Symmetry of the Relativistic Two-Body Bound State

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Abstract: We show that in a relativistically covariant formulation of the two-body problem, the bound state spectrum is in agreement, up to relativistic corrections, with the non-relativistic bound-state spectrum. This solution is achieved by solving the problem with support of the wave functions in an $O(2,1)$ invariant submanifold of the Minkowski spacetime. The $O(3,1)$ invariance of the differential equation requires, however, that the solutions provide a representation of $O(3,1)$. Such solutions are obtained by means of the method of induced representations, providing a basic insight into the subject of the symmetries of relativistic dynamics.

Keywords: relativistic quantum mechanics; bound states; symmetries; spectrum; covariant two-body central force problem

1. Introduction

In the non-relativistic Newtonian-Galilean view, two particles may be thought of as interacting through a potential function $V(\mathbf{x}_1(t), \mathbf{x}_2(t))$; for Galilean invariance, V must be a scalar function of the difference, i.e., $V(\mathbf{x}_1(t) - \mathbf{x}_2(t))$. In such a potential model, $\mathbf{x}_1$ and $\mathbf{x}_2$ are taken to be at equal time, corresponding to a correlation between the two particles consistent with the Newtonian-Galilean picture.

For the relativistic theory, two world lines with action at a distance interaction between two points x_1^μ and x_2^μ cannot be correlated by the variable t in every frame.

The Stueckelberg (SHP) theory [1] provides an effective andsystematic way of dealing with the N body problem, and has been applied in describing relativistic fluid mechanics [2], the Gibbs ensembles in statistical mechanics and the Boltzmann equation [3], systems of many identical particles [4], and other applications.

The basic idea of the SHP theory is the parametrization of the world lines of particles with a universal parameter τ [5] (see also [6,7]). Stueckelberg [8] described *classical* pair annihilation with a world line that proceeds, in τ, in the positive direction of the time t (the observable time of Einstein [9]) and then passes to a motion in the negative direction of time for τ proceeding in its monotonic development, precisely as postulated by Newton [10,11]. The transition is caused by interaction, such as emission of a photon. Although this process was considered to be classical, it occurs in a diagram in Feynman's perturbative expansion of the S-matrix [12].

Stueckelberg [8] then considered the symplectic manifold of $\{x^\mu, p_\mu\}$, with $\mu, \nu = (0,1,2,3)$ with diagonal metric $\eta_{\mu\nu} = (-,+,+,+)$ (raising and lowering indices). Here, $x^\mu = \{x^0, x^1, x^2, x^3\}$, where $x^0 = ct$ (For $c \to \infty$, ct may remain finite for $t \to 0$ and can be taken to be an arbitrary constant), $p^0 = E/c$. We shall generally write $c = 1$ but note that in the non-relativistic (NR) limit, $c \to \infty$,

so that $p^0 \to 0$ for finite energy E. Stueckelberg then wrote an invariant Hamiltonian of the form (for $V(x)$ scalar)

$$K = \frac{p^\mu p_\mu}{2M} + V(x), \tag{1}$$

which goes over to the usual NR Hamiltonian for in the NR limit.

He assumed the equations of motion

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \quad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu} \equiv \frac{dp_\mu}{d\tau} \tag{2}$$

It then follows from (1) that the proper time $ds^2 = -dx^\mu dx_\mu$ satisfies

$$\frac{ds^2}{d\tau^2} = -\frac{p^\mu p_\mu}{M^2} = \frac{m^2}{M^2}. \tag{3}$$

The theory implies that the particle mass m is a dynamical variable, reflecting the fact that the Einstein time t is an observable, and therefore that $E = \pm\sqrt{\mathbf{p}^2 + m^2}$, conjugate to t, must be an observable as well [5]. For $m^2 = M^2$, (3) implies that the square of the proper time interval is equal to $(d\tau)^2$, but in general, this relation cannot be maintained for non-trivial interaction.

The Poisson bracket structure then follows from (2). The τ derivative of a function of x, p is given by

$$\frac{d}{d\tau} F(x,p) = \frac{\partial F}{\partial x^\mu} \frac{dx^\mu}{d\tau} + \frac{\partial F}{\partial p_\mu} \frac{dp_\mu}{d\tau} = \frac{\partial F}{\partial x^\mu} \frac{\partial K}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial K}{\partial x^\mu} \equiv [F,K]_{PB}; \tag{4}$$

With this, we see that

$$[x^\mu, p_\nu]_{PB} = \delta^\mu{}_\nu. \tag{5}$$

Following Dirac [13], it is assumed that the operator commutation relations, following the group action of translation implied by the Poisson bracket,

$$[x^\mu, p_\nu] = i\hbar \delta^\mu{}_\nu \tag{6}$$

as the basis for the construction of the quantum theory [5].

The corresponding Stueckelberg-Schrödinger equation is then taken to be, derived from the unitary evolution of the wave function $\psi_\tau(x)$,

$$i\hbar \frac{\partial \psi_\tau(x)}{\partial \tau} = K\psi_\tau(x), \tag{7}$$

with the operators p_μ in K represented as $-i\hbar \frac{\partial}{\partial x^\mu}$, self-adjoint in the scalar product $(\psi, \chi) = \int d^4x \psi_\tau{}^*(x)\chi_\tau(x)$.

Equation (7) corresponds to the quantum one particle problem. We now proceed to discuss the two-body problem.

2. The Two-Body Bound State

We review here the relativistic two-body problem with invariant action at a distance potential, for bound states.

As a candidate for an invariant action at a distance potential for the two-body relativistic bound state we take for the potential V the function $V(\rho)$, for

$$\rho^2 = (\mathbf{x}_1 - \mathbf{x}_2)^2 - (t_1 - t_2)^2 \equiv \mathbf{x}^2 - t^2, \tag{8}$$

where x_1^μ and x_2^μ are taken at equal τ, acting as a correlation parameter as well as the global generating parameter of evolution. This "relative coordinate" (squared) reduces to $(\mathbf{x}_1 - \mathbf{x}_2)^2 \equiv \mathbf{x}^2$ at equal time

for the two particles in the non-relativistic limit, so that ρ becomes r in this limit (for simultaneous t_1 and t_2). Clearly, the solutions of a problem with this potential must then reduce to the solutions of the corresponding non-relativistic problem in that limit.

The two-body Stueckelberg Hamiltonian, is

$$K = \frac{p_1{}^\mu p_{1\mu}}{2M_1} + \frac{p_2{}^\mu p_{2\mu}}{2M_2} + V(x). \tag{9}$$

Since K does not depend on the total (spacetime) "center of mass"

$$X^\mu = \frac{M_1 x_1^\mu + M_2 x_2^\mu}{M_1 + M_2}, \tag{10}$$

the two-body Hamiltonian can be separated into the sum of two Hamiltonians, one for the "center of mass" motion and the second for the relative motion, by defining the total momentum, which is absolutely conserved,

$$P^\mu = p_1^\mu + p_2^\mu \tag{11}$$

and the relative motion momentum

$$p^\mu = \frac{M_2 p_1^\mu - M_1 p_2^\mu}{M_1 + M_2}. \tag{12}$$

The pairs P^μ, X^μ and p^μ, x^μ satisfy separately the canonical Poisson bracket (classically) and commutation relations (quantum mechanically), and commute with each other. Then

$$K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(x), \equiv K_{CM} + K_{rel}, \tag{13}$$

where $M = M_1 + M_2$, $m = M_1 M_2/(M_1 + M_2)$, and $x = x_1 - x_2$. Both K_{CM} and K_{rel} are constants of the motion; the total and relative momenta for the quantum case may be represented by partial derivatives with respect to the corresponding coordinates. This problem was solved explicitly for the classical case by Horwitz and Piron [5], where it was shown that there is no precession of the type predicted by Sommerfeld [14], who used the non-relativistic form $1/r$ for the potential (and obtained a period for the precession of Mercury that does not fit the data).

The corresponding quantum problem was solved by Cook [15], with support for the wave functions in the full space-like region; however, he obtained a spectrum of the form $1/(n + \frac{1}{2})^2$, with n an integer, which does not agree with the Balmer spectrum for hydrogen. Zmuidzinas [16], brought to our attention by P. Winternitz [17]), however, proved that there *is no complete orthogonal set of functions* in the full space-like region, and separated the space-like region into two submanifolds, in each of which there could be complete orthogonal sets. The region for which $\mathbf{x}^2 > t^2$, in particular, permits the solution of the differential equations corresponding to the problem posed by (9) by separation of variables and provides spectra that coincide, up to relativistic corrections, with the corresponding non-relativistic problems with potentials depending on r alone. We shall call this sector the RMS (reduced Minkowski space) [18,19].

We may see, moreover, that the RMS carries an important physical interpretation for the nature of the solutions of the differential equations by examining the appropriate variables describing the full space-like and RMS regions. The full space-like region is spanned by

$$x^0 = \rho \sinh\beta, \quad x^1 = \rho \cosh\beta \cos\phi \sin\theta \, x^2 = \rho \cosh\beta \sin\phi \sin\theta, \quad x^3 = \rho \cosh\beta \cos\theta \tag{14}$$

overall ρ from 0 to ∞, β in $(-\infty, \infty)$, ϕ in $(0, 2\pi)$ and θ in $(0, \pi)$. Separation of variables in this choice, however, leaves the variable β for last; the quantum number (separation constant) obtained in this

way has no obvious physical interpretation. Moreover, as found by Cook [15], the resulting spectrum for the Coulomb type potential (proportional to $1/\rho$) does not agree with the Balmer series.

On the other hand, the set of variables describing the RMS, running over the same range of parameters [16],

$$x^0 = \rho \sin\theta \sinh\beta, \quad x^1 = \rho \sin\theta \cosh\beta \cos\phi x^2 = \rho \sin\theta \cosh\beta \sin\phi, \quad x^3 = \rho \cos\theta, \quad (15)$$

cover the entire space within the RMS (for $x_1^2 + x_2^2 > t^2$). In this coordinatization, the separation constant for θ (at the last stage), which enters the radial equation and determines the corresponding spectrum, has the interpretation of the angular momentum quantum number $\ell(\ell+1)$.

As for (14), for $\beta \to 0$, these coordinates become the standard spherical representation of the three-dimensional space (at the "simultaneity" point $t = 0$, where ρ becomes r). Independently of the form of the potential $V(\rho)$, one obtains the same radial equation (in ρ) as for the non-relativistic Schrödinger equation (in r), and therefore, the same spectra (the two-body *mass* squared) for the reduced Hamiltonian. We shall discuss the relation of these results to the energy spectrum after writing the solutions. We summarize in the following the basic mathematical steps.

Assuming the total wavefunction (for $P \to P'$, a point on the continuum of the spectrum of the conserved operator P)

$$\Psi_{P'\tau}(X, x) = e^{iP'^\mu X_\mu} \psi_{P'\tau}(x), \quad (16)$$

the evolution equation for each value of the total energy momentum of the system is then

$$i\frac{\partial}{\partial\tau}\Psi_{P'\tau}(X, x) = (K_{CM} + K_{rel})\Psi_{P'\tau}(X, x) = \left[\frac{P'^2}{2M} + K_{rel}\right]\Psi_{P'\tau}(X, x). \quad (17)$$

For the case of discrete eigenvalues K_a of K_{rel}.

We then have the eigenvalue equation (cancelling the center of mass wave function factor and K_{CM} on both sides)

$$K_{rel}\psi^{(a)}(x) = K_a\psi^{(a)}(x) = (-(1/2m)\partial_\mu\partial^\mu + V(\rho))\psi^{(a)}(x). \quad (18)$$

Using the $O(3,1)$ Casimir operator, in a way quite analogous to the use of the square of the total angular momentum operator, the Casimir operator of the rotation group $O(3)$ in the non-relativistic case, we may separate the angular and hyperbolic angular degrees of freedom from the ρ dependence. There are two Casimir operators defining the representations of $O(3,1)$ [20–22]. The first Casimir operator is

$$\Lambda = \frac{1}{2}M_{\mu\nu}M^{\mu\nu}; \quad (19)$$

the second Casimir operator $\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}M_{\mu\nu}M_{\lambda\sigma}$ is identically zero for two particles without spin. Recalling that our separation into center of mass and relative motion is canonical, and that

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu; \quad (20)$$

using the canonical commutation relations, one finds that

$$\Lambda = x^2 p^2 + 2ix \cdot p - (x \cdot p)^2. \quad (21)$$

Since

$$x \cdot p \equiv x^\mu p_\mu = -i\rho\frac{\partial}{\partial\rho}, \quad (22)$$

so that

$$\Lambda = -\rho^2\partial^\mu\partial_\mu + 3\rho\frac{\partial}{\partial\rho} + \rho^2\frac{\partial^2}{\partial\rho^2},$$

or

$$-\partial_\mu \partial^\mu = -\frac{\partial^2}{\partial\rho^2} - \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{\Lambda}{\rho^2}. \tag{23}$$

Equation (18) can then be written as

$$K_a \psi^{(a)}(x) = \left\{ \frac{1}{2m}\left[-\frac{\partial^2}{\partial\rho^2} - \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{\Lambda}{\rho^2}\right] + V(\rho) \right\}\psi^{(a)}(x). \tag{24}$$

Choosing the RMS variables as we have defined them in (15), and with

$$L_i = \frac{1}{2}\epsilon_{ijk}(x^j p^k - x^k p^j), \tag{25}$$

corresponding to the definition of the non-relativistic angular momentum **L**, and

$$A^i = x^0 p^i - x^i p^0, \tag{26}$$

corresponding to the boost generator **A**,

$$\Lambda = \mathbf{L}^2 - \mathbf{A}^2. \tag{27}$$

We then find that

$$\Lambda = -\frac{\partial^2}{\partial\theta^2} - 2\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}N^2, \tag{28}$$

where

$$N^2 = L_3^2 - A_1^2 - A_2^2 \tag{29}$$

is the Casimir operator of the $O(2,1)$ subgroup of $O(3,1)$ leaving the z axis (and the RMS submanifold) invariant [18]. In terms of the RMS variables that we have defined above,

$$N^2 = \frac{\partial^2}{\partial\beta^2} + 2\tanh\beta\frac{\partial}{\partial\beta} - \frac{1}{\cosh^2\beta}\frac{\partial^2}{\partial\phi^2}. \tag{30}$$

We now proceed to separate variables and find the eigenfunctions. The solution of the general eigenvalue problem (24) can be written

$$\psi(x) = R(\rho)\Theta(\theta)B(\beta)\Phi(\phi), \tag{31}$$

with invariant measure in the $L^2(R^4)$ of the RMS

$$d\mu = \rho^3 \sin^2\theta \cosh\beta d\rho d\phi d\beta d\theta. \tag{32}$$

To satisfy the ϕ derivatives in (2.23), it is necessary to take

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{i[m+\frac{1}{2}]\phi}, \quad 0 \le \phi < 2\pi, \tag{33}$$

where we have indexed the solutions by the separation constant m. For the case m an integer, this is a double valued function. To be compatible with the conditions on the other factors, this is the necessary choice; one must use, in fact, $\Phi_m(\phi)$ for $m \ge 0$ and $\Phi_m^*(\phi)$ for $m < 0$.

It has been suggested by M. Bacry [23] that the occurrence of the half-integer in the phase is associated with the fact that the RMS is a connected, but not simply connected manifold. One can see this by considering the projective form of the restrictions

$$x^2 + y^2 + z^2 - t^2 > 0 \tag{34}$$

assuring that the events are relatively space-like, and

$$x^2 + y^2 - t^2 > 0, \tag{35}$$

assuring, in addition, that the relative coordinates lie in the RMS. Dividing (34) and (35) by t^2, and calling the corresponding projective variables X, Y, Z, we have from (34)

$$X^2 + Y^2 + Z^2 > 1, \tag{36}$$

the exterior of the unit sphere in the projective space, and from (35),

$$X^2 + Y^2 > 1, \tag{37}$$

the exterior of the unit cylinder along the z-axis. Identifying the points at infinity of the cylinder, we see that this corresponds to a torus with the unit sphere imbedded in the torus at the origin. Such a topological structure is associated with half-integer phase (e.g., [24]).

We now continue with our discussion of the structure of the solutions.

The operator Λ contains the $O(2,1)$ Casimir N^2; with our solution (38), we then have

$$N^2 B_{mn}(\beta) = \left[\frac{\partial^2}{\partial \beta^2} + 2\tanh\beta \frac{\partial}{\partial \beta} + \frac{(m + \frac{1}{2})^2}{\cosh^2 \beta}\right] B_{mn}(\beta) \equiv (n^2 - \frac{1}{4}) B_{mn}(\beta), \tag{38}$$

where n^2 is the separation constant for the variable β. The term $(m + \frac{1}{2})^2$ must be replaced by $(m - \frac{1}{2})^2 = (|m| + \frac{1}{2})^2$ for $m < 0$. We study only the case $m \geq 0$ in what follows. The remaining equation for Λ is then

$$\Lambda\Theta(\theta) = \left[-\frac{\partial^2}{\partial \theta^2} - 2\cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta}(n^2 - \frac{1}{4})\right]\Theta(\theta). \tag{39}$$

For the treatment of Equation (38), it is convenient to make the substitution

$$\zeta = \tanh\beta, \tag{40}$$

so that $-1 \leq \zeta \leq 1$. One then finds that for

$$B_{mn}(\beta) = (1 - \zeta^2)^{1/4} \hat{B}_{mn}(\zeta), \tag{41}$$

(38) becomes

$$(1 - \zeta^2)\frac{\partial^2 \hat{B}_{mn}(\zeta)}{\partial \zeta^2} - 2\zeta\frac{\partial \hat{B}_{mn}(\zeta)}{\partial \zeta} + \left[m(m+1) - \frac{n^2}{1 - \zeta^2}\right]\hat{B}_{mn}(\zeta) = 0. \tag{42}$$

The solutions are the associated Legendre functions of the first and second kind (Gel'fand [21]; see also Merzbacher [25]), $P_m^n(\zeta)$ and $Q_m^n(\zeta)$. The normalization condition on these solutions, with the measure (42) is

$$\int \cosh\beta |B(\beta)|^2 < \infty,$$

or, in terms of the variable ζ

$$\int_{-1}^{1} (1 - \zeta^2)^{-1} |\hat{B}(\zeta)|^2 d\zeta < \infty. \tag{43}$$

The second kind Legendre functions do not satisfy this condition. For the condition on the $P_m^n(\zeta)$, it is simplest to write the known result [26]

$$\int_{-1}^{1}(1-\zeta^2)^{-1}|P_{\mu+\nu}^{-\nu}(\zeta)|^2d\zeta = \frac{1}{\nu}\frac{\Gamma(1+\mu)}{\Gamma(1+\mu+2\nu)} \tag{44}$$

The normalized solutions (it is sufficient to consider $n \geq 0$) may be written as

$$\hat{B}_{mn}(\zeta) = \sqrt{n}\sqrt{[\Gamma(1+m+n)/\Gamma(1+m-n)]} \times P_m^{-n}(\zeta), \tag{45}$$

where $m \geq n$.

The case $n = 0$ must be treated with special care; it requires a regularization. For $n = 0$, the associated Legendre functions become the Legendre polynomials $P_m(\zeta)$. In terms of the integration on β, the factor $\cosh\beta = (1-\zeta^2)^{-1/2}$ in the measure is cancelled by the square of the factor $(1-\zeta^2)^{1/4}$ in the norm, so that the integration appears as

$$\int_{-\infty}^{\infty}|\hat{B}_m(\zeta)|^2d\beta.$$

The Legendre polynomials do not vanish at $\zeta = \pm 1$, so if $\hat{B}_m$ and P_m are related by a finite coefficient, the integral would diverge. When n goes to zero, associated with the ground state, the wave function spreads along the hyperbola labelled by ρ, going asymptotically to the light plane; the probability density with respect to intervals of β becomes constant for large β. The (regularized) expectation values reproduce the distribution of the Schrödinger bound states, although the spacetime wave function approaches that of a generalized eigenfunction.

To carry out the regularization, we take the limit as n goes continuously to zero after computation of scalar products. Thus, we assume the form

$$\hat{B}_m(\zeta) = \sqrt{\epsilon}(1-\zeta^2)^{\epsilon/2}P_m(\zeta), \tag{46}$$

with $\epsilon \to 0$ after computation of scalar products. This formula is essentially a residue of the Rodrigues formula

$$P_m^{-n}(\zeta) = (-1)^n(1-\zeta^2)^{n/2}\frac{d^n}{d\zeta^n}P_m(\zeta) \tag{47}$$

for $n \to 0$.

The operator for the differential Equation (2.17) for the eigenvalue of the reduced motion is invariant under the action of the Lorentz group. It follows from acting on the equation with the unitary representation of the Lorentz group that the eigenfunctions must be representations of that group [27] for each value of the eigenvalue. However, as one can easily see, the solutions that we found are, in fact, irreducible representations of $O(2,1)$, not, *a priori*, representations of the Lorentz group $O(3,1)$. We discuss below how to construct such a representation.

We have required that the wave functions be eigenfunctions of the Casimir operator (29) of the $O(2,1)$ subgroup. For the generators of $O(2,1)$, we note that

$$
\begin{aligned}
H_{\pm} \equiv A_1 \pm iA_2 &= e^{\pm i\phi}\left(-i\frac{\partial}{\partial\beta} \pm \tanh\beta\frac{\partial}{\partial\phi}\right), \\
L_3 &= -i\frac{\partial}{\partial\phi}, \\
A_3 &= -i\left(\cot\theta\cosh\beta\frac{\partial}{\partial\beta} - \sinh\beta\frac{\partial}{\partial\theta}\right) \\
L_{\pm} &= L_1 \pm iL_2 \\
&= e^{\pm i\phi}\left(\pm\cosh\beta\frac{\partial}{\partial\theta} - \sinh\beta\cot\theta\frac{\partial}{\partial\beta}\right. \\
&\quad + \left. i\frac{\cot\theta}{\cosh\beta}\frac{\partial}{\partial\phi}\right).
\end{aligned}
\tag{48}
$$

It then follows that $H_\pm$ are raising and lowering operators for m on the functions

$$
\begin{aligned}
\zeta_{n+k}^{-n}(\zeta,\phi) &\equiv B_{n+k,n}(\beta)\Phi_{n+k}(\phi) \\
&= (1-\zeta^2)^{1/4}\hat{B}_{n+k,n}(\zeta)\Phi_{n+l}(\phi),
\end{aligned}
\tag{49}
$$

where it is convenient to replace m by $n+k$. With the relation

$$
[L_3, H_\pm] = \pm H_\pm
\tag{50}
$$

one can show [19] that

$$
H_+\chi_{n+k}^{-n}(\zeta,\phi) = i\sqrt{(k+1)(2n+k+1)}\chi_{n+k+1}^{-n}(\zeta,\phi)
\tag{51}
$$

and that

$$
H_-\chi_{n+k+1}^{-n}(\zeta,\phi) = -i\sqrt{(k+1)(2n+k+1)}\chi_{n+k}^{-n}(\zeta,\phi).
\tag{52}
$$

The complex conjugate of χ_{n+k}^{-n} transforms in a similar way, resulting in a second (inequivalent) representation of $O(2,1)$ with the same value of the $O(2,1)$ Casimir operator (these states correspond to replacement of $m+\frac{1}{2}$ by $m-\frac{1}{2}$ for $m<0$, and are the result of charge conjugation. Since the operators A_1, A_2 and L_3 are Hermitian, complex conjugation is equivalent to the transpose. Replacing these operators by their negative transpose (defined by C), leaves the commutation relations invariant. Thus, the action on the complex conjugate states involves

$$
H_-^C = -H_+^* = H_-, \quad H_+^C = -H_-^* = H_+, L_3^C = -L_3^* = L_3;
\tag{53}
$$

These are precisely the operators under which the complex conjugate states transform, and this operation therefore corresponds to charge conjugation.

The wave functions we have obtained are irreducible representations of $O(2,1)$, determined by the differential equations with solutions restricted to support in a particular choice of orientation of the RMS. To construct representations of $O(3,1)$, let us consider first the well-established method which is effective in constructing representations of $O(3,1)$ from representations of $O(3)$, a group that we would have found if we were working with solutions in the time-like region [20,26], called the *ladder representation*. It follows from the Lie algebra of $O(3,1)$ that the $O(3)$ subgroup Casimir operators $\ell(\ell+1)$ are stepped by $\ell \to \ell \pm 1$ under the action of the boost from $O(3,1)$. The whole set of representations of $O(3)$, from $\ell = 0$ to ∞ form a representation of $O(3,1)$. Each of the representations of $O(3)$ entering this tower are trivially normalizable, since they are of dimension $(2\ell+1)$. However, attempting to apply this method to the representations of $O(2,1)$ fails because the application of the Lie algebra to this set connects the lowest state of the tower with the ground state which, as we have shown, requires regularization. The action of the algebra does not provide such a regularization, and therefore the method is inapplicable.

We therefore turn to the method of *induced representations* [27]. We may apply this method to constructing the representations of $O(3,1)$ based on an induced representation with the $O(2,1)$ "little group", based on a space-like vector corresponding to the choice of the z axis. We shall discuss his method in detail below.

We first record the solutions of the Equation (18).

Defining

$$
\xi = \cos\theta
\tag{54}
$$

and the functions

$$
\hat{\Theta}(\theta) = (1-\xi^2)^{1/4}\Theta(\theta),
\tag{55}
$$

Equation (39) becomes

$$\frac{d}{d\xi}\left((1 - \xi^2)\frac{d}{d\xi}\hat{\Theta}(\theta)\right) + \left(\ell(\ell+1) - \frac{n^2}{1 - \xi^2}\right)\hat{\Theta}(\theta) = 0, \tag{56}$$

where we have defined

$$\Lambda = \ell(\ell+1) - \frac{1}{4}. \tag{57}$$

The solutions are proportional to the associated Legendre functions of the first or second kind, $P_\ell^n(\xi)$ or $Q_\ell^n(\xi)$. For $n \neq 0$, the second kind functions are not normalizable. We therefore reject these.

The normalizable irreducible representations of $O(2,1)$ are single or double valued, and hence m must be integer or half integer. As we have seen, k is integer valued, and therefore n must be integer or half integer also. Normalizability conditions on the associated Legendre functions then require that ℓ be respectively, positive half-integer or integer. The lowest mass state, as we shall see from the spectral results, corresponds to $\ell = 0$, and hence we shall consider only integer values of ℓ. Therefore, n and m must be integer.

We now turn to the solution of the radial equations, containing the spectral content of the theory. With the evaluation of Λ in (57), we may write the radial equation as

$$\left[\frac{1}{2m}\left(-\frac{\partial^2}{\partial\rho^2} - \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{\ell(\ell+1) - \frac{3}{4}}{\rho^2}\right) + V(\rho)\right]R^{(a)}(\rho) = K_a R^{(a)}(\rho). \tag{58}$$

If we put

$$R^{(a)}(\rho) = \frac{1}{\sqrt{\rho}}\hat{R}^{(a)}(\rho), \tag{59}$$

Equation (58) becomes precisely the non-relativistic Schrödinger equation for $\hat{R}^{(a)}$ in the variable ρ, with potential $V(\rho)$ (the measure for these functions is, from (32), just $\rho^2 d\rho$, as for the non-relativistic theory)

$$\frac{d^2\hat{R}^{(a)}(\rho)}{d\rho^2} + \frac{2}{\rho}\frac{d\hat{R}^{(a)}(\rho)}{d\rho} - \frac{\ell(\ell+1)}{\rho^2}\hat{R}^{(a)}(\rho) + 2m(K_a - V(\rho))\hat{R}^{(a)}(\rho) = 0. \tag{60}$$

3. The Spectrum

The lowest eigenvalue K_a, as for the energy in the non-relativistic Schrödinger equation, corresponds to the $\ell = 0$ state of the sequence $\ell = 0, 1, 2, 3, ...$, and therefore the quantum number ℓ plays a role analogous to the orbital angular momentum. This energy is of a lower value than achievable with wave functions with support in the full space-like region [15] and the relaxation of the system to wave functions with support in the RMS may be thought of, in this sense, as a spontaneous symmetry breaking (we thank A. Ashtekar for his remark on this point [28]).

The value of the full generator K is then determined by these eigenvalues and the value of the center of mass total mass squared operator, i.e.,

$$K = \frac{P^\mu P_\mu}{2M} + K_a. \tag{61}$$

The first term corresponds to the total effective rest mass of the system. In particular, the invariant mass squared of the system is given by (sometimes called the Mandelstam variable s [29])

$$s_a \equiv -P_a^2 = 2M(K_a - K). \tag{62}$$

This total center of mass momentum is observed in the laboratory in scattering and decay processes, where it is defined as the sum of the outgoing momenta squared. In the case of two particles, it would

be given by $-(p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu})$, as we have defined it in (62). This quantity is given in terms of total energy and momentum by

$$s_a = E_T^2 - \mathbf{P}_T^2, \tag{63}$$

and in the center of momentum frame, for $\mathbf{P} = 0$, is just E_T^2.

To extract information about the *energy* spectrum, we must therefore make some assumption on the value of the conserved quantity K. In the case of a potential that vanishes for large ρ, we may consider the two particles to be asymptotically free, so the effective Hamiltonian in this asymptotic region

$$K \cong \frac{p_1^\mu p_{1\mu}}{2M_1} + \frac{p_2^\mu p_{2\mu}}{2M_2}. \tag{64}$$

Furthermore, assuming that the two particles at very large distances, in accordance with our experience, undergo a relaxation to their mass shells, so that $p_i^2 \cong -M_i^2$. In this case, K would be assigned the value

$$K \cong -\frac{M_1}{2} - \frac{M_2}{2} = -\frac{M}{2}. \tag{65}$$

The two particles in this asymptotic state would, for the bound-state problem, be at the ionization point. If these assumptions are approximately valid, we find for the total energy, which we now label E_a,

$$E_a/c \cong \sqrt{M^2 c^2 + 2MK_a}, \tag{66}$$

where we have restored the factors c.

In the case of excitations small compared to the total mass of the system, we may factor out Mc and represent the result in a power series expansion

$$E_a \cong Mc^2 + K_a - \frac{1}{2}\frac{K_a}{Mc^2} + \dots, \tag{67}$$

so that the energy spectrum is just the set $\{K_a\}$ up to relativistic corrections. Thus, the spectrum for the $1/\rho$ potential is just that of the non-relativistic hydrogen problem up to relativistic corrections, of order $1/c^2$.

If the spectral set $\{K_a\}$ includes large negative values, the result (66) could become imaginary, indicating the possible onset of instability. However, the asymptotic condition imposed on the evaluation of K must be re-examined in this case. If the potential grows very rapidly as $\rho \to 0$, then at large space-like distances, where the hyperbolic surfaces $\rho = const$ approach the light cone, the Euclidean measure d^4x (thought of, in this context, as small but finite) on the R^4 of spacetime starts to cover very singular values and the expectation values of the Hamiltonian at large space-like distances may not permit the contribution of the potential to become negligible; it may have an effectively very long range. This effect can occur in the transverse direction to the z axis along the tangent to the light cone; the hyperbolas cannot reach the light cone in the z direction, which may play an important role in the modelling the behavior of the transverse scattering amplitudes in high energy scattering studied, for example, by Hagedorn [30].

4. Some Examples

In this section, we give the examples of the Coulomb potential and the oscillator.

For the analog of the Coulomb potential, we take

$$V(\rho) = -\frac{Ze^2}{\rho}. \tag{68}$$

As we have remarked above, for $c \to \infty$, this potential reduces to $1/r$, the usual Coulomb, and therefore the spectrum must reduce to the usual Balmer series in this limit.

In this case, the spectrum, according to the solutions above, is given by

$$K_a = -\frac{Z^2 m e^4}{2\hbar^2 (\ell + 1 + n_a)^2}, \tag{69}$$

where $n_a = 0, 1, 2, 3.....$ The wave functions $\hat{R}(\rho)^a$ are the usual hydrogen functions

$$\hat{R}_{n_a \ell}(\rho) = \sqrt{\frac{Z n_a!}{(n_a + \ell + 1)^2 (n_a + 2\ell + 1)}} e^{-x/2} x^{\ell+1} L_{n_a}^{2\ell+1}(x), \tag{70}$$

where $L_{n_a}^{2\ell+1}$ are the Laguerre polynomials, and the variable x is defined by

$$x = \frac{(2Z\rho/a_0)}{(n_a + \ell + 1)}, \tag{71}$$

and $a_0 = \hbar^2/me^2$. The *size* of the bound state, which is related to the atomic form factor, is measured according to the variable ρ [31]. For the lowest level (using the regularized functions) $n_a = \ell = 0$,

$$< \rho >_{n_a = \ell = 0} = \frac{3}{2} a_0. \tag{72}$$

The total mass spectrum, given by (62), is then

$$s_{n_a, \ell} \cong M^2 c^2 - \frac{m M Z^2 e^4}{\hbar^2 (n_a + \ell + 1)^2}. \tag{73}$$

For the case that the non-relativistic spectrum has value small compared to the sum of the particle rest masses, we may use the approximate relation (66) to obtain

$$E_{a,\ell} \cong M c^2 - \frac{Z^2 m e^4}{2\hbar^2 (n_a + \ell + 1)^2} - \frac{1}{8} \frac{Z^4 m^2 e^8}{M c^2 \hbar^4 (n_a + \ell + 1)^4} + ... \tag{74}$$

The lowest order relativistic correction to the rest energy of the two body system with Coulomb-like potential is then

$$\frac{\Delta(E_{a,\ell} - M c^2)}{E_{a,\ell} - M c^2} = \frac{Z \alpha^2}{4} \left(\frac{m}{M}\right) \frac{1}{(n_a + \ell + 1)^2}. \tag{75}$$

For positronium, $\Delta(E - M c^2) \sim 2 \times 10^{-5}$ eV it is about one part in 10^5, about 2% of the positronium hyperfine splitting of 8.4×10^{-4} eV [32]. We see quantitatively that the relativistic theory gives results that are consistent with the known data on these experimentally well studied bound-state systems.

For the four-dimensional oscillator, with $V(\rho) = \frac{1}{2} m \omega^2 \rho^2$, Equation (60) takes the form

$$\frac{d^2 \hat{R}^{(a)}(\rho)}{d\rho^2} + \frac{2}{\rho} \frac{d\hat{R}^{(a)}(\rho)}{d\rho} - \frac{\ell(\ell+1)}{\rho^2} \hat{R}^{(a)}(\rho) + 2m \left(K_a - \frac{m^2 \omega^2}{\hbar^2} \rho^2 - \frac{\ell(\ell+1)}{\rho^2}\right) \hat{R}^{(a)}(\rho) = 0. \tag{76}$$

With the transformation

$$\hat{R}^{(a)}(\rho) = x^{\ell/2} e^{-x/2} w^{(a)}(x), \tag{77}$$

for

$$x = \frac{m\omega}{\hbar} \rho^2, \tag{78}$$

we obtain the equation

$$x \frac{d^2 w^{(a)}}{dx^2} + \left(\ell + \frac{3}{2} - x\right) \frac{dw^{(a)}}{dx} + \frac{1}{2} \left(\ell + \frac{3}{2} - \frac{K_a}{\hbar \omega}\right) w^{(a)} = 0 \tag{79}$$

Normalizable solutions, the Laguerre polynomials $L_{n_a}^{\ell+1/2}(x)$, exist [Landau (1965)] when the coefficient of $w^{(a)}(x)$ is a negative integer, so that the eigenvalues are

$$K_a = \hbar\omega(\ell + 2n_a + \frac{3}{2}),\tag{80}$$

where $n_a = 0, 1, 2, 3, \ldots$ The total mass spectrum is given by (62) as

$$s_{n_a,\ell} = -2MK + 2M\hbar\omega(\ell + 2n_a + \frac{3}{2}),\tag{81}$$

Please note that the "zero point" term is $\frac{3}{2}$, indicating that in the RMS, in the covariant equations there are effectively three intrinsic degrees of freedom, as for the non-relativistic oscillator.

The choice of K is arbitrary here, since there is no ionization point for the oscillator, and no *a priori* way of assigning it a value; setting $K = -\frac{Mc^2}{2}$ as for the Coulomb problem (a choice that may be justified by setting the spring constant equal to zero and adiabatically increasing it to its final value), one obtains, for small excitations relative to the particle masses,

$$E_a \cong Mc^2 + \hbar\omega\left(\ell + 2n_a + \frac{3}{2}\right) - \frac{1}{2}\frac{\hbar^2\omega^2(\ell + 2n_a + \frac{3}{2})^2}{Mc^2} + \ldots\tag{82}$$

Feynman, Kislinger and Ravndal [33], Kim and Noz [34] and Leutwyler and Stern [35] have studied the relativistic oscillator and obtained a positive spectrum by imposing a subsidiary condition suppressing time-like excitations, which lead, in the formalism of annihilation-creation operators to generate the spectrum, to negative norm states ("ghosts"). There are no ghost states in the covariant treatment we discuss here, and no extra constraints invoked in finding the spectrum. The solutions are given in terms of Laguerre polynomials, but unlike the case of the standard treatment of the 4D oscillator, in which $x^\mu \pm ip^\mu$ are considered annihilation-creation operators, the spectrum generating algebra (for example, Dothan [36]) for the covariant SHP oscillator has been elusive [37].

5. The Induced Representation

We have remarked that the solutions of the invariant two-body problem results in solutions that are irreducible representations of $O(2,1)$, in fact, the complex representations of its covering group $SU(1,1)$, and pointed out that the ladder representations generated by the action of the Lorentz group on these states cannot be used to obtain representations of the full Lorentz group $O(3,1)$ or its covering $SL(2,C)$. Since the differential equations defining the physical states are covariant under the action of $O(3,1)$, the solutions must be representations of $O(3,1)$. To solve this problem, one observes [1] that the $O(2,1)$ solutions are constructed in the RMS which is referred to the space-like z axis. Under a Lorentz boost, the entire RMS turns, leaving the light cone invariant. After this transformation, the new RMS is constructed on the basis of a new space-like direction which we call here m^μ. However, the differential equations remain identically the same since the operator form of these equations is invariant. The change of coordinates to RMS variables has the same form as well, and therefore the set of solutions of these equations have the same structure. These functions are now related to the new "z" axis. Under the action of the full Lorentz group the wave functions undergo a transformation involving a linear combination of the set of eigenfunctions found in the previous section; this action does not change the value of the $SU(1,1)$ (or $O(2,1)$) Casimir operator; together with the change in direction of the vector m^μ, they provide an induced representation of $SL(2,C)$ (or $O(3,1)$ with little group $SU(1,1)$ in the same way that relativistic spin is a representation of $SL(2,C)$ with $SU(2)$ little group [27]. The coefficients in this superposition then play the role of the Wigner D functions in the induced representation of relativistic particles with spin.

Let us define the coordinates $\{y_\mu\}$, isomorphic to the set $\{x_\mu\}$, defined in an accompanying frame for the RMS(m_μ)), with y_3 along the axis m_μ. Along with infinitesimal operators of the $O(2,1)$

generating changes within the RMS(m_μ)), there are generators on $O(3,1)$ which change the direction of m_μ; as for the induced representations for systems with spin [27], the Lorentz group contains these two actions, and therefore both Casimir operators are essential to defining the representations, i.e., both

$$c_1 \equiv \mathbf{L}(m)^2 - \mathbf{A}(m)^2 \tag{83}$$

and

$$c_2 \equiv \mathbf{L}(m) \cdot \mathbf{A}(m), \tag{84}$$

which is not identically zero, and commutes with c_1.

In the following, we construct functions on the *orbit* of the $SU(1,1)$ little group representing the full Lorentz group; along with the designation of the point on the orbit, labelled by m_μ, these functions constitute a description of the physical state of the system.

It is a quite general result that the induced representation of a non-compact group contains all of the irreducible representations. We decompose the functions along the orbit into basis sets corresponding to eigenfunctions for the $O(3)$ subgroup Casimir operator $\mathbf{L}(m)^2 \to L(L+1)$ and $L_1 \to q$ that take on values that persist along the orbit; these solutions correspond to the *principal series* of Gel'fand [21]. These quantum numbers for the induced representation do not correspond directly to the observed angular momenta of the system. The values that correspond to spectra and wavefunctions with non-relativistic limit coinciding with those of the non-relativistic problem, are those with L *half-integer* for the lowest Gel'fand L level. The partial wave expansions in scattering theory, which we discuss in a later chapter (for the continuous spectrum of K_{rel}), depend on the quantum number ℓ of the $O(3,1)$ defined on the whole space, defined by the quantum form of (95), and a magnetic quantum number, which we shall call n, associated with the Casimir of the $SU(1,1)$ discussed above, then playing the role of the magnetic quantum number, as discussed in the previous section for the bound-state problem. In fact, in the Gel'fand classification, the two Casimir operators take on the values $c_1 = L_0^2 + L_1^2 - 1$, $c_2 = -iL_0L_1$, where L_1 is pure imaginary and, in general, L_0 is integer or half-integer. In the non-relativistic limit, the action of the group on the relative coordinates becomes deformed in such a way that the $O(3,1)$ goes into the non-relativistic $O(3)$, and the $O(2,1)$ into the $O(2)$ subgroup in the initial configuration of the RMS based on the z axis.

The representations that we shall obtain, in the principal series of Gel'fand [21], are unitary in a Hilbert space with scalar product product that is defined by an integration invariant under the full $SL(2,C)$, including an integration over the measure space of $SU(1,1)$, carried out in the scalar product in $L^2(R^4 \subseteq \text{RMS}(m_\mu))$, for each m_μ (corresponding to the orientation of the new z axis, and an integration over the measure of the coset space $SL(2,C)/SU(1,1)$; the complete measure is $d^4y d^4m \delta(m^2 - 1)$, i.e., a probability measure on R^7, where $y_\mu \in \text{RMS}(m_\mu)$. The coordinate description of the quantum state therefore corresponds to an ensemble of (relatively defined) events lying in a set of RMS(m_μ)'s over all possible space-like $\{m_\mu\}$.

A coordinate system oriented with its z axis along the direction m_μ, as referred to above, can be constructed by means of a coordinate transformation of Lorentz type (here m represents the space-like orientation of the transformed RMS, not to be confused with a magnetic quantum number),

$$y_\mu = L(m)_\mu{}^\nu x_\nu. \tag{85}$$

For example, if we take a vector x_μ parallel to m_μ, with $x_\mu = \lambda m_\mu$, then the corresponding y_μ is λm_μ^0, with m_μ^0 in the direction of the initial orientation of the orbit, say, the z axis. This definition may be replaced by another by right multiplication of an element of the stability group of m_μ and left multiplication by an element of the stability group of m_μ^0, constituting an isomorphism in the RMS.

The variables y_μ may be parametrized by the same trigonometric and hyperbolic functions as in (15) since they span the RMS, and provide a complete characterization of the configuration space in

the RMS(m_μ) that is universal in the sense that it is the same in every Lorentz frame. It is convenient to define the functions

$$\psi_m(y) = \phi_m(L^T(m)y) = \phi_m(x) \tag{86}$$

We can then define the map of the Hilbert spaces associate with each m_μ in the foliation $\mathcal{H}_m \to \mathcal{H}_{\Lambda m}$ such that the state vectors are related by the norm preserving transformation

$$\Psi^\Lambda_{\Lambda m} = U(\Lambda)\Psi_m. \tag{87}$$

In the new Lorentz frame (with $y = L(\Lambda m)x$),

$$\phi^\Lambda_{\Lambda m}(x) = {}_{\Lambda m} < x|\Psi^\Lambda_{\Lambda m} > = {}_{\Lambda m} < x|U(\Lambda)\Psi_m > = \phi^\Lambda_{\Lambda m}(L^T(\Lambda m)y) = \psi^\Lambda_{\Lambda m}(y). \tag{88}$$

If $\phi_m(x)$ is scalar under Lorentz transformation, so that (we assume no additional phase)

$$\phi^\Lambda_{\Lambda m}(\Lambda x) = \phi_m(x), \tag{89}$$

it follows from (88) that

$$U(\Lambda)|x >_m = |\Lambda x >_{\Lambda m}. \tag{90}$$

The wave function $\phi^\Lambda_{\Lambda m}(x)$ describes a system in a Lorentz frame in motion with respect to the frame in which the state is described by $\phi_m(x)$, and for which the support is in the RMS($(\Lambda m)_\mu$). The value of this function at x in the new frame is determined by its value at $\Lambda^{-1}x$ in the original frame; moreover, the sub-ensemble associated with values of m_μ over the orbit in the new frame is determined by the sub-ensemble associated with the values of $(\Lambda^{-1}m)_\mu$ in the old frame. We define the description of the state of the system in the new frame in terms of the set (over $\{m_\mu\}$) of transformed wave functions

$$\psi^\Lambda_m(y) \equiv \phi_{\Lambda^{-1}m}(\Lambda^{-1}x) = \psi^\Lambda_m(D^{-1}(\Lambda, m)y) \tag{91}$$

where we have used (88) (the transformed function has support oriented with m_μ) and defined the (pseudo) orthogonal matrix (we define a "matrix" A as $\{A_\mu{}^\nu\}$)

$$D(\Lambda, m) = L(m)\Lambda L^T(\Lambda^{-1}m). \tag{92}$$

The transformation $D^{-1}(\Lambda, m)$ stabilizes m^0_μ, and is therefore in the $O(2, 1)$ subgroup that leaves the RMS of the original system invariant. Equation (91) defines an induced representation of $SL(2, C)$, the double covering of $O(3, 1)$.

Classification of the orbits of the induced representation are determined by the Casimir operators of $SL(2, C)$, defined as differential operators on the functions $\psi_m(y)$ of (86), i.e., the operators defined in (83) and (84). To define these variables as differential operators on the space $\{y\}$, we study the infinitesimal Lorentz transformations

$$\Lambda \cong 1 + \lambda, \tag{93}$$

for which

$$\psi^{1+\lambda}{}_m(y) = \psi_{m-\lambda m}(D^{-1}(1+\lambda, n)y), \tag{94}$$

and λ is an infinitesimal Lorentz transformation (antisymmetric). To first order, the little group transformation is

$$D^{-1}(1+\lambda, n) \cong 1 - (d_m(\lambda)L(m))L^T(m) - L(m)\lambda L^T(m), \tag{95}$$

where d_m is a derivative with respect to m_μ holding y_μ fixed,

$$d_m(\lambda) = \lambda_\mu{}^\nu m_\nu \frac{\partial}{\partial n_\mu}. \tag{96}$$

From the property $L(m)L^T(m) = 1$, it follows that

$$(d_m(\lambda)L(m))L^T(m) = -L(m)(d_m(\lambda)L^T(m)), \tag{97}$$

so that (95) can be written as

$$D^{-1}(1+\lambda, n) \cong 1 + L(m)(d_n(\lambda)L^T(m) - \lambda L^T(m)) \equiv 1 - G_m(\lambda). \tag{98}$$

For the transformation of ψ_m we then obtain

$$\psi^{1+\lambda}{}_m(y) \cong \psi_m(y) - d_m(\lambda + g_m(\lambda))\psi_m(y), \tag{99}$$

where

$$g_m(\lambda) = G_m(\lambda)_\mu{}^\nu y_\nu \frac{\partial}{\partial y_\mu}. \tag{100}$$

Equation (99) displays explicitly the effect of the transformation along the orbit and the transformation within the little group.

The algebra of these generators of the Lorentz group are investigated in [1]; the closure of this algebra follows from the remarkable property of compensation for the derivatives of the little group generators along the orbit (behaving as a covariant derivative in differential geometry). The general structure we have exhibited here is a type of fiber bundle, sometimes called a *Hilbert bundle*, consisting of a set of Hilbert spaces on the base space of the orbit; in this case, the fibers, corresponding to these Hilbert spaces, transform under the little group $O(2,1)$.

There are functions on the orbit with definite values of the two Casimir operators, as well as $\mathbf{L}(m)^2$ and $L_1(m)$; one finds the Gel'fand Naimark canonical representation with decomposition over the $SU(2)$ subgroup of $SL(2,C)$, enabling an identification of the angular momentum content of the representations [17]. With a consistency relation between the Casimir operators (for the solution of the finite set of equations involving functions on the hyperbolic parameters of the space-like four vector m_μ), we find that we are dealing with the principal series of Gel'fand [20,21].

6. Conclusions

We have reviewed and discussed the symmetry of the two-body central potential problem in the relativistically covariant framework of the SHP theory. The solutions of the Stuekelberg-Schrödinger equation with support in the full space-like region of the Minkowski space provide a spectrum that does not agree with the solutions of the non-relativistic Schrödinger equation. Guided by the work of Zmuidzinas [15] we used variables in the Minkowski configuration space that span a space-like $O(2,1)$ invariant subspace of the full Minkowski space for the relative coordinates. In this subspace the spectrum agrees, up to relativistic corrections, with the non-relativistic Schrödinger spectrum.

In this subspace, which we call the RMS (reduced Minkowski space), the eigenfunctions of the stationary Stueckelberg-Schrödiger equation form representations of the orthogonal group $O(2,1)$. However, the Hamiltonian operator is $O(3,1)$ invariant, which implies that the solutions must be representations of $O(3,1)$. Extending the $O(2,1)$ representations by the method of stepping to get a ladder representation leads to a non-normalizable state, and we therefore turned to an *induced representation* [19]. This representation was constructed following Wigner's method [27] for dealing with spin in a relativistic framework, but with the non-compact $O(2,1)$ little group instead of the $O(3)$ little group used by Wigner to describe spin. One might think of the reduced symmetry $O(2,1)$, as suggested by Ashtekar [28] as a spontaneous symmetry breaking (the ground state has lower energy than for the solutions in the full space-like region). This construction leads to eigenfunctions for the two-body problem that have the intrinsic spinorial property of being double valued, perhaps a reflection of the topological properties of the $O(2,1)$ invariant submanifold [23].

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