $\|\cdot\|_{\left(L_{p,q}(A),H\right)_{\vartheta,2}}$.

Furthermore, we will deal with the problem of estimating the best approximations of elements of the Hilbert space H by invariant subspaces $L_{p,q}(A)$ of the operator A.

To estimate these best approximation errors, we apply (see, e.g., [1] (Chapter 7)), the so-called approximation E-functional $E_{q,p}(t, x; L_{p,q}(A), H)$ with $x \in L_{p,q}(A)$ and $t > 0$ in the following form:

$$\begin{aligned} E_{q,p}(t,x) &:= E_{q,p}(t, x; L_{p,q}(A), H) \\ &= \inf\left\{ \|x - x_0\|_H : x \in L_{p,q}(A),\ \|x_0\|_{L_{p,q}(A)} < t\right\}. \end{aligned} \tag{8}$$

For each index $0 < \vartheta < 1$, we assign the quadratic approximation subspace

$$\mathcal{E}_{2\vartheta}\left(L_{p,q}(A), H\right) \subset L_{p,q}(A) + H$$

endowed with the quasi-norm $\|\cdot\|_{\mathcal{E}_{2\vartheta}}$, where

$$\begin{gathered} \mathcal{E}_{2\vartheta} := \mathcal{E}_{2\vartheta}\left(L_{p,q}(A), H\right) = \left\{x \in L_{p,q}(A) + H \colon \|a\|_{\mathcal{E}_{2\vartheta}} < \infty\right\}, \\ \|x\|_{\mathcal{E}_{2\vartheta}} := \left(\int_0^\infty \left[t^{-1+1/\vartheta} E_{q,p}(t,x)\right]^{2\vartheta} \frac{dt}{t}\right)^{1/2\vartheta}. \end{gathered} \tag{9}$$

Following ([1] (Exercise B.5)) (see also [8] (Appendix B, p. 329)), we use the normalization factor

$$N_{\vartheta,2} := \left(\int_0^\infty t^{1-2\vartheta}/(1+t^2)\, dt\right)^{-1/2} = ((2/\pi)\sin(\pi\vartheta))^{1/2}.$$

The following theorem contains the main result:

Theorem 3. *(a) The following isomorphism with equivalent quasinorms*

$$\mathcal{E}_{2\vartheta}\left(L_{p,q}(A), H\right) = \left(L_{p,q}(A), H\right)^{1/\vartheta}_{\vartheta,2} \tag{10}$$

holds, where $\left(L_{p,q}(A), H\right)^{1/2\vartheta}_{\vartheta,2}$ *means the real interpolation space* $\left(L_{p,q}(A), H\right)_{\vartheta,2}$ *endowed with the quasi-norm* $\|x\|^{1/2\vartheta}_{\left(L_{p,q}(A),H\right)_{\vartheta,2}}$.

(b) The following estimation of best spectral approximation errors

$$E_{p,q}(t,x) \le t^{1-1/\vartheta}\big((1/\vartheta\pi)\sin(\vartheta\pi)\big)^{1/2\vartheta}\|x\|_{\mathcal{E}_{2\vartheta}} \quad t>0 \tag{11}$$

is achieved for all elements $x \in \mathcal{E}_{2\vartheta}\left(L_{p,q}(A), H\right)$.

Proof. (a) First, note that $t^{-1+1/\vartheta}E_{p,q}(t,x) \to 0$ as $t \to 0$ and as $t \to \infty$ (see [1] (Section 7.1)). Let us define

$$K_\infty(t,x) := \inf_{x=x_0+x_1} \max\left(\|x_0\|_{L_{p,q}(A)}, t\|x_1\|_H\right), \quad t>0.$$

Then, similarly as the above, $t^{-\vartheta}K_\infty(t,x) \to 0$ as $t \to 0$ and as $t \to \infty$ (see [1] (Section 7.1)). Integrating by parts with the change of variables $v = t/E_{p,q}(t,x)$, we similarly to [4] get that

$$\begin{aligned}\int_0^\infty \left(v^{-\vartheta}K_\infty(v,x)\right)^2\frac{dv}{v} &= -\frac{1}{2\vartheta}\int_0^\infty K_\infty(v,x)^2 dv^{-2\vartheta}\\ &= \frac{1}{2\vartheta}\int_0^\infty v^{-2\vartheta}dK_\infty(v,x)^2 = \frac{1}{2\vartheta}\int_0^\infty \left(t/E_{p,q}(t,x)\right)^{-2\vartheta}dt^2\\ &= \frac{1}{4\vartheta}\int_0^\infty \left(t^{-1+1/\vartheta}E_{p,q}(t,x)\right)^{2\vartheta}\frac{dt}{t}.\end{aligned} \tag{12}$$

The following inequalities are a consequence of definitions K_∞ and $K_{p,q}$ (see [2] (3.1)),

$$K_\infty(t,x) \le K_{p,q}(t,x) \le 2^{1/2}K_\infty(t,x). \tag{13}$$

According to the above equality (12) and the left inequality from (13), we have

$$\begin{aligned}\frac{1}{4\vartheta}\|x\|^{2\vartheta}_{\mathcal{E}_{2\vartheta}} &= \frac{1}{4\vartheta}\int_0^\infty \left(t^{-1+1/\vartheta}E_{p,q}(t,x)\right)^{2\vartheta}\frac{dt}{t}\\ &= \int_0^\infty \left(v^{-\vartheta}K_\infty(v,x)\right)^2\frac{dv}{v}\\ &\le \int_0^\infty \left(v^{-\vartheta}K_{p,q}(v,x)\right)^2\frac{dv}{v} = \|x\|^2_{\left(L_{p,q}(A),H\right)_{\vartheta,2}}.\end{aligned} \tag{14}$$

On the other hand, from the right inequality (13), it follows that

$$\begin{aligned}\|x\|^2_{\left(L_{p,q}(A),H\right)_{\vartheta,2}} &= \int_0^\infty \left(v^{-\vartheta}K_{p,q}(v,x)\right)^2\frac{dv}{v} \le 2\int_0^\infty \left(v^{-\vartheta}K_\infty(v,x)\right)^2\frac{dv}{v}\\ &= \frac{2}{4\vartheta}\int_0^\infty \left(t^{-1+1/\vartheta}E_{p,q}(t,x)\right)^{2\vartheta}\frac{dt}{t} = \frac{2}{4\vartheta}\|x\|^{2\vartheta}_{\mathcal{E}_{2\vartheta}}.\end{aligned}$$

As a result, from the previous inequalities, we get

$$\|x\|^2_{\left(L_{p,q}(A),H\right)_{\vartheta,2}} \le 2(4\vartheta)^{-1}\|x\|^{2\vartheta}_{\mathcal{E}_{2\vartheta}} \le 2\|x\|^2_{\left(L_{p,q}(A),H\right)_{\vartheta,2}} \tag{15}$$

for all $x \in L_{p,q}(A)$. Hence, the isomophism (10) holds. The statement (a) is proved.

(b) Let us use the auxiliary function

$$f(v/t) = (v/t)\Big(1 + (v/t)^2\Big)^{-1/2}, \quad t, v > 0.$$

By integrating both sides of $f(v/t)^2 K_{p,q}(t,x)^2 \leq K_{p,q}(v,x)^2$, we find

$$\int_0^\infty \Big(v^{-\vartheta} f(v/t)\Big)^2 \frac{dv}{v} K_{p,q}(t,x)^q \leq \int_0^\infty \Big(v^{-\vartheta} K_{p,q}(v,x)\Big)^2 \frac{dv}{v} = \|x\|^2_{(L_{p,q}(A),H)_{\vartheta,2}},$$

as well as

$$\int_0^\infty v^{-2\vartheta}(v/t)^2 \Big(1 + (v/t)^2\Big)^{-1} \frac{dv}{v} = \int_0^\infty \Big(v^{-\vartheta} f(v/t)\Big)^2 \frac{dv}{v} = (t^\vartheta N_{\vartheta,2})^{-2}.$$

It follows that

$$\left(\int_0^\infty \Big(v^{-\vartheta} f(v/t)\Big)^2 \frac{dv}{v} K_{p,q}(t,x)^2\right)^{1/2} = \frac{K_{p,q}(t,x)}{t^\vartheta N_{\vartheta,2}} \leq \|x\|_{(L_{p,q}(A),H)_{\vartheta,2}}.$$

Thus, $K_{p,q}(t,x) \leq t^\vartheta N_{\vartheta,2} \|x\|_{(L_{p,q}(A),H)_{\vartheta,2}}$ and, taking into account (13), we have

$$K_\infty(t,x) \leq t^\vartheta N_{\vartheta,2} \|x\|_{(L_{p,q}(A),H)_{\vartheta,2}}.$$

Applying the known inequality from [1] (Lemma 7.1.2), we get that, for a given $v > 0$, there exists $t > 0$ such that

$$v^{1-\vartheta} E_{p,q}(v,x)^\vartheta \leq t^{-\vartheta} K_\infty(t,x) \leq N_{\vartheta,2} \|x\|_{(L_{p,q}(A),H)_{\vartheta,2}}.$$

As a result, from (15), we obtain $v^{1-\vartheta} E_{p,q}(v,x)^\vartheta \leq 2^{1/2}(4\vartheta)^{-1/2} N_{\vartheta,2} \|x\|^\vartheta_{\mathcal{E}_{2\vartheta}}$ or

$$E_{p,q}(v,x) \leq v^{-1+1/\vartheta} 2^{1/2\vartheta} (4\vartheta)^{-1/2\vartheta} N_{\vartheta,2}^{1/\vartheta} \|x\|_{\mathcal{E}_{2\vartheta}}.$$

Substituting values of the normalisation factor $N_{\vartheta,2}$, we get the inequality (11). □

5. Conclusions

The motivation of a given publication is to present precise estimates of best quadratic spectral approximations for self-adjoint operators in Hilbert space. The solution to this problem is included in the main Theorem 3. This is our first quick publication in this direction. In the future, we plan to analyze the connection of our results with various already known studies in the near areas of the best spectral approximations theory. At the moment, the analysis of such connections is not yet complete—in particular, towards the research presented in publications from recent years [10,11].

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